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Addition laws in introductory physics*

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Abstract
We propose a unified approach to addition of some physical quantities (among which resistors and capacitors are the most well known) that are usually encountered in introductory physics such that the formulae required to solve problems are always simply additive. This approach has the advantage of being consistent with the intuition of students. To demonstrate the effectiveness of our approach, we propose and solve several problems. We hope that this paper can serve as a resource paper for problems on the subject.

1. Introduction

All introductory physics textbooks, with or without calculus, cover the addition of both resistances and capacitances in series and in parallel. The formulae for adding resistances

$$R = R_1 + R_2 + \cdots, \qquad \text{in series,} \tag{1}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots, \qquad \text{in parallel,} \tag{2}$$

and capacitances

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots, \qquad \text{in series,} \tag{3}$$

$$C = C_1 + C_2 + \cdots, \qquad \text{in parallel,} \tag{4}$$

are well known and well studied in all the books.

In books with calculus there are often end-of-chapter problems in which students must find R and C using continuous versions of equations (1) and (4) [3, 9, 11–13]. However, we have found *none* which includes problems that make use of continuous versions of equations (2) and (3) [3, 4, 7, 9, 11–13]. Students who can understand and solve the first class of problems should be able to handle the second class of problems, as well. We feel that continuous problems that make use of all four equations should be shown to the students in order to give

* This paper is an abridged version of paper [2] posted on the physics archive of www.arXiv.org.

them a global picture of how calculus is applied to physical problems. Physics contains much more than mathematics. When integrating quantities in physics, *the way we integrate them is motivated by the underlying physics*. Students often forget the physical reasoning and they tend to add (integrate) quantities only in one way.

In this paper, we introduce an approach to solving continuous versions of equations (2) and (3) that is as straightforward and logical for the students as solving continuous versions of equations (1) and (4). We then extend the logic to the addition of other quantities encountered in undergraduate introductory physics. This demonstrates that the method is not specific to resistors and capacitors but general and includes all quantities obeying similar addition laws.

The organization of this paper is as follows: section 2 discusses many physical quantities taken from introductory physics that obey similar addition laws. Among these quantities, resistance and capacitance are the most well known to students. Inductance is also known but probably not mastered at the level it should be. Elasticity is somewhat known, but thermal resistance, diffusion resistance and viscous resistance are almost unknown to students. As a result, for resistance and capacitance we only remind readers of the basic formulae, while for the rest of the quantities we expand the discussion to some length so the students will become familiar with the physical background. In section 3, we present basic applications of the addition formulae. This section is meant to demonstrate in a simple way how one chooses the correct addition formula (in series or in parallel), given a problem. In section 4 we solve several additional problems that make use of the main idea. In each problem, we have chosen one representative quantity. However, the reader must realize that the same problem can be stated for any of the quantities given in section 2, not just the chosen quantity.

We hope that this paper will motivate teachers to explain to students the subtle points between 'straight integration' as taught in calculus and 'physical integration' to find a physical quantity.

2. Basic formulae

2.1. Resistance

The basic formula to compute resistance R is the formula of a uniform cylindrical resistor,

$$R = \rho \frac{L}{A}$$

where ρ is the resistivity of the material, L is the length of the conductor and A the cross-sectional area. Written as conductance[1], this is

$$G = \frac{1}{R},$$

and

$$G = \sigma \frac{A}{L},$$

where $\sigma = 1/\rho$ is the conductivity of the material.

2.2. Capacitance

The basic formula to compute capacitance C is that of a parallel-plate capacitor filled with a uniform dielectric material,

$$C = \varepsilon_0 \kappa \frac{A}{d},$$

[1] Often the term *conductivity* is used for G. However, the term *conductance* is in uniform linguistic agreement with the rest of the terminology.

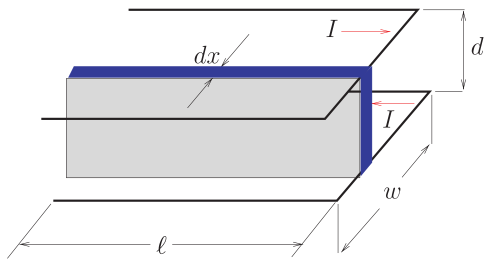

Figure 1. A parallel-plate inductor. The figure also shows a partition into infinitesimal inductors connected in parallel.

where κ is the dielectric constant, A is the area of the plates, and d is the distance between the plates. We will call the inverse of the capacitance

$$D = \frac{1}{C},$$

the *incapacitance* of the capacitor. For a parallel-plate capacitor

$$D = \frac{1}{\varepsilon_0 \kappa} \frac{d}{A}.$$

2.3. Inductance

Calculation of inductance is usually based on the definition

$$L = \frac{\Phi_B}{I}, \tag{5}$$

where Φ_B is the flux of the magnetic field produced by the current I; therefore some discussion is necessary regarding our point of view.

It is well known that the inductance for a parallel-plate inductor is given by

$$L = \mu_0 \frac{A}{w}, \tag{6}$$

where $A = \ell d$ is the area of the cross-section and w the width of the parallel wires (see figure 1).

We will call the inverse of the inductance

$$K = \frac{1}{L},$$

the *deductance* of the inductor. Therefore, the deductance of a parallel-plate inductor is

$$K = \frac{1}{\mu_0} \frac{w}{A}.$$

2.4. Transport phenomena

Transport phenomena are irreversible processes that occur in systems that are not in statistical equilibrium. In these systems, there is a net transfer of energy, matter or momentum.

Imagine a cylinder made from a uniform conducting material whose bases are kept at different temperatures. Then, due to the temperature difference ΔT between the bases, heat will flow from one base to the other. The rate at which heat flows, i.e.

$$I_{\rm th} = \frac{\Delta Q}{\Delta t},$$

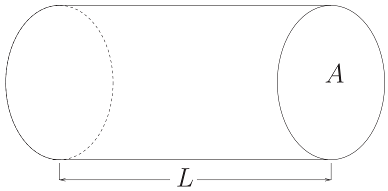

Figure 2. A uniform cylinder of length/height L and base area A comprises the central idea in our calculations. See table 1 for the formulae of all quantities for such a cylinder (d or w are also used as alternative symbols for L in that table).

is called the *thermal current*. It is known, see for example [1, 11], that

$$I_{\mathrm{th}} = \sigma_{\mathrm{th}} A \frac{\Delta T}{L}, \tag{7}$$

where L is the length of the cylinder, A is its cross-section, and σ_{th} is a constant characteristic of the material, called the *thermal conductivity*. We define the inverse of the thermal conductivity $\rho_{\mathrm{th}} = 1/\sigma_{\mathrm{th}}$ to be the *thermal resistivity* of the material. Equation (7) is sometimes referred to as *Fourier's law* for the flow of energy.

The *thermal resistance* of the cylinder is then defined by

$$\mathcal{R}_{\mathrm{th}} = \frac{\Delta T}{I_{\mathrm{th}}}. \tag{8}$$

Note the analogy with the standard resistance: $R = \Delta V/I$. Potential difference is the reason electric current flows. Here, temperature difference is the reason behind the thermal current. Comparing the two formulae (7) and (8) we have written above, we arrive at

$$\mathcal{R}_{\mathrm{th}} = \rho_{\mathrm{th}} \frac{L}{A}, \tag{9}$$

an expression almost identical to that of the electric resistance for the cylinder.

We define the *thermal conductance* as

$$\mathcal{G}_{\mathrm{th}} = \frac{1}{\mathcal{R}_{\mathrm{th}}}.$$

This implies that for the uniform cylinder

$$\mathcal{G}_{\mathrm{th}} = \sigma_{\mathrm{th}} \frac{A}{L} \tag{10}$$

Now, imagine a cylinder filled with gas such that the particle densities n_1 and n_2 of the gas at the bases are kept constant at different values. Then, due to the density difference Δn between the bases, particles will flow from one base to the other. The rate at which particles flow, i.e.

$$I_{\mathrm{diff}} = \frac{\Delta n}{\Delta t},$$

is called the *particle current*. It is known, see for example [1], that

$$I_{\mathrm{diff}} = \sigma_{\mathrm{diff}} A \frac{\Delta n}{L}, \tag{11}$$

where L is the length of the cylinder, A is its cross-section, and σ_{diff} is a constant characteristic of the material called the *diffusion coefficient*. Another name, in the spirit of what we have been discussing, would be *diffusion conductivity*. The inverse of σ_{diff}, $\rho_{\mathrm{diff}} = 1/\sigma_{\mathrm{diff}}$, is named the *diffusion resistivity* of the material. Equation (11) is sometimes referred to as *Fick's law*. The *diffusive resistance* of the cylinder is then defined by

$$\mathcal{R}_{\text{diff}} = \frac{\Delta n}{I_{\text{diff}}} = \rho_{\text{diff}} \frac{L}{A}. \tag{12}$$

Its inverse gives the *diffusive conductance*:

$$\mathcal{G}_{\text{diff}} = \frac{1}{\mathcal{R}_{\text{diff}}} = \sigma_{\text{diff}} \frac{A}{L}.$$

Now imagine that the thermal agitation (speed) of the molecules at the two bases of the cylinder is different. Then, due to the speed difference Δv between the bases, momentum will flow from one base to the other. The rate at which speed flows, i.e.

$$I_{\text{vis}} = \frac{\Delta v}{\Delta t},$$

is called the *momentum current*. It is known, see for example [1], that

$$I_{\text{vis}} = \sigma_{\text{vis}} A \frac{\Delta v}{L}, \tag{13}$$

where L is the length of the cylinder, A is its cross-section, and σ_{vis} is a constant characteristic of the material called the *coefficient of viscosity*. Another name, again in the spirit of what we have been discussing, would be *viscous conductivity*. The inverse of σ_{vis}, $\rho_{\text{vis}} = 1/\sigma_{\text{vis}}$, is the *viscous resistivity* of the material. The *viscous resistance* of the cylinder is then defined by

$$\mathcal{R}_{\text{vis}} = \frac{\Delta v}{I} = \rho_{\text{vis}} \frac{L}{A}. \tag{14}$$

Its inverse gives the *viscous conductance*:

$$\mathcal{G}_{\text{vis}} = \frac{1}{\mathcal{R}_{\text{vis}}} = \sigma_{\text{vis}} \frac{A}{L}.$$

Formulae (9) and (10), derived in the previous section, are also applicable in the present cases with the appropriate index changes.

Before we close this section, it is worth mentioning that the linear laws (Fourier's law, Fick's law and equation (13)) discussed above are elementary versions of what is found in a more general context in (linear) non-equilibrium thermodynamics. The interested reader may want to consult papers [10] and [6] for additional information.

2.5. Elasticity

The concept of elasticity is more than a mere definition. The behaviour of a rubber band or the behaviour of a rod or a cable under stress is basically analogous to that of many springs connected together.

Let us imagine a uniform rod of length L and cross-sectional area A. We focus on an infinitesimal piece of length $\mathrm{d}x$ at distance x from one base as seen in figure 3. If $\mathrm{d}\xi$ is the infinitesimal extension of this piece under the force $F(x)$, then Hooke's law states that

$$\mathrm{d}\xi = -F(x)\,\mathrm{d}\ell,$$

where $\mathrm{d}\ell$ is the elasticity constant for the piece $\mathrm{d}x$. We can write the above relation as

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = -F(x)\frac{\mathrm{d}\ell}{\mathrm{d}x},$$

where $\lambda = \mathrm{d}\ell/\mathrm{d}x$ is the elasticity per unit length and $\epsilon = \mathrm{d}\xi/\mathrm{d}x$ is the extension of the system per unit length, known as *linear strain*. It is known that approximately [1]

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = -\frac{1}{YA}F(x),$$

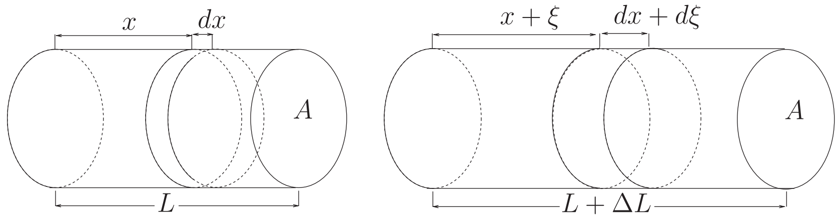

Figure 3. An undeformed and deformed rod.

where A is the cross-section and Y is a constant characteristic of the material known as *Young's modulus*. Combining the last expressions we conclude that

$$\lambda = \frac{\mathrm{d}\ell}{\mathrm{d}x} = \frac{1}{YA}.$$

If λ is constant then the elasticity constant is

$$\ell = \frac{1}{Y}\frac{L}{A},$$

where L is the length of the system. The inverse gives the *stiffness*:

$$k = Y\frac{A}{L}.$$

We have thus obtained basic formulae similar to those of resistors and capacitors that allow the computation of k and ℓ in any geometry. Most probably these formulae are well known to engineers, but they are not well known among physicists. However, once written down, they look familiar and natural.

3. Addition formulae

Suppose that there is a (discrete or continuous) sequence either of resistors, capacitors, inductors, springs or transport conductors (thermal, diffusion or viscous). Either from introductory physics, or as a straightforward exercise of the definitions, the reader can persuade himself that when the objects are connected in series

$$\begin{aligned} P &= \sum_i P_i, &\qquad &\text{discrete case,} \\ P &= \int \mathrm{d}P, &\qquad &\text{continuous case,} \end{aligned} \tag{15}$$

where

1. if the elements are connected in series P stands for any of R, D, L, ℓ, $\mathcal{R}$;
2. if the elements are connected in parallel P stands for any of G, C, K, k, $\mathcal{G}$;

each symbol having the meaning given in section 2.

In the next section, we will demonstrate the application of the addition formulae by examining specific examples from resistors and capacitors. Along with the solution, several comments are made to help the reader understand some of the implicit assumptions and other details in the solution. In the section that follows, more problems are discussed for the readers who wish to master the technique.

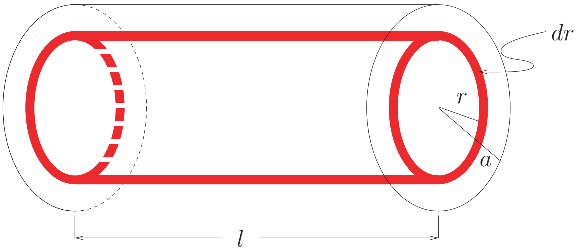

Figure 4. The figure shows a cylindrical wire of radius a. A potential difference is applied between the bases of the cylinder and therefore electric current is running parallel to the axis of the cylinder.

Table 1. This table summarizes the additive physical quantities in the most common cases encountered in introductory physics. The quantities that are not usually defined in the introductory books are the conductance $G = 1/R$, the incapacitance $D = 1/C$, the deductance $K = 1/L$, the elasticity constant $\ell = 1/k$, and the the thermal conductance $\mathcal{G}_{\mathrm{tr}} = 1/\mathcal{R}_{\mathrm{tr}}$. The index tr stands for *transport* and it should be interpreted as a generic name for any of the three cases of thermal conductance, diffusion or viscosity. The basic formulae refer to the geometry of figure 2.

Connection		Resistors	Capacitors	Inductors	Springs	Transport conductors
		Resistance	Incapacitance	Inductance	Elasticity	Thermal resistance
Series	Definition	$R = \frac{\Delta V}{I}$	$D = \frac{\Delta V}{Q}$	$L = \frac{\Phi_B}{I}$	$\ell = \frac{\xi}{F}$	$\mathcal{R}_{\mathrm{tr}} = \frac{\Delta T_{\mathrm{tr}}}{I_{\mathrm{tr}}}$
	Basic formula	$R = \rho \frac{L}{A}$	$D = \frac{1}{\varepsilon_0 \kappa} \frac{d}{A}$	$L = \mu_0 \frac{A}{w}$	$\ell = \frac{1}{Y} \frac{L}{A}$	$\mathcal{R}_{\mathrm{tr}} = \rho_{\mathrm{tr}} \frac{L}{A}$
		Conductance	Capacitance	Deductance	Stiffness	Thermal conductance
Parallel	Definition	$G = \frac{I}{\Delta V}$	$C = \frac{Q}{\Delta V}$	$K = \frac{I}{\Phi_B}$	$k = \frac{F}{\xi}$	$\mathcal{G}_{\mathrm{tr}} = \frac{I_{\mathrm{tr}}}{\Delta T_{\mathrm{tr}}}$
	Basic formula	$G = \sigma \frac{A}{L}$	$C = \varepsilon_0 \kappa \frac{A}{d}$	$K = \frac{1}{\mu_0} \frac{w}{A}$	$k = Y \frac{A}{L}$	$\mathcal{G}_{\mathrm{tr}} = \sigma_{\mathrm{tr}} \frac{A}{L}$

4. Sample problems with solutions

In this section, we pose and solve[2] a few problems that will help the reader become fluid in the application of the quantities in table 1. Some of the problems are well-known, standard ones found in all textbooks; these problems are re-examined and solved using our technique.

4.1. Cylindrical resistor with voltage applied to its bases

The cylindrical resistor shown in figure 4 is made such that the resistivity ρ is a function of the distance r from the axis. What is the total resistance R of the resistor?

Solution. *We divide the cylindrical resistor into infinitesimal resistors in the form of cylindrical shells of thickness* dr*. One of these shells is seen in figure 4.*

When the current is flowing along the axis of the cylinder, the infinitesimal resistors are not connected in series. Instead, all of the infinitesimal cylindrical shells of thickness dr *are connected at the same end points and, therefore, have the same applied potential. In other words, the shells are connected in parallel and it is the conductance that is important. Specifically,*

$$G = \int_{\text{cylinder}} \mathrm{d}G.$$

[2] For an extended version of this paper with additional solved problems, the reader should look at [2].

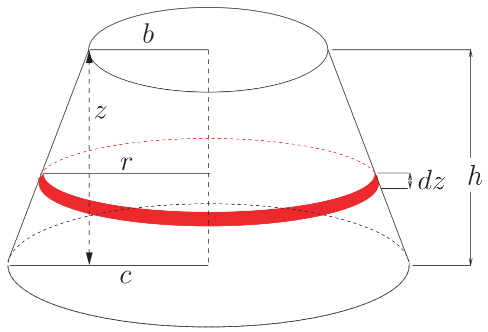

Figure 5. A truncated cone which has been sliced into infinitesimal cylinders of height dz.

For the infinitesimal shell

$$\mathrm{d}G = \sigma(r)\frac{2\pi r\,\mathrm{d}r}{\ell}.$$

Therefore

$$G = \int_{\text{cylinder}} \mathrm{d}G = \frac{2\pi}{\ell}\int_0^a \sigma(r) r\,\mathrm{d}r.$$

For example, if $\sigma(r) = \sigma_0 \frac{a}{r}$, *then*

$$G = 2\sigma_0 \frac{\pi a^2}{\ell},$$

where $\sigma_0 = 1/\rho_0$. *The resistance is therefore*

$$R = \frac{\rho_0}{2}\frac{\ell}{\pi a^2}.$$

As another example, we may take $\sigma(r) = \sigma_0 \frac{r}{a}$. *We can easily find that in this case the conductance of the resistor would then be*

$$G = \frac{2\pi\sigma_0 a^2}{3\ell}, \tag{16}$$

and the corresponding resistance would be

$$R = \frac{3\rho_0}{2}\frac{\ell}{\pi a^2}.$$

4.2. Truncated-cone resistor with potential difference applied between its bases

A resistor is made from a truncated cone of material with uniform resistivity ρ. What is the total resistance R of the resistor when a potential difference is applied between the two bases of the cone?

Solution. *This is a well-known problem found in many of the introductory physics textbooks (p 631 in [3], p 856 in [9], p 821 in [11], p 708 in [12], p 977 in [13]). We can partition the cone into infinitesimal cylindrical resistors of length* dz. *One representative resistor at distance* z *from the top base is seen in figure 5. The area of the resistor is* $A = \pi r^2$ *and therefore its infinitesimal resistance is given by*

$$\mathrm{d}R = \rho\frac{\mathrm{d}z}{\pi r^2}.$$

From the figure we can see that

$$\frac{z}{h} = \frac{r-b}{c-b} \Rightarrow \mathrm{d}z = \frac{h}{c-b}\,\mathrm{d}r.$$

The infinitesimal resistors are connected in series and therefore

$$R = \int_{\text{cone}} \mathrm{d}R = \rho \frac{h}{\pi(c-b)} \int_b^c \frac{\mathrm{d}r}{r^2} = \rho \frac{h}{\pi bc}. \tag{17}$$

Comment 1. However, this solution, which is common in textbooks [3, 9, 11–13], tacitly assumes that the discs used in the partition of the truncated cone are equipotential surfaces. This is of course not true, as can be seen quite easily. If they were equipotential surfaces, then the electric field lines would be straight lines, parallel to the axis of the cone. However, this cannot be the case since, close to the lateral surface of the cone, it would mean that the current goes through the lateral surface and does not remain inside the resistor. Therefore, the discs are not equipotential surfaces. One way out of this subtlety is to assume that the discs are approximate equipotential surfaces as suggested in [12]. This is the attitude we adopt in this paper as our intention is not to discuss the validity of the partitions used in each problem, but to emphasize the unified description of resistances and capacitances as additive quantities. Similar questions can be raised and studied in the majority of the problems mentioned in the present paper. A reader with serious interest in electricity is referred to the paper of Romano and Price [8] where the conical resistor is studied. Once that paper is understood, the reader can attempt to generalize it to the rest of the problems of our paper.

Comment 2. The reader may have noted that the results for a particular geometry are not specific to the quantity computed, but they can be transferred to other quantities among those discussed in table 1 due to the similarity of formulae. For the case at hand, think of a capacitor which is made of two circular discs of radii b and c respectively placed at a distance h such that the line that joins their centres is perpendicular to the discs. To find the capacitance of this arrangement, we partition the capacitor into infinitesimal parallel-plate capacitors of distance $\mathrm{d}z$ and plate area $A = \pi r^2$ exactly as seen in figure 5. These infinitesimal capacitors are connected in series and therefore the incapacitance is the relevant additive quantity:

$$\mathrm{d}D = \frac{1}{\varepsilon_0} \frac{\mathrm{d}z}{\pi r^2}.$$

Note that the computation is identical to that of R with the final result:

$$D = \frac{1}{\varepsilon_0} \frac{h}{\pi bc} \Rightarrow C = \varepsilon_0 \frac{\pi bc}{h}. \tag{18}$$

When $b = c$, we recover the result of the parallel-plate capacitor.

4.3. Hollow cylindrical capacitor composed of coaxial cylinders

In introductory physics, the capacitance of a cylindrical capacitor is found using the definition $C = Q/\Delta V$, where Q is the charge on the positive plate of the capacitor and ΔV the absolute value of the potential difference between the two plates. However, this problem requires us to compute the capacitance using only the formulae giving the capacitance of a parallel-plate, plane capacitor. In fact, we will discuss more generally the case in which the capacitor is filled with a dielectric constant $\kappa(r) = cr^n$, where r is the distance from the axis and c, n are constants.

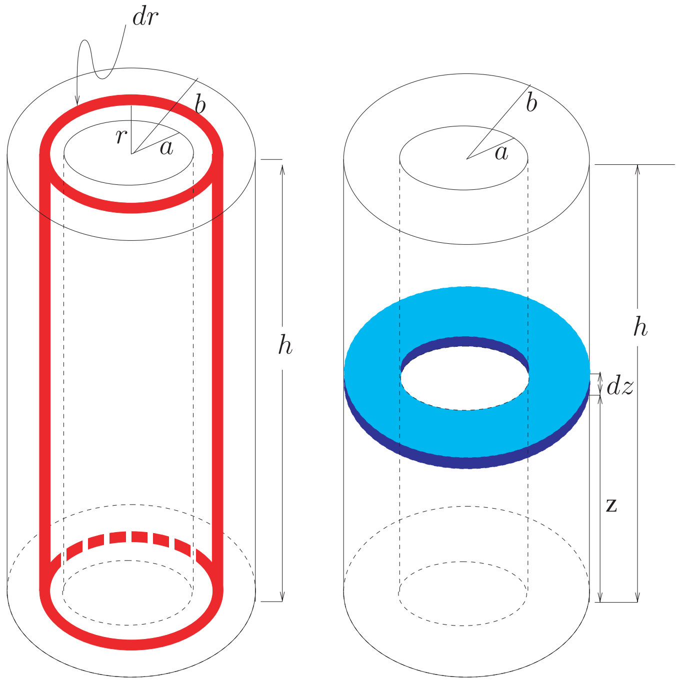

Figure 6. A cylindrical capacitor with radii a and b and height h. In the left picture, we have sliced it in infinitesimal cylindrical shells, while in the right picture we have sliced it in infinitesimal annuli.

Solution. *As shown in the left side of figure 6, the capacitor is partitioned into small cylindrical capacitors for which the distance between the plates is* $\mathrm{d}r$*. For such small capacitors, the formula of a parallel-plate capacitor is valid. We note though that all infinitesimal capacitors are connected in series. Therefore*

$$\mathrm{d}D = \frac{1}{\varepsilon_0 \kappa(r)} \frac{\mathrm{d}r}{2\pi r h}$$

and

$$D = \int_{\text{cylinder}} \mathrm{d}D = \frac{1}{2\pi\varepsilon_0 hc} \int_a^b \frac{\mathrm{d}r}{r^{n+1}}.$$

When $n = 0$*, the capacitor is filled with a uniform dielectric. This is the standard case of a capacitor filled with air found in any book:*

$$D = \frac{1}{2\pi\varepsilon_0 hc} \int_a^b \frac{\mathrm{d}r}{r} = \frac{1}{2\pi\varepsilon_0 hc} \ln \frac{b}{a}.$$

When $n \neq 0$*, the capacitor is filled with a non-uniform dielectric and*

$$D = \frac{1}{2\pi\varepsilon_0 hc} \int_a^b \frac{\mathrm{d}r}{r^{n+1}} = \frac{1}{2\pi\varepsilon_0 hc} \frac{b^n - a^n}{n a^n b^n}.$$

Therefore, the total capacitance is

$$C = \frac{1}{D} = \frac{2\pi\varepsilon_0 h}{\ln \frac{b}{a}},$$

if $n = 0$ *and*

$$C = \frac{1}{D} = 2\pi\varepsilon_0 hcn \frac{a^n b^n}{b^n - a^n},$$

if $n \neq 0$.

Comment 1. One might be tempted to partition the cylindrical capacitor into infinitesimal capacitors as seen in the figure to the right. Such capacitors look simpler than the infinitesimal cylindrical shell we used above. Furthermore, they are connected in parallel (note that each capacitor is carrying an infinitesimal charge $\mathrm{d}Q$ and $\int_{\text{cylinder}} \mathrm{d}Q = Q$) and therefore it is enough to deal with capacitance, $C = \int_{\text{cylinder}} \mathrm{d}C$, and not incapacitance D.

However, with a minute's reflection the reader will see that in order to use the parallel-plate capacitor formula in the infinitesimal case, the distance between the plates must be infinitesimal which indicates that the infinitesimal capacitors must be connected in series. In the proposed slicing, the distance between the plates of the infinitesimal capacitor is finite, namely $b - a$. The infinitesimal capacitor is still a cylindrical capacitor of infinitesimal height and therefore its capacitance should be expressed in a form that is not known before the problem is solved.

Comment 2. The geometry of the problem at hand is identical to that of a cylindrical conductor from which a smaller coaxial cylindrical piece has been removed and potential difference applied between the inner and outer surfaces. Then we can slice the conductor in infinitesimal cylindrical shells as shown in figure 6 with infinitesimal resistance

$$\mathrm{d}R = \rho \frac{\mathrm{d}r}{2\pi hr}.$$

The total resistance will then be

$$R = \frac{\rho}{2\pi h} \int_a^b \frac{\mathrm{d}r}{r} = \frac{\rho}{2\pi h} \ln \frac{b}{a},$$

a well-known result, too.

4.4. Hollow truncated-cone capacitor composed of two annular bases

(a) Two metallic flat annuli are placed such that they form a capacitor with the shape of a hollow truncated cone as seen in figure 7. Partition the capacitor into infinitesimal capacitors and show that the capacitance is given by

$$C = 2\pi\varepsilon_0 \frac{a(c-b)}{h} \left[\ln \frac{c-a}{c+a} - \ln \frac{b-a}{b+a} \right]^{-1}.$$

Show that this result reduces to that of a cylindrical capacitor for $c = b$. Also, show that it agrees with the result for a parallel-plate capacitor with $a = 0$.

(b) Now, fill the two bases with discs of radius a and argue that the capacitance of the hollow truncated cone equals that of the truncated cone minus the capacitance of the parallel-plate capacitor that we have removed. This means that the capacitance of the hollow truncated cone should equal

$$C = \pi\varepsilon_0 \frac{bc - a^2}{h}.$$

How is it possible that this result does not agree with that of part (a)?

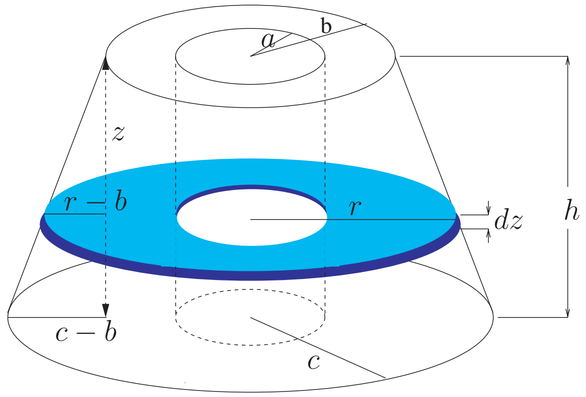

Figure 7. A partition of the truncated cone in infinitesimal slices.

Solution. *(a) We divide the truncated cone into annuli of height* dz. *These are parellel-plate capacitors connected in series. Therefore*

$$\mathrm{d}D = \frac{1}{\varepsilon_0}\frac{\mathrm{d}z}{\pi(r^2 - a^2)}.$$

As in example 4.2,

$$\mathrm{d}z = \frac{h}{c-b}\,\mathrm{d}r. \tag{19}$$

Therefore

$$\mathrm{d}D = \frac{1}{\varepsilon_0\pi}\frac{h}{c-b}\frac{\mathrm{d}r}{r^2-a^2} = \frac{h}{2\pi\varepsilon_0 a(c-b)}\left(\frac{1}{r-a} - \frac{1}{r+a}\right)\mathrm{d}r,$$

and

$$D = \frac{h}{2\pi\varepsilon_0 a(c-b)}\int_b^c\left(\frac{1}{r-a} - \frac{1}{r+a}\right)\mathrm{d}r$$
$$= \frac{h}{2\pi\varepsilon_0 a(c-b)}\left(\ln\frac{c-a}{c+a} - \ln\frac{b-a}{b+a}\right).$$

When $c = b$ *we apply de L'Hospital's rule to get*

$$D = \frac{h}{\pi\varepsilon_0}\frac{1}{b^2-a^2}.$$

This is just $C = \varepsilon_0 A/h$ *for a parallel-plate capacitor with plate area* $A = \pi(b^2 - a^2)$.

The case $a = 0$ *is obtained in the same way and yields*

$$D = \frac{h}{\pi\varepsilon_0 cb},$$

as found previously.

(b) We use a parallel-plate capacitor with circular plates of radius a at a distance h to fill the plates of our capacitor (figure 8). This capacitor has capacitance

$$C_{\mathrm{add}} = \varepsilon_0\frac{\pi a^2}{h}.$$

The conical capacitor we have thus created has capacitance

$$C_{\mathrm{total}} = \varepsilon_0\frac{\pi cb}{h}.$$

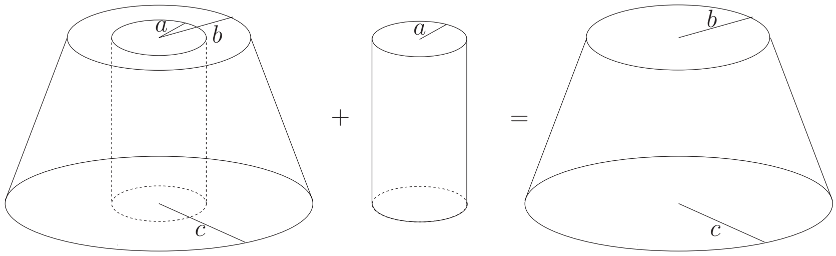

Figure 8. The truncated cone is the sum of the hollow truncated cone plus a cylinder.

The original capacitor and the one we added are connected in parallel since the same voltage is applied at their plates. Therefore, according to the superposition principle

$$C_{\text{cylinder}} = C + C_{\text{add}} \Rightarrow C = \varepsilon_0 \frac{\pi(cb - a^2)}{h}.$$

Apparently, this result does not agree with that of part (a). The reason is subtle but easy to find. The superposition principle states that if a problem in electricity can be split in two other problems, then the solution to the original problem is the sum of the solutions of the partial problems. But is our problem the exact sum of the two partial ones?

Let us assume that each plate of the truncated cone has a charge of absolute value Q and constant charge density equal to $\sigma = Q/\pi b^2$ on the top plate and equal to $\sigma' = -Q/\pi c^2$ on the bottom plate.

Q splits into Q_1 and Q_2 on the plates of the hollow truncated cone and the cylinder, respectively, in proportion to the areas of the plates.

On the top plate of the hollow truncated cone we have $Q_1 = \sigma\pi(b^2 - a^2)$ and on the top plate of the cylinder $Q_2 = \sigma\pi a^2$.

On the bottom plate of the hollow truncated cone we have $Q_1' = \sigma'\pi(c^2 - a^2)$ and on the bottom plate of the cylinder $Q_2' = \sigma'\pi a^2$. However, Q_1' and Q_2' are not $-Q_1$ and $-Q_2$ (except when $b = c$). The only way to ensure this is to change the charge densities on the plates. But then the problem is not a simple addition of two other problems.

4.5. Spherical capacitor composed of concentric spheres

Re-derive the well-known expression for the capacitance of a spherical capacitor

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a},$$

(where a, b are the radii of the spheres with $b > a$) by partitioning it into infinitesimal capacitors.

Solution. *We partition the capacitor into spherical shells of thickness* dr (figure 9). *The infinitesimal shells are connected in series and therefore their incapacitance is*

$$\mathrm{d}D = \frac{1}{\varepsilon_0}\frac{\mathrm{d}r}{4\pi r^2}.$$

The total incapacitance is thus

$$D = \int_{\text{sphere}} \mathrm{d}D = \frac{1}{4\pi\varepsilon_0}\int_{\text{sphere}} \frac{\mathrm{d}r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{b-a}{ab},$$

from which we find the well-known formula for the capacitance:

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$

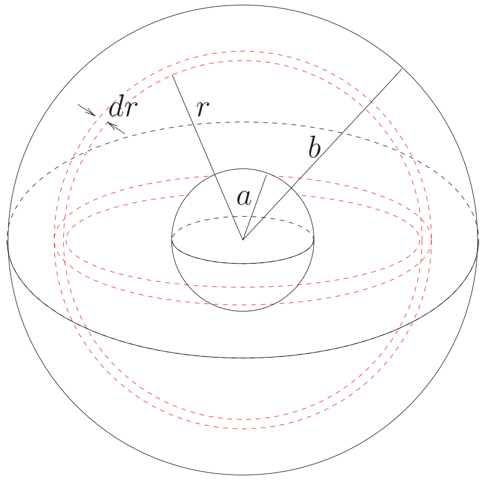

Figure 9. The partition for a spherical capacitor into infinitesimal shells.

4.6. Parallel-plate inductor

Split the parallel-plate inductor into convenient infinitesimal inductors. Then make use of equation (5) to derive equation (6).

Solution. *The inductor is split in parallel infinitesimal slices as seen in figure 1. Each slice is similar to a turn of a solenoid; it is carrying an infinitesimal current* $\mathrm{d}I = J_s\,\mathrm{d}x$*. The infinitesimal slices have a deductance of*

$$\mathrm{d}K = \frac{\mathrm{d}I}{\Phi_B},$$

where $\Phi_B = BLd = \mu_0 J_s Ld$*. Therefore*

$$\mathrm{d}K = \frac{1}{\mu_0 Ld}\,\mathrm{d}x.$$

The total deductance is then

$$K = \int_0^w \frac{1}{\mu_0 Ld}\,\mathrm{d}x = \frac{w}{\mu_0 Ld},$$

and

$$L = \frac{1}{K} = \mu_0 \frac{Ld}{w}.$$

4.7. Cylindrical thermal conductor

A long cylindrical straight pipe of radius a and length ℓ is carrying lubricating oil at some temperature. To minimize heat loss, the pipe is covered with a cylindrical insulator of radius b and thermal resistivity ρ_{th}. Compute the total thermal resistance of the insulator.

Solution. *Ignoring the edge effects at the bases, heat can only flow radially. We can then split the insulator in infinitesimal thermal insulators in the shape of cylindrical shells as in figure 6. Each such infinitesimal insulator will present a thermal resistance of*

$$\mathrm{d}R_{\mathrm{th}} = \rho_{\mathrm{th}} \frac{\mathrm{d}r}{2\pi\ell r}.$$

Then the total thermal resistance will be

$$R_{\text{th}} = \frac{\rho_{\text{th}}}{2\pi\ell} \int_a^b \frac{\mathrm{d}r}{r} = \frac{\rho_{\text{th}}}{2\pi\ell} \ln\frac{b}{a}.$$

Note that the computation is identical to that of the electrical resistance when the current flows radially.

5. Conclusions

There is probably no need for additional problems. The reader has certainly uncovered the pattern. All the quantities we have used follow the simple additive law (15). For elements in the shape of a uniform cylinder of length L and cross-section A whose material is characterized by the constant p (corresponding to quantity P), we have

if P stands for $R, D, L, \ell, \mathcal{R}$,

$$P = p\frac{L}{A}, \quad \text{(discrete case)}, \qquad \mathrm{d}P = p\frac{\mathrm{d}L}{A}, \quad \text{(continuous case)},$$

if P stands for $G, C, K, k, \mathcal{G}$,

$$P = \frac{1}{p}\frac{A}{L}, \quad \text{(discrete case)}, \qquad \mathrm{d}P = \frac{1}{p}\frac{\mathrm{d}A}{L}, \quad \text{(continuous case)}.$$

In all cases of identical geometry, the results will be identical. In fact, in many instances above we could have saved some computations but we avoided doing so in order to present the big picture first. Once the reader is aware of the global picture, she can easily use it to transfer a result for a quantity P in some geometry to a quantity P' in a similar geometry. An example of this is as follows.

Example. A cylindrical cable of radius a and length ℓ is made of a huge number of small filaments such that the cable may be considered continuous. The filaments have been arranged in such a way that Young's modulus for the cable is $Y(r) = Y_0\frac{r}{a}$, where r is the distance from the centre of the cable and c some constant. What is the stiffness of the cable?

The stiffness of an infinitesimal cylindrical shell of radius r is given by

$$\mathrm{d}k = Y(r)\frac{2\pi r\,\mathrm{d}r}{\ell}.$$

The stiffness of the cable would then be

$$k = \frac{2\pi Y_0 a^2}{3\ell}.$$

Please note that the geometry is identical to that of the resistance problem in section 4.1. Knowing this, one could have transferred the result (16) in this case too.

The reader is invited to construct similar problems for quantities in table 1.

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