states for two particles," Phys. Rev. Lett. **72**, 1951 (1994).

[7]H. P. Stapp, *Mind, Matter and Quantum Mechanics* (Springer-Verlag, New York, 1993), pp. 5–9.

[8]T. F. Jordan, "Testing Einstein–Podolsky–Rosen assumptions without inequalities with two photons or particles with spin-$\frac{1}{2}$," Phys. Rev. A **50**, 62–66 (1994).

[9]T. F. Jordan, "Quantum mysteries explored," Am. J. Phys. **62**, 874–880 (1994).

[10]N. D. Mermin, "Quantum mysteries refined," Am. J. Phys. **62**, 880–887 (1994).

[11]U. Fano, "Pairs of two-level systems," Rev. Mod. Phys. **55**, 855–874 (1983).

[12]See, for example, E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), p. 279.

[13]We use the Einstein summation convention for repeated indices throughout the text.

[14]The idea of using the relation $\rho^2=\rho$ to simplify the description of a pure state has been exploited in another context by J. Bohn, "Observable characteristics of pure quantum states," Phys. Rev. Lett. **66**, 1547–1550 (1991).

[15]The precise correspondence is as follows:

$|\Psi\rangle=1/\sqrt{2}[(|\beta_1\rangle|\beta_2\rangle+|\alpha_1\rangle|\alpha_2\rangle)(C_{zz'}=1,C_{xx'}=1,C_{yy'}=-1)$;

$|\Psi\rangle=1/\sqrt{2}[(|\beta_1\rangle|\beta_2\rangle-|\alpha_1\rangle|\alpha_2\rangle)(C_{zz'}=1,C_{xx'}=-1,C_{yy'}=1)$;

$|\Psi\rangle=1/\sqrt{2}[(|\beta_1\rangle|\alpha_2\rangle+|\alpha_1\rangle|\beta_2\rangle)(C_{zz'}=-1,C_{xx'}=1,C_{yy'}=1)$;

$|\Psi\rangle=1/\sqrt{2}[(|\beta_1\rangle|\alpha_2\rangle-|\alpha_1\rangle|\beta_2\rangle)](C_{zz'}=-1,C_{xx'}=-1,C_{yy'}=-1)$.

The above states, which form an orthonormal and complete set, are often referred to as the "Bell basis;" see, for example, S. L. Braunstein, A. Mann, and M. Revzen, "Maximal violation of Bell inequalities for mixed states," Phys. Rev. Lett. **68**, 3259–3261 (1994).

[16]A. Einstein, B. Podolsky, and N. Rosen, "Can quantum mechanical description of physical reality be considered complete?," Phys. Rev. **47**, 777–780 (1935).

[17]Well, almost any angles. The choice $\theta_a=0$ must be excluded because it does not lead to a distinct set of directions for **a**, **a**$'$, **b**, and **b**$'$.

[18]We can even identify a suitable Hardy state if all four directions are specified arbitrarily, as noted by Jordan in Ref. 8. To do this we take $\phi_a = \phi'_a = 0$ and $\phi_b = \phi'_b = \pi$ and are left with the task of fixing the four polar angles and Ω. Since $\theta_a - \theta'_a$ and $\theta_b - \theta'_b$ are now fixed we have just three unknowns, which we may take to be θ_a, θ_b, and Ω. The three equations (24)–(26) suffice to determine these unknowns, but the equations are complicated enough that they can be solved only numerically.

[19]J. Torgerson, D. Branning, and L. Mandel, "A method for demonstrating violation of local realism with a two-photon downconverter without use of Bell inequalities," Appl. Phys. B **60**, 267–269 (1995).

The conical resistor conundrum: A potential solution

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A truncated cone, made of material of uniform resistivity, is given in many introductory physics texts as a nontrivial problem in the computation of resistance. The intended method and answer are incorrect and the problem cannot be solved by elementary means. In this paper, we (i) discuss the physics of current flow in a nonconstant cross-section conductor, (ii) examine the flaws in the "standard" solution for the truncated cone, (iii) present a computed resistance found from a numerically generated solution for the electrical potential in the truncated cone, and (iv) consider whether any problem exists to which the standard solution applies.

In introductory courses students are taught that a solid of electrical resistivity ρ, length L, and constant cross-sectional area A, has a resistance

$$R=\rho L/A. \tag{1}$$

Many (most?) texts[1] for calculus-based courses try to introduce an application of calculus by including a problem in which a resistor has a nonconstant cross section. In these books (usually in Chap. 26±2) the student is asked to find the resistance of a truncated cone, as in Fig. 1. The reader is meant to assume that the resistor is connected via "wires" that end with perfectly conducting disks attached to the end faces of the truncated cone.

The method that is intended can be inferred from the answer. (In the text by Wolfson and Pasachoff[1] the student is explicitly instructed to use this method.) The student is meant to break the cone into isolated differential slabs of resistance (each could be called a pièce de résistance). The slab at location x would have differential thickness dx, and would have a resistance $dR=\rho\, dx/(\pi r^2)$, where r is the radius of the cone's cross section at x. Since $dx/dr=L/(b-a)$ we have $dR=\rho[L/(b-a)](dr/\pi r^2)$. The cone is then

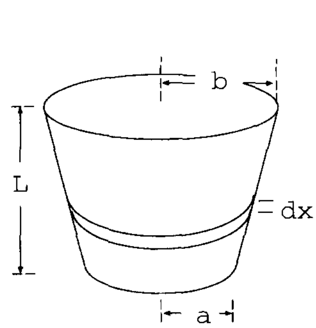

Fig. 1. The standard truncated cone problem.

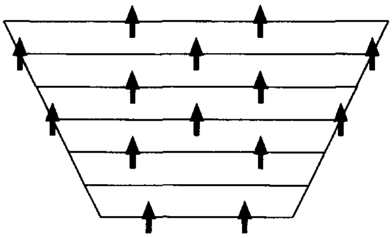

Fig. 2. Planar equipotentials and the implied electric field.

treated as if it is nothing more than a stack of these slabs, and the total resistance is found by summing the resistance of all the slabs:

$$R=\int dR=\rho\,\frac{L}{b-a}\int_a^b \frac{dr}{\pi r^2}=\frac{\rho L}{\pi ab}. \tag{2}$$

This seems plausible at first—the role of A in (1) is played in (2) by the geometric mean of the area of the disks bounding the cone—but the plausibility does not stand up to a close second look. When the slabs are added "in series," it is required that their planar faces be equipotentials. The stack of slabs is electrically equivalent to the cone only if the equipotentials in the current-carrying cone consist of planes perpendicular to the axis, as is shown in Fig. 2. This is impossible. If these *were* the equipotentials then the electrical field (orthogonal to the equipotentials) would be parallel to the axis, as shown in the figure. The current (parallel to the electric field in an Ohmic conductor) would then also be parallel to the cone axis, and therefore not parallel to the sides of the resistor. This would imply that current is flowing in through the sides! The actual equipotentials, therefore, must be curved in order to be perpendicular to the sides of the resistor. A few of the texts suggest that some sort of approximation is being made (about which more below); only Wolfson and Pasachoff[1] point out that the assumption of the planar equipotentials is being made, and that the assumption is wrong.

To discuss what is right, let us first note that there is no charge density in the resistor. If there were charge density, $\nabla\cdot\mathbf{E}$ would be nonzero and, through Ohm's law, this would imply that the divergence of the current flow $\nabla\cdot\mathbf{J}$ is nonzero, and that charge is building up. The mathematical problem of finding the field inside the resistor therefore amounts to finding the solution of Laplace's equation $\nabla^2\Phi=0$ for the electrostatic potential Φ. This equation is to be solved with the following boundary conditions: (1) Φ must be constant, say at values Φ_a and Φ_b, on the disks at the end of the resistor; (2) the gradient $\nabla\Phi$ must be parallel to the resistor sides. Once the solution is found, the current flow is known to be $\mathbf{J}=-\nabla\Phi/\rho$, and integrating the normal component of $\mathbf{J}$ over the area of either of the bounding disks gives the total current I. The resistance is then $|(\Phi_a-\Phi_b)/I|$.

Casting this as a numerical problem is reasonably straightforward, and the solution of that problem, while not trivial, is not daunting with modern computational tools. Our approach has been to formulate the problem in a coordinate system convenient for the boundary conditions, and in which numerical errors are expected to be minimized. The equations based on this coordinate system were solved using the public-domain package *Sparse*.[2] For those who are interested in this sort of thing, details are provided in the Appendix. Here, we want to emphasize the results.

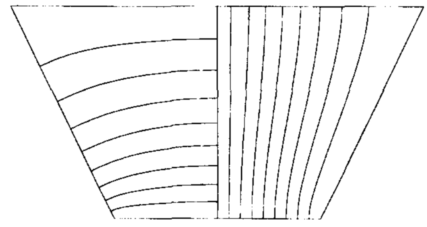

Fig. 3. Equipotentials and flow lines for computed resistance.

We present results first for a resistor with $a/L=1/2$, $b/L=1$. A picture of the current flow lines and the equipotentials are shown in Fig. 3. (Half the lateral cross section is shown for each.) All expected features are evident in these figures. In particular, current does not flow through the resistor's sides and the equipotentials are perpendicular to the sides. At the corners the $\mathbf{E}$ field is required to be perpendicular to the end faces and to be parallel to the sides. To satisfy these incompatible constraints, $\mathbf{E}$ vanishes.

How wrong is the "stack of slabs" solution? Intuition suggests (correctly) that the correct solution will correspond to higher resistance since the correct flows are more constrained than those of the slab solution. (The corner regions, for example, must have reduced current flow.) From the numerical results we find, for our reference case ($a/L=1/2$, $b/L=1$) in Fig. 3, that the resistance is $R=0.692\rho/L$, larger by 9% than the textbook answer given in (2).

Several of the textbooks[3] in the list in Ref. 1 tell the student to assume that the current is uniformly spread over the cross section. We can see no strong case for this assumption being ineluctably linked with the intended method. The current could, for example, be a uniform flow on straight lines that can be traced back to the apex of the cone, as in Fig. 4. This would give a flow pattern that satisfied the correct boundary conditions on the sides of the resistor, but not on the disks that truncate it. This flow pattern would be precisely correct if the resistor were formed by the intersection of a cone with a spherical shell. The resistor would then be truncated not by disks, but by the spherical caps shown dashed in the figure. This replacement might in principle (though not plausibly) be taken by the student to be the natural interpretation of the instructions to assume uniform flow. This same implausibly perverse student would compute a resistance of

$$R=\frac{\rho}{2\pi ab}\,\frac{(b-a)^2}{\sqrt{L^2+(b-a)^2}-L}. \tag{3}$$

This calculation is a straightforward application of geometry and does not involve calculus. It gives, furthermore, an answer $R=0.674\rho/L$ for our reference problem (that of Fig. 3)

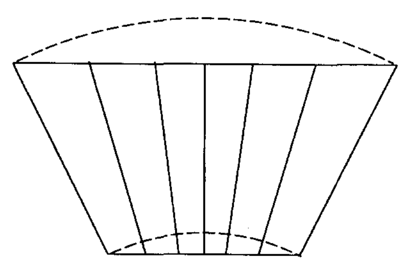

Fig. 4. Another type of "uniform" flow.

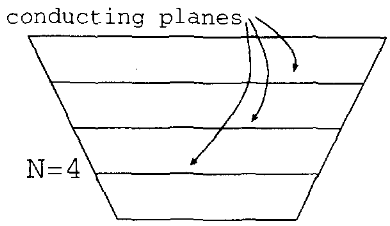

Fig. 5. Resistor segmented by conducting planes.

which is less than 3% off from the correct (numerical) answer, while the expected textbook method is off by almost 9%.

One can make other approximations using a portion of a spherical shell. In particular one could use a spherical shell which is just barely completely enclosed within the truncated cone, or one could use a shell which just barely encloses the truncated cone. The former approximation, which provides a lower bound on the resistance, gives $R=0.595\rho/L$ for our reference case, while the latter approximation, which provides an upper bound, gives $R=0.833\rho/L$.

The textbook by Keller, Gettys, and Skove[1] suggests, in its presentation of the problem, that the answer expected from the student will apply only if the resistor does not taper very much, i.e., for $(b-a)\ll L$. Intuition suggests (and numerical models confirm) that the textbook formula becomes a good approximation in this limit. This does provide some justification for the stack of slabs approach as an easy way of deriving an approximation, but it should not be taken seriously as unique or efficient. An example of an alternative approach which is equally justified is that of (3). In the limit $(b-a)\ll L$ this becomes $\rho L/\pi ab$ and, as we have seen, gives a better approximation, at least for some range of parameters.

We have asked ourselves what might be the problem for which the textbook solution is the correct answer. The result of this inverse problem solving is to imagine thin conducting planes used to break the resistor into N segments, as in Fig. 5. (These segments are different from the slabs of the textbook method. For one thing, the segments are tapered; the slabs are not.) We can compute the resistance of a segmented cone by using our numerical program to compute the resistance of each segment. Table I shows the results of segmentation for a resistor with the geometry ($a/L=1/2$, $b/L=1$) of that in Fig. 3. As the number of segments increases from $N=1$ (the unsegmented resistor) to $N=8$, the excess of the computed resistance over the textbook answer decreases. In the limit of an infinite number of segments, the textbook answer would be reached.

The transverse conducting planes do not give us a very satisfying physical problem, but they do point us in the direction of an interesting physical problem: a truncated conical resistor made of material with *anisotropic* resistivity. The resistivity parallel to the cone axis should be ρ, while the resistivity in the transverse directions vanishes. For such a resistor the method and the answer of the textbooks is correct. This, of course, is a very inappropriate problem for an introductory course, but the same, or worse, must be said for the problem as it presently appears in textbooks.

Table I. Resistance of a cone with N segments.

N	Computed R/Textbook R
1	1.087
2	1.064
4	1.040
8	1.022

ACKNOWLEDGMENTS

We wish to thank Christopher Johnson for directing us to an appropriate package for numerically solving our equations and Carlton DeTar for helping with its use. We thank anonymous referees for useful suggestions, in particular the idea of upper and lower bounds on the resistance using spherical shell approximations. This work was partially supported by the National Science Foundation under Grant No. PHY9207225.

APPENDIX

We choose to have the apex defining the conical resistor to be at the origin of a system of cylindrical oordinates r,ϕ,z, in terms of which we define new coordinates η, ζ by

$$\eta\equiv\frac{1}{2}\frac{r^2}{z^2}\quad \zeta\equiv\ln\left(\frac{z}{L}\right). \tag{4}$$

The surfaces of constant η are conical surfaces with the same apex (like the flow lines shown in Fig. 4). The squared form of r/z in (4) was chosen to eliminate the usual r^{-1} factor that enters the axi-symmetric (ϕ-independent) Laplace equation. The logarithmic form of z in (4) was chosen to eliminate the z-dependent factors that arise from the definition of η.

The boundaries of the resistor are constant coordinate surfaces. The equipotential disks truncating the resistor are at $\zeta=\zeta_a\equiv\ln[a/(b-a)]$ and $\zeta=\zeta_b\equiv\ln[b/(b-a)]$. The sides of the resistor are at a $\eta=\eta_0\equiv\frac{1}{2}[(b-a)/L]^2$. In terms of these coordinates the axisymmetric Laplace equation takes the form

$$\frac{\partial^2\Phi}{\partial\zeta^2}-\frac{\partial\Phi}{\partial\zeta}+2\eta(1+2\eta)\frac{\partial^2\Phi}{\partial\eta^2}+(2+6\eta)\frac{\partial\Phi}{\partial\eta}-4\eta\frac{\partial^2\Phi}{\partial\zeta\,\partial\eta}=0. \tag{5}$$

The boundary conditions on the disks are straightforward: At ζ_a and ζ_b the potentials are taken to be $\Phi=0$ and $\Phi=1$. At the resistor sides, $\eta=\eta_0$, the condition is that $\nabla\Phi$ is parallel to the surface $\eta=\eta_0$. Since the η, ζ coordinates are not orthogonal, some care must be used in computing the gradient of Φ and in evaluating the condition of parallelism at the side. The result is that at $\eta=\eta_0$ we must have

$$(1+2\eta)\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\zeta}=0. \tag{6}$$

Although the axis is not a physical boundary, $\eta=0$ is a boundary of our coordinate region. The appropriate condition at the axis is simply the Laplace equation (5) with $\eta=0$.

The range of η and ζ was discretized into an $n\times n$ square grid, and the Laplace equation, and boundary conditions at $\zeta=\zeta_a,\zeta_b$ and $\eta=0$, η_0, were written as difference equations on this grid. These difference equations form an $n\times(n-2)$ set of linear equations in which the unknowns are the values of Φ at the grid points. This set of equations was solved numerically with the *Sparse* package[2] which solves explicitly (rather than iteratively). From the computed solution for Φ near the bounding disks, the current was computed as described in the text, and a value for the resistance was found.

To estimate and improve the accuracy of the result for the computed resistance we used a version of Richardson extrapolation. The resistance computed for different grid size n, with n ranging from 20 to 100, was found to follow the pattern $R(n) = A + B/n$. The resistor value $R(\infty)$ was taken to be A. The uncertainty in A depended on the scatter in the values of B found for different values of n (due to round-off error, $1/n^2$ truncation terms, etc.). From this we can say with some confidence that the computed resistances were accurate to around 0.1%. This uncertainty is much smaller than the differences (several percent) between our numerically computed resistances and the textbook values given by (2).

[1]We do not claim to have made an exhaustive search of the texts. Of 13 calculus-based introductory texts we looked into, the truncated cone problem was found in the following nine: W. P. Crummett and A. B. Western, *University Physics* (Brown, Dubuque, 1994), Chap. 26, Prob. 49; P. M. Fishbane, S. Gasiorowicz, and S. T. Thornton, *Physics for Scientists and Engineers* (Prentice-Hall, Englewood Cliffs, NJ, 1993), Chap. 27, Prob. 55; D. Halliday and R. Resnick, *Fundamentals of Physics* (Wiley, New York, 1981), Chap. 28, Prob. 25; A. Hudson and R. Nelson, *University Physics* (Harcourt Brace Jovanovich, New York, 1982), Chap. 24, Prob. 24C-1; F. J. Keller, W. E. Gettys, and M. J. Skove, *Physics* (McGraw-Hill, New York, 1993), Chap. 24, Prob. 5; H. C. Ohanian, *Physics* (Norton, New York, 1985), Chap. 28, Prob. 25; R. A. Serway, *Physics for Scientists and Engineers* (Saunders, Philadelphia, 1992), 3rd ed., Chap. 27, Prob. 68; P. A. Tipler, *Physics for Scientists and Engineers* (Worth, New York, 1991), Chap. 22, Prob. 69; R. Wolfson and J. A. Pasachoff, *Physics* (Harper Collins, New York, 1995), Chap. 27, Prob. 69.

[2]K. S. Kundert and A. Sangiovanni-Vincentelli, *Sparse* (University of California, Berkeley, 1988).

[3]Of those listed in Ref. 1, texts which instruct the student to assume uniform current flow are those by: Crummett and Western; Halliday and Resnick; Hudson and Nelson; Serway. In the Keller, Gettys and Skove text, the student is told that uniform current flow follows from the assumption of a small taper angle for the cone.

Lagrangian field theories and energy-momentum tensors

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We discuss the implications of Poincaré invariance within the context of Lagrangian field theories. It is shown that the correct implementation of this invariance leads in a straightforward manner to a conserved energy-momentum tensor which is both symmetric and gauge invariant. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Few things are more frustrating to students than to be led through a long, formal argument only to be told at the end that the result obtained is incorrect and must somehow be fixed by an auxiliary procedure. This is particularly harmful if the formal argument involved turns out to be one of the mathematical cornerstones of modern physics. Unless the discussion includes a re-examination of the analysis to find out exactly what went wrong, the students will be left with the paradoxical feeling that a supposedly very general theorem produces unacceptable answers when applied to certain specific situations. Quite understandably, later on they will be reluctant to think about any physical problem in terms of the tools provided by such a theorem.

The case we have in mind is the typical derivation of the energy-momentum tensor for relativistic field theories. Most textbooks begin the discussion by writing down a Lagrangian $\mathscr{L}$, and arguing that this Lagrangian must be invariant under a spacetime translation. The details of the derivation vary from text to text, but the end result invariably yields the energy-momentum tensor (our conventions are the same as Jackson's;[1] see the Appendix):

$$T^{\mu}_{\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_{\mu} A_{\lambda})} \partial_{\nu} A_{\lambda} - \delta^{\mu}_{\nu} \mathscr{L} = -\frac{1}{4\pi} F^{\mu\lambda} \partial_{\nu} A_{\lambda} + \frac{1}{16\pi} \delta^{\mu}_{\nu} F^{\rho\lambda} F_{\rho\lambda}, \tag{1}$$

when applied to the free electromagnetic field, $\mathscr{L} = -(1/16\pi) F^{\rho\lambda} F_{\rho\lambda}$.[1–4] It is then pointed out that this so-called *canonical* energy-momentum (or stress) tensor is unacceptable for a number of reasons. First, it is clearly not a gauge-invariant quantity. Second, it is not symmetric, thereby ruining conservation of angular momentum. Third, its components do not reproduce the standard definitions of energy density and momentum density. Fourth, it is not traceless, which contradicts the fact that our starting Lagrangian is conformally invariant (i.e., photons are massless). Some technical details of the derivations add to the confusion: In some instances, it is apparently crucial to assume a Lagrangian with an explicit dependence on the spacetime position[2]—a dependence which is clearly absent in $\mathscr{L} = -(1/16\pi) F^{\rho\lambda} F_{\rho\lambda}$—while in other cases a *local* translation $x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + a^{\mu}(x)$ seems necessary,[3] even though in special relativity we deal exclusively with *global* translations.

