# The Area Function 

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The tangent line at $\left(x^{\prime}, R\left(x^{\prime}\right)\right)$ is

$$
\begin{equation*}
y=R^{\prime}\left(x^{\prime}\right)\left(x-x^{\prime}\right)+R\left(x^{\prime}\right) \tag{1}
\end{equation*}
$$

so that

$$
\begin{equation*}
x_{0}-x^{\prime}=-\frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)} \tag{2}
\end{equation*}
$$

The height of the arc at $x^{\prime \prime}\left(x \leq x^{\prime \prime} \leq x^{\prime}\right)$ is given by

$$
\begin{align*}
\left(x^{\prime \prime}-x_{0}\right)^{2}+h\left(x^{\prime \prime}\right)^{2} & =\left(x^{\prime}-x_{0}\right)^{2}+R\left(x^{\prime}\right)^{2}  \tag{3}\\
& =\left(\frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)}\right)^{2}+R\left(x^{\prime}\right)^{2} \tag{4}
\end{align*}
$$

or

$$
\begin{align*}
h\left(x^{\prime \prime}\right) & =\sqrt{\left(\frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)}\right)^{2}+R\left(x^{\prime}\right)^{2}-\left(x^{\prime \prime}+\frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)}\right)^{2}}  \tag{5}\\
& =\sqrt{R\left(x^{\prime}\right)^{2}-2 \frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)}-\left(x^{\prime \prime}\right)^{2}} . \tag{6}
\end{align*}
$$

$x$ is where $h=0$ :

$$
\begin{equation*}
x=\sqrt{R\left(x^{\prime}\right)^{2}-2 \frac{R\left(x^{\prime}\right)}{R^{\prime}\left(x^{\prime}\right)}} . \tag{7}
\end{equation*}
$$

Then $A(x)$ is the area of the surface of revolution:

$$
\begin{equation*}
A(x)=2 \pi \int_{x}^{x^{\prime}} h\left(x^{\prime \prime}\right) \sqrt{1+\left(\frac{d h}{d x^{\prime \prime}}\right)^{2}} d x^{\prime \prime} \tag{8}
\end{equation*}
$$

The limit $R^{\prime}\left(x^{\prime}\right) \rightarrow 0$ is well-defined and in that case $A(x)$ is the cross-sectional area of the cylinder

$$
\begin{equation*}
\pi R(x)^{2} \tag{9}
\end{equation*}
$$

