Maximum dissipation principle in numerical simulation of complex fluids

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Outline

- Complex Fluids
- Energetic Variational Approaches on Simple Fluid
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- Numerical Results in Finite Element Methods
- Remarks

Complex Fluids

- Macroscopic fluids with microscopic configuration/structure
- Competition between the kinetic energy and the internal "elastic" energy
- Intrinsically multiscale-multiphysics model, or multicomponent

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Complex Fluids : Examples

- Viscoelastic Fluids (Polymeric Materials, Liquid Crystal,...)
- Electrorheological (ER) Fluids (Ion Dynamics(Electrolyte Solutions) in Biology, Semiconductor,...)
- Magnetohydrodynamical (MHD) Fluids
- Multi-phase Flows (Fluid-Fluid Mixture)

Complex Fluids

- The analysis of complex fluids is intrinsically difficult.
- Numerical computations are highly demanded to understand the complex fluids.
- The starting point of energetic variational approaches: Energy dissipation law for the whole coupled system,

$$\frac{d}{dt}E^{total} = -\triangle$$

where E^{total} is the total energy, $\triangle$ is the dissipation of the system.

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Energetic Variational Approaches: Flow Map

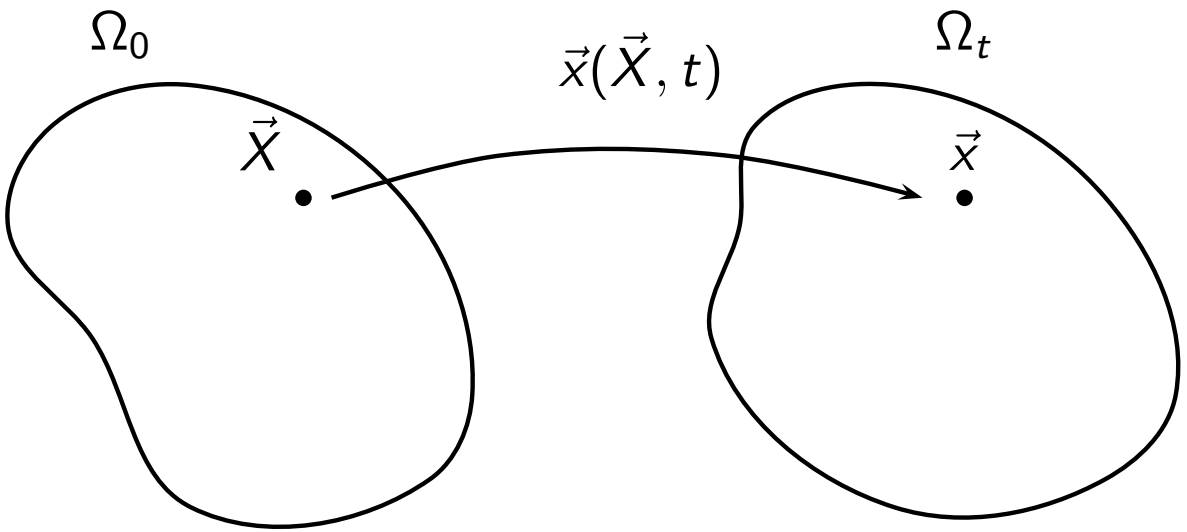

Figure: The flow map from the reference domain, Ω_0 to the current domain, Ω_t.

- Basic Mechanics
  - Flow Map (Trajectory) : $\vec{x}_t(\vec{X},t) = \vec{u}(\vec{x}(\vec{X},t),t), \quad \vec{x}(\vec{X},0) = \vec{X}.$
  - The deformation tensor (strain) of the flow map : $F(\vec{x}(\vec{X},t),t) = \frac{\partial \vec{x}(\vec{X},t)}{\partial \vec{X}}$ satisfying

$$F_t + \vec{u}\cdot\nabla F = \nabla\vec{u}F. \tag{1}$$

Energetic Variational Approaches

- Least Action Principle (Hamiltons Principle; Principle of Virtual Work) : the reversible part of the system, conservative force (f_c)

$$\delta \mathcal{A} = f_c \cdot \delta \vec{x}$$

  where $\mathcal{A}$ is the action functional defined from the energy.

- Maximum Dissipation Principle[1] (Onsager's Principle) : the irreversible part of the system, the dissipative force (f_d)

$$\frac{1}{2}\delta \triangle = f_d \cdot \delta \vec{u}.$$

- Force Balance Law through the energy law

[1] L. Onsager, *Reciprocal Relations in Irreversible Processes I*, Phys. Rev. 37(1931), pp.405–426, *Reciprocal Relations in Irreversible Processes II*, Phys. Rev. 38(1931), pp. 2265–2279, L. Onsager, and S. Machlup, *Fluctuations and Irreversible Processes*, Phys. Rev. 91(1953), pp. 1505–1512.

- Simple Fluid : the fluid described by the incompressible Navier-Stokes equations

$$\begin{aligned} \rho(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) + \nabla p &= \mu\Delta\vec{u}, \quad \text{(momentum conservation)} \\ \nabla\cdot\vec{u} &= 0, \quad \text{(incompressibility)} \\ \rho_t + \nabla\cdot(\rho\vec{u}) &= 0. \quad \text{(mass conservation)} \end{aligned} \tag{2}$$

  where ρ is mass, $\vec{u}$ is velocity field, p is pressure, and μ is the viscosity.

- The energy equation for incompressible Navier-Stokes equation:

$$\frac{d}{dt}\int \frac{1}{2}\rho|\vec{u}|^2\, d\vec{x} = -\int \mu|\nabla\vec{u}|^2\, d\vec{x}. \tag{3}$$

- The total energy, the dissipation :

$$E^{total} = \int \frac{1}{2}\rho|\vec{u}|^2\, d\vec{x}, \quad \triangle = \int \mu|\nabla\vec{u}|^2\, d\vec{x}, \tag{4}$$

Energetic Variational Approaches : Simple Fluid

- Define Action Functional, $\mathcal{A}$, for Least Action Principle (LAP)

$$\mathcal{A} = \int_0^T \int_{\Omega_t} \frac{1}{2}\rho|\vec{u}|^2 \, d\vec{x}dt. \tag{5}$$

- Pull back the current domain, Ω_t, to the reference one, Ω_0, through the flow map, $\vec{x}(\vec{X}, t)$

$$\mathcal{A}(\vec{x}) = \int_0^T \int_{\Omega_0} \frac{1}{2}\frac{\rho_0(\vec{X})}{\det F}|\vec{x}_t|^2 \det F \, d\vec{X}dt \tag{6}$$

  where $\rho_0(\vec{X}) = \rho(\vec{X}, t)|_{t=0}$ is the initial mass. Then $\frac{\delta\mathcal{A}(\vec{x})}{\delta\vec{x}} = 0$ gives the Hamiltonian (energy conserved) part under the incompressibility condition, $\nabla \cdot \vec{u} = 0$, i.e., $\det(F) = 1$.
- The resulting equation is the Euler equation : the total energy conservation

$$\begin{aligned} \rho(\vec{u}_t + \vec{u} \cdot \nabla\vec{u}) &= -\nabla p \\ \nabla \cdot \vec{u} &= 0. \end{aligned} \tag{7}$$

- Applying Maximum Dissipation Principle (MDP) for the dissipation in the form of the quadratic of the "rate" functions,

$$\frac{1}{2}\frac{\delta \triangle}{\delta \vec{u}}\Big|_{\varepsilon=0} = \int (\nabla \vec{u} + \varepsilon \nabla \vec{v}) \cdot \nabla \vec{v}\, d\vec{x}\Big|_{\varepsilon=0} = 0,$$

  then we obtain the Stokes equation,

$$\begin{aligned} \mu \Delta \vec{u} &= \nabla \tilde{p}, \\ \nabla \cdot \vec{u} &= 0. \end{aligned} \tag{8}$$

- Combine (7) and (8) then we have the incompressible Navier-Stokes equation (2)

Energetic Variational Approaches : Framework

1. Define an appropriate energy equation for describing the physical phenomenon.
2. Applying energetic variational approaches, LAP, MDP.
3. Combine the conservative part from LAP and the dissipative part from MDP through the force balance to satisfy the energy law.(The system of PDE is uniquely determined.)
4. Numerical Computations, developing numerical methods, existence, uniqueness of solutions

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- Immiscible Two-phase Flow[2]:
  - Let ϕ be a phase function $\phi(\vec{x}, t) = \pm 1$ in the incompressible fluids
  - Let $\Gamma_t = \{x : \phi(\vec{x}, t) = 0\}$ be the interface of mixture
  - In the Eulerian description, the immiscibility of fluids implies the pure transport of ϕ.

$$\phi_t + \vec{u} \cdot \nabla \phi = 0. \tag{9}$$

[2] C. Liu, and J. Shen, *A Phase Field Model for the Mixture of Two Incompressible Fluids and Its Approximation by a Fourier-Spectral Method*, Physica D, 179(2003), pp.211–228.

- Immiscible Two-phase Flow:
  - Ginzburg-Landau mixing energy, represents a competition between two fluids with their (hydro-) philic and (hydro-) phobic properties:

$$W(\phi, \nabla\phi) = \frac{1}{2}|\nabla\phi|^2 + \frac{1}{4\eta^2}(\phi^2 - 1)^2.$$

    The equilibrium profile of interface is tanh-like function as $\eta \to 0$.

  - The total energy is defined by the combination of the kinetic energy and the internal energy as follows:

$$E^{total} = \int_{\Omega_t} \left\{ \frac{1}{2}|\vec{u}|^2 + \lambda W(\phi, \nabla\phi) \right\} d\vec{x}. \tag{10}$$

    where λ is the contribution coefficient of the mixing energy to the total energy.

- Immiscible Two-phase Flow:
  - The action functional $\mathcal{A}$ from the total energy (10):

$$\mathcal{A} = \int_0^T \int_{\Omega_t} \left\{ \frac{1}{2} |\vec{x}_t|^2 - \lambda W(\phi, \nabla \phi) \right\} d\vec{x} dt. \tag{11}$$

  - The action functional $\mathcal{A}$ in terms of the flow map:

$$\mathcal{A}(\vec{x}) = \int_0^T \int_{\Omega_0} \left\{ \frac{1}{2} |\vec{x}_t|^2 - \lambda \left(\frac{1}{2} |F^{-1} \nabla_{\vec{X}} \phi|^2 + f(\phi_0) \right) \det F \right\} d\vec{X} dt. \tag{12}$$

where $f(\phi_0) = \dfrac{1}{4\eta^2} |\phi_0^2 - 1|^2$.

- Immiscible Two-phase Flow without Dissipation:
  - the LAP leads us to the following Hamiltonian system:

$$\begin{aligned} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \nabla\tilde{p} &= -\lambda\nabla\cdot(\nabla\phi\otimes\nabla\phi) && (13)\\ \nabla\cdot\vec{u} &= 0 && (14)\\ \phi_t + \vec{u}\cdot\nabla\phi &= 0. && (15) \end{aligned}$$

  - The above system (18) – (20) converges to the "sharp interface model" and its energy equation is

$$\frac{d}{dt}\int\left\{\frac{1}{2}|\vec{u}|^2 + \lambda\left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4\eta^2}(\phi^2-1)^2\right)\right\}d\vec{x} = 0. \qquad (16)$$

    : the total energy conservation, a Hamiltonian system, a conservative force with no dissipation of system.

- Immiscible Two-phase Flow:
  - Consider the following dissipation with the viscosity of fluids and the dissipation within the diffusive interface:

$$\triangle = \int \left(\mu |\nabla \vec{u}|^2 + \lambda\gamma \left| \Delta\phi - \frac{1}{\eta^2}(\phi^2 - 1)\phi \right|^2 \right) d\vec{x}. \tag{17}$$

where γ is the relation time. Since the dissipation of the diffusive interface is not a rate square, the maximum dissipation principle gives only the viscosity term, $\mu\Delta\vec{u}$.

- Immiscible Two-phase Flow:
  - The system with dissipation:

$$\begin{aligned} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \nabla\tilde{p} &= \mu\Delta\vec{u} - \lambda\nabla\cdot(\nabla\phi\otimes\nabla\phi) && (18)\\ \nabla\cdot\vec{u} &= 0 && (19)\\ \phi_t + \vec{u}\cdot\nabla\phi &= \gamma\left(\Delta\phi - \frac{1}{\eta^2}\left(\phi^2-1\right)\phi\right). && (20) \end{aligned}$$

  - The dissipative energy law is

$$\begin{aligned} &\frac{d}{dt}\int\left\{\frac{1}{2}|\vec{u}|^2 + \lambda\left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4\eta^2}(\phi^2-1)^2\right)\right\}d\vec{x} \\ &\qquad = -\int\left(\mu|\nabla\vec{u}|^2 + \lambda\gamma\left|\Delta\phi - \frac{1}{\eta^2}(\phi^2-1)\phi\right|^2\right)d\vec{x}. \end{aligned} \tag{21}$$

  - In numerical computation applying finite element methods one might need at least H^2 finite element space to satisfying the energy law.

- Immiscible Two-phase Flow with Dissipation:

  Reducing the computational cost in numerical experiment through the Maximum dissipation principle.

  - Manipulate the dissipation (17) in terms of a rate

$$\triangle = \int \left(\mu |\nabla \vec{u}|^2 + \frac{\lambda}{\gamma} |\phi_t + \vec{u} \cdot \nabla \phi|^2 \right) d\vec{x}. \tag{22}$$

  - Onsarger's principle (MDP) with incompressibility of flow, $\nabla \cdot \vec{u} = 0$ implies

$$\frac{1}{2} \frac{\delta \triangle}{\delta \vec{u}} \bigg|_{\varepsilon=0} = - \int \left\{ \mu \Delta \vec{u} - \frac{\lambda}{\gamma} (\phi_t + \vec{u} \cdot \nabla \phi) \nabla \phi \right\} \cdot \vec{v} \, d\vec{x} = 0. \tag{23}$$

- Immiscible Two-phase Flow:
  - The resulting system obtained by LAP and MDP with the manipulation on the dissipation is

$$\begin{aligned} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \nabla p &= \mu\Delta\vec{u} - \frac{\lambda}{\gamma}(\phi_t + \vec{u}\cdot\nabla\phi)\nabla\phi && (24)\\ \nabla\cdot\vec{u} &= 0 && (25)\\ \phi_t + \vec{u}\cdot\nabla\phi &= \gamma\left(\Delta\phi - \frac{1}{\eta^2}(\phi^2-1)\phi\right). && (26) \end{aligned}$$

  - The dissipative energy law:

$$\begin{aligned} &\frac{d}{dt}\int\left\{\frac{1}{2}|\vec{u}|^2 + \lambda\left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4\eta^2}(\phi^2-1)^2\right)\right\}d\vec{x} \\ &= -\int\left(\mu|\nabla\vec{u}|^2 + \frac{\lambda}{\gamma}|\phi_t + \vec{u}\cdot\nabla\phi|^2\right)d\vec{x}. \end{aligned} \qquad (27)$$

- Remark:
  - The dissipative force term in (24),

$$-\frac{\lambda}{\gamma}(\phi_t + \vec{u}\cdot\nabla\phi)\nabla\phi$$

    is exactly same as the conservative force in (18),

$$\nabla\cdot(\nabla\phi\otimes\nabla\phi - W(\phi,\nabla\phi)I)$$

  - As $\eta \to 0$, this is exactly the surface tension force on the interface[3]

[3] X.F. Yang, J.J. Feng, C. Liu, and J. Shen, *Numerical Simulations of Jet Pinching-off and Drop Formation Using An Energetic Variational Phase-Field Method*, J. Comput. Phys., 218(2006), pp.417–428.

- Immiscible Two-phase Flow (continue):
  - Computational Viewpoints:

$$\begin{aligned}\frac{d}{dt}\int\left\{\frac{1}{2}|\vec{u}|^2+\lambda\left(\frac{1}{2}|\nabla\phi|^2+\frac{1}{4\eta^2}(\phi^2-1)^2\right)\right\}d\vec{x}\\ =-\int\left(\mu|\nabla\vec{u}|^2+\frac{\lambda}{\gamma}|\phi_t+\vec{u}\cdot\nabla\phi|^2\right)d\vec{x}.\end{aligned}\tag{28}$$

    : Lower Order Approximation in Numerical Simulations

$$\begin{aligned}\frac{d}{dt}\int\left\{\frac{1}{2}|\vec{u}|^2+\lambda\left(\frac{1}{2}|\nabla\phi|^2+\frac{1}{4\eta^2}(\phi^2-1)^2\right)\right\}d\vec{x}\\ =-\int\left(\mu|\nabla\vec{u}|^2+\lambda\gamma|\Delta\phi-\frac{1}{\eta^2}(\phi^2-1)\phi|^2\right)d\vec{x}.\end{aligned}\tag{29}$$

    : High Order Approximation in Numerical Simulations

- Another Example: an incompressible viscoelastic complex fluid model[4] with the elastic energy, $W(F) = \frac{1}{2}|F|^2$

  - The energy law:

$$\frac{d}{dt}\int \left(\frac{1}{2}|\vec{u}|^2 + \frac{1}{2}|F|^2\right) d\vec{x} = -\int \mu|\nabla\vec{u}|^2\, d\vec{x}. \tag{30}$$

  - The system of equations satisfies the energy law (30):

$$\begin{aligned} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \nabla p &= \mu\Delta\vec{u} + \nabla\cdot\left(W_F F^{-T}\right) && (31)\\ \nabla\cdot\vec{u} &= 0 && (32)\\ F_t + \vec{u}\cdot\nabla F &= \nabla\vec{u}F && (33) \end{aligned}$$

    : $W_F F^{-T}$ is conservative force.

[4] F.H. Lin, C. Liu, and P. Zhang, *On Hydrodynamics of Viscoelastic Fluids*, Comm. Pure Appl. Math., LVIII(2005), pp. 1–35

- Another Example (continue): Add an artificial term ΔF into (33)

$$F_t + \vec{u} \cdot \nabla F = \nabla \vec{u} F + \varepsilon^2 \Delta F. \tag{34}$$

  - The energy law of the system (31), (32), (34):

$$\frac{d}{dt} \int \left(\frac{1}{2} |\vec{u}|^2 + \frac{1}{2} |F|^2 \right) d\vec{x} = - \int \left(\mu |\nabla \vec{u}|^2 + \varepsilon^2 |\nabla F|^2 \right) d\vec{x}. \tag{35}$$

  - Apply MDP: the resulting energy equation is

$$\begin{aligned} &\frac{d}{dt} \int \left(\frac{1}{2} |\vec{u}|^2 + \frac{1}{2} |F|^2 \right) d\vec{x} \\ &\quad = - \int \left(\mu |\nabla \vec{u}|^2 + \varepsilon^2 \mathscr{R} [F_t + \vec{u} \cdot \nabla F - \nabla \vec{u} F]^2 \right) d\vec{x}. \end{aligned} \tag{36}$$

  $W_F F^{-T}$ can also be derived from MDP.

- Finite Element Space :

$$\vec{u}_h \in \mathbf{V}_h = (P_1 \oplus \text{bubble})^2 \quad (37)$$
$$p_h \in W_h = P_1 \quad (38)$$
$$\phi_h \in Q_h = P_1 \quad (39)$$

for the finite dimensional solution pair $(\vec{u}_h, p_h, \phi_h)$.

- Numerical Method : The explicit-implicit scheme[5]

$$
\begin{aligned}
&(\tilde{\vec{u}}_{h,t}^{n+1}, \vec{v}_h) + \left(\left(\frac{3\vec{u}_h^n - \vec{u}_h^{n-1}}{2} \cdot \nabla\right) \vec{u}_h^{\frac{n+1}{2}}, \vec{v}_h\right) + \left(\frac{1}{2}\left(\nabla \cdot \frac{3\vec{u}_h^n - \vec{u}_h^{n-1}}{2}\right) \vec{u}_h^{\frac{n+1}{2}}, \vec{v}_h\right) \\
&-(p_h^{\frac{n+1}{2}}, \nabla \cdot \vec{v}_h) = -(\mu \nabla \vec{u}_h^{\frac{n+1}{2}}, \nabla \vec{v}_h) - \frac{\lambda}{\gamma}\left(\tilde{\phi}_{h,t}^{n+1} \nabla\left(\frac{3\phi_h^n - \phi_h^{n-1}}{2}\right), \vec{v}_h\right) \qquad (40) \\
&-\frac{\lambda}{\gamma}\left(\vec{u}_h^{\frac{n+1}{2}}, \nabla\left(\frac{3\phi_h^n - \phi_h^{n-1}}{2}\right)\right)\left(\nabla\left(\frac{3\phi_h^n - \phi_h^{n-1}}{2}\right), \vec{v}_h\right) \quad \text{for all } \vec{v}_h \in \mathbf{V}_h, \\
&\left(\nabla \cdot \vec{u}_h^{\frac{n+1}{2}}, w_h\right) + \varepsilon\left(p_h^{\frac{n+1}{2}}, w_h\right) = 0 \quad \text{for all } w_h \in W_h, \qquad (41) \\
&(\tilde{\phi}_{h,t}^{n+1}, q_h) + \left(\vec{u}_h^{\frac{n+1}{2}}, \left(\frac{3\phi_h^n - \phi_h^{n-1}}{2}\right) q_h\right) \\
&= -\gamma\left(\nabla \phi_h^{\frac{n+1}{2}}, \nabla q_h\right) - \frac{\gamma}{\eta^2}(f_h(\phi_h^n, \phi_h^{n+1}), q_h) \quad \text{for all } q_h \in Q_h \qquad (42)
\end{aligned}
$$

[5] P. Lin, and C. Liu, *Simulation of Singularity Dynamics in Liquid Crystal Flows: a C^0 Finite Element Approach*, J. Comput. Phys., **215(1)** (2006), 348–362. .

where

$$f_h(\phi_h^n, \phi_h^{n+1}) = \left\{ \frac{(|\phi_h^{n+1}|^2 - 1) + (|\phi_h^n|^2 - 1)}{2} \right\} \phi_h^{\frac{n+1}{2}},$$

$$\tilde{\vec{u}}_{h,t}^{n+1} = \frac{\vec{u}_h^{n+1} - \vec{u}_h^n}{\Delta t},\ \vec{u}_h^{\frac{n+1}{2}} = \frac{\vec{u}_h^{n+1} + \vec{u}_h^n}{2},\ \tilde{\phi}_{h,t}^{n+1} = \frac{\phi_h^{n+1} - \phi_h^n}{\Delta t},\ \phi_h^{\frac{n+1}{2}} = \frac{\phi_h^{n+1} + \phi_h^n}{2},$$

$(\cdot, \cdot)$ is the inner product operator, and Δt is the time step for simulations and ε is a small positive constant, $0 < \varepsilon << 1$.

The scheme satisfies the energy dissipation law:

$$\begin{aligned} &\left[\int \left\{ \frac{1}{2} |\vec{u}_h^{n+1}|^2 + \lambda \left(\frac{1}{2} |\nabla \phi_h^{n+1}|^2 + \frac{1}{4\eta^2} |\phi_h^{n+1\,2} - 1|^2 \right) \right\} d\vec{x} \right]_{h,t} = \\ &- \int \left\{ \mu |\nabla \vec{u}_h^{n+1}|^2 + \varepsilon |\phi_h^{\frac{n+1}{2}}|^2 + \frac{\lambda}{\gamma} \left| \tilde{\phi}_{h,t}^{n+1} + (\vec{u}_h^{\frac{n+1}{2}} \cdot \nabla) \left(\frac{3\phi_h^n - \phi_h^{n-1}}{2} \right) \right|^2 \right\} d\vec{x}. \end{aligned} \tag{43}$$

- Simulation 1 - Initial Data :

$$\vec{u}_0 = \vec{0}, \quad \phi_0 = \tanh\left(\frac{d_1(x,y)}{\eta\sqrt{2}}\right) + \tanh\left(\frac{d_2(x,y)}{\eta\sqrt{2}}\right) - 1.0 \tag{44}$$

with

$$\begin{aligned} d_1(x,y) &= \sqrt{(x-0.38)^2 + (y-0.5)^2} - 0.11, \\ d_2(x,y) &= \sqrt{(x-0.62)^2 + (y-0.5)^2} - 0.11, \end{aligned}$$

$\mu = 1.0,\ \eta = 0.01,\ \lambda = 0.01.$

Numerical Simulations

- Simulation 1.- Initial Data, ϕ_0 :

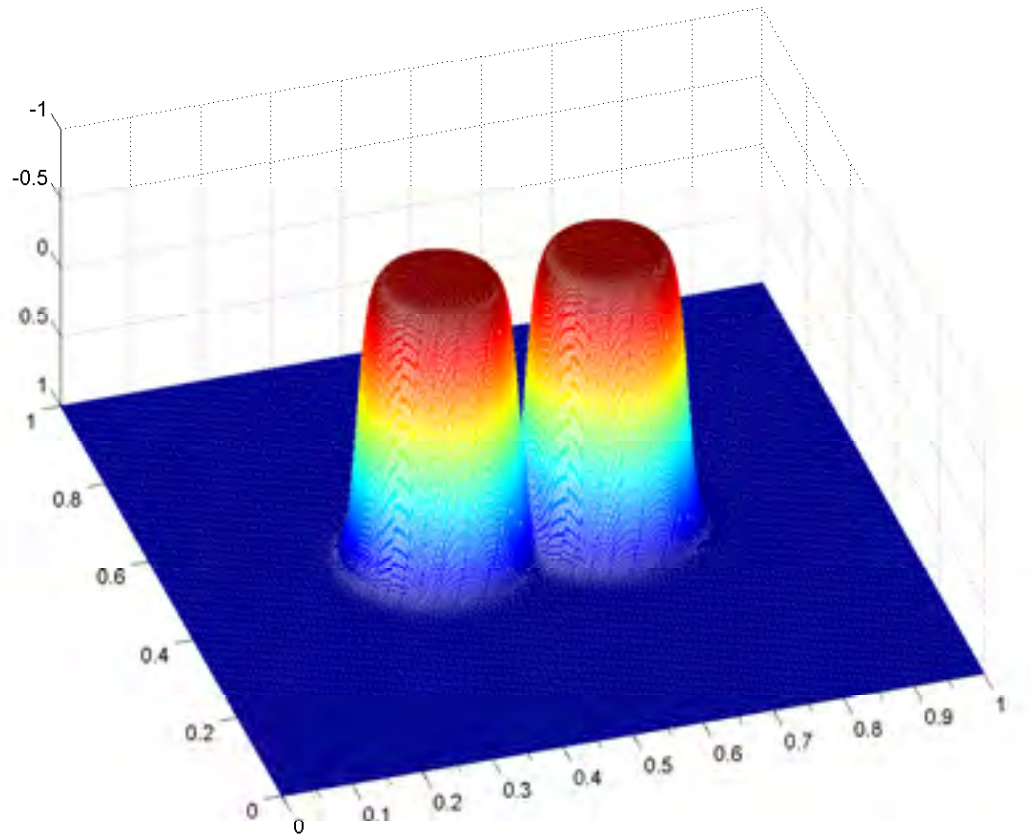


Figure: Initial Phase Field for Simulation.

Numerical Simulations

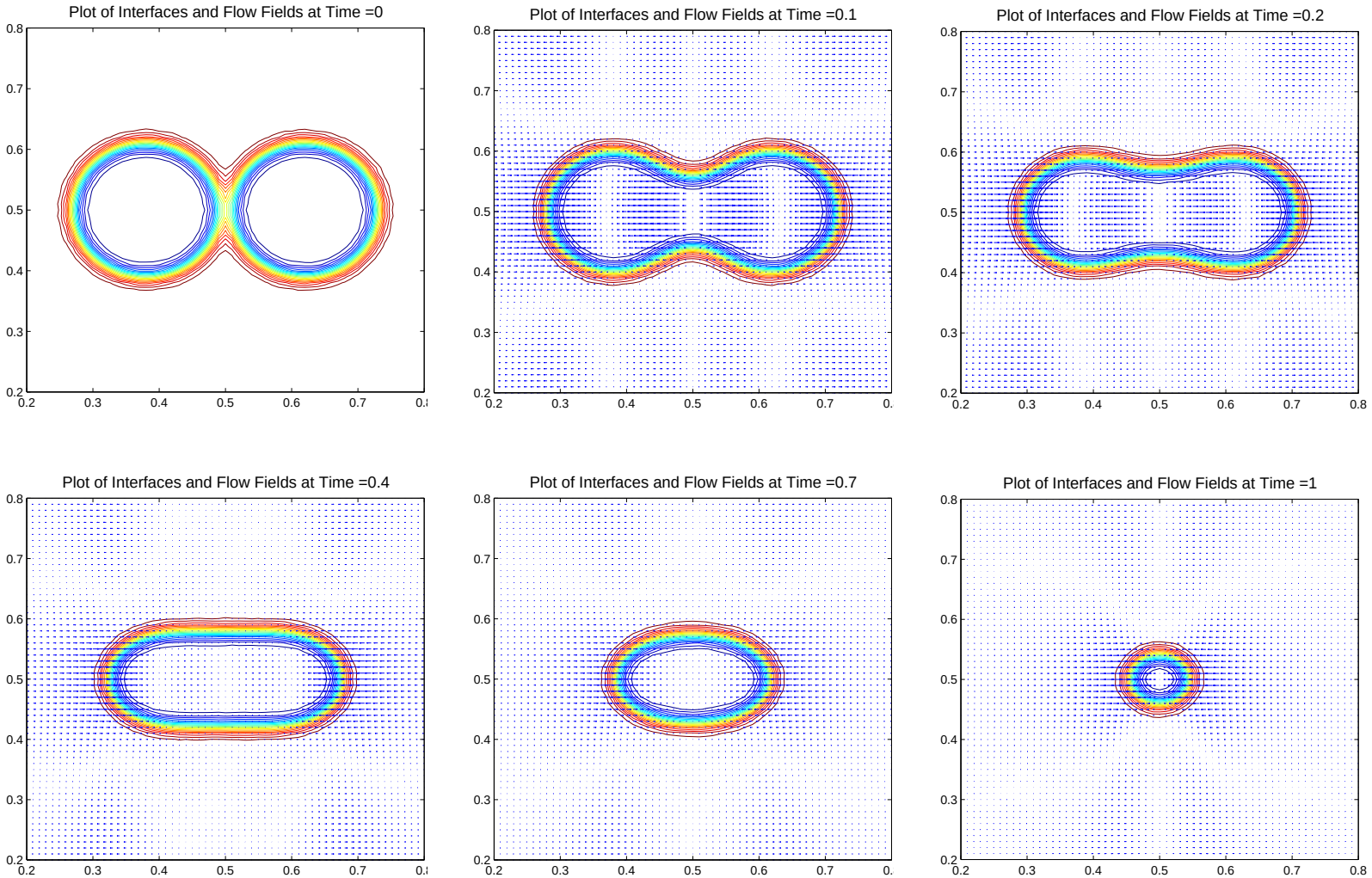


Figure: The time evolution results (merging effect) of phase field and velocity field from left to right and top to bottom ($t = 0.0, 0.1, 0.2, 0.4, 0.7, 1.0$ and $\Delta t = 0.001$).

Numerical Simulations

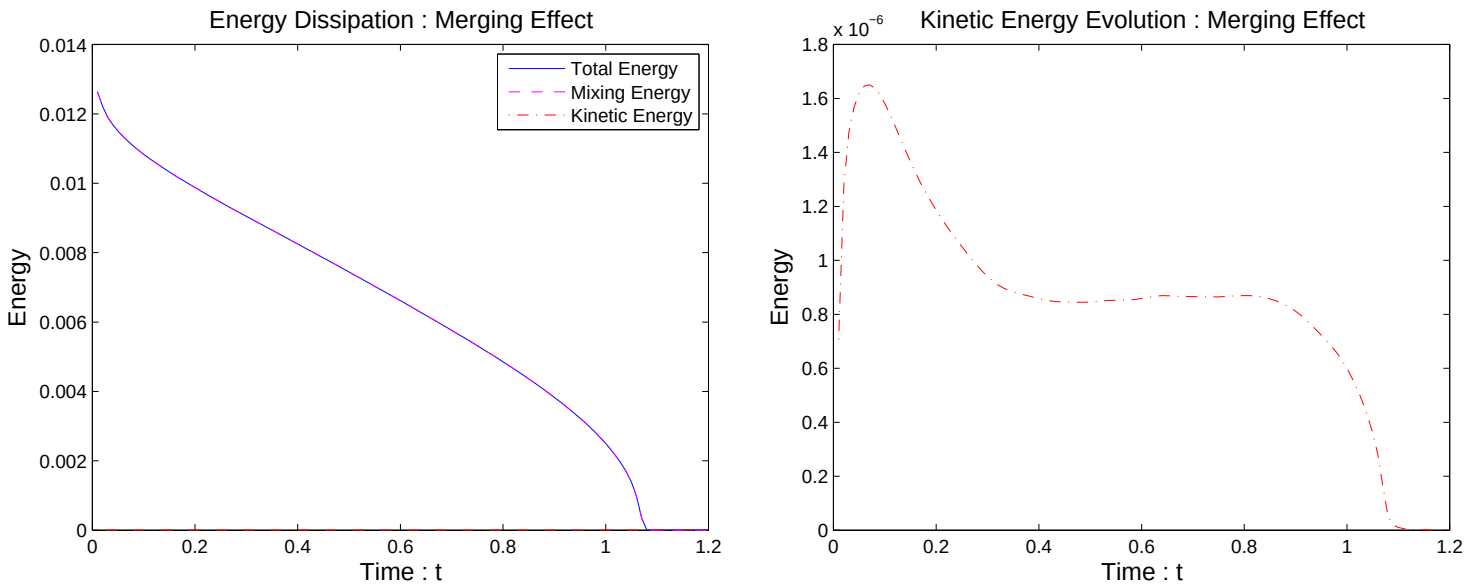


Figure: The total energy dissipation (left) and the kinetic energy (right) of merging phenomena in two-phase interface model.

- Simulation 2 - Initial Data :

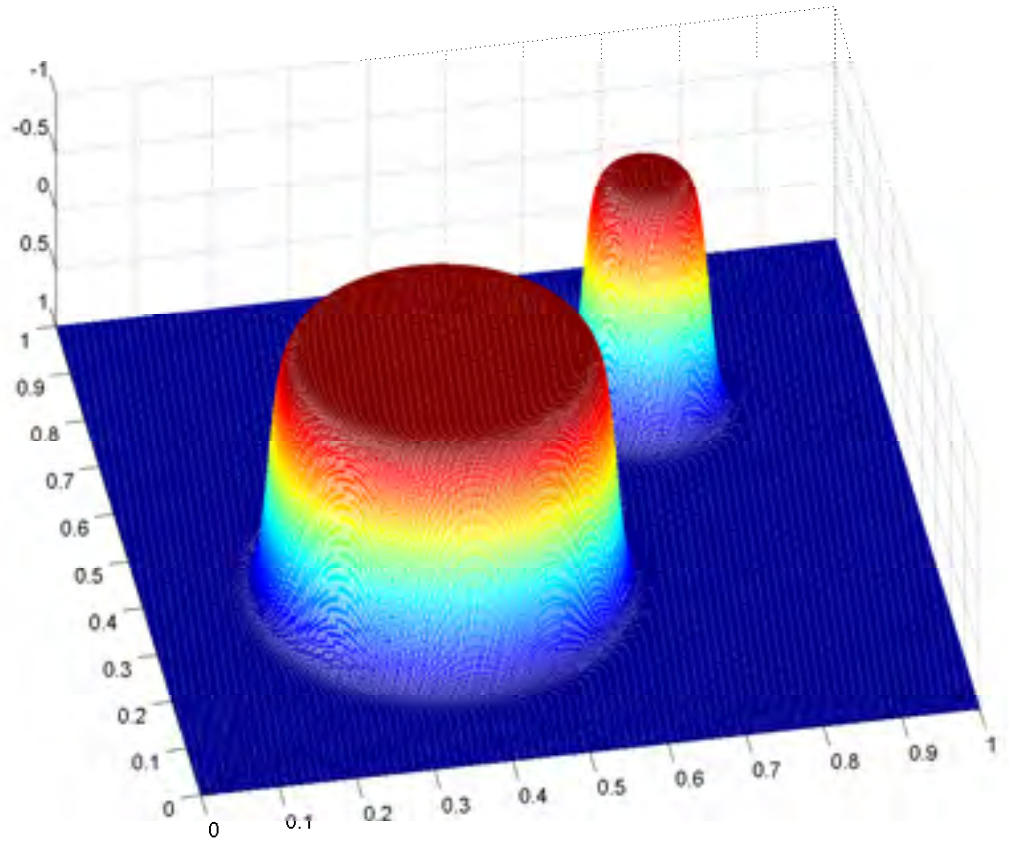
$$\vec{u}_0 = \vec{0}, \quad \phi_0 = \tanh\left(\frac{d_1(x,y)}{\eta\sqrt{2}}\right) + \tanh\left(\frac{d_2(x,y)}{\eta\sqrt{2}}\right) - 1.0 \tag{45}$$

with

$$\begin{aligned} d_1(x,y) &= \sqrt{(x-0.38)^2 + (y-0.38)^2} - 0.22, \\ d_2(x,y) &= \sqrt{(x-0.70)^2 + (y-0.70)^2} - 0.08, \end{aligned}$$

$\mu = 1.0,\ \eta = 0.01,\ \lambda = 0.01.$

- Simulation 2.- Initial Data, ϕ_0 :



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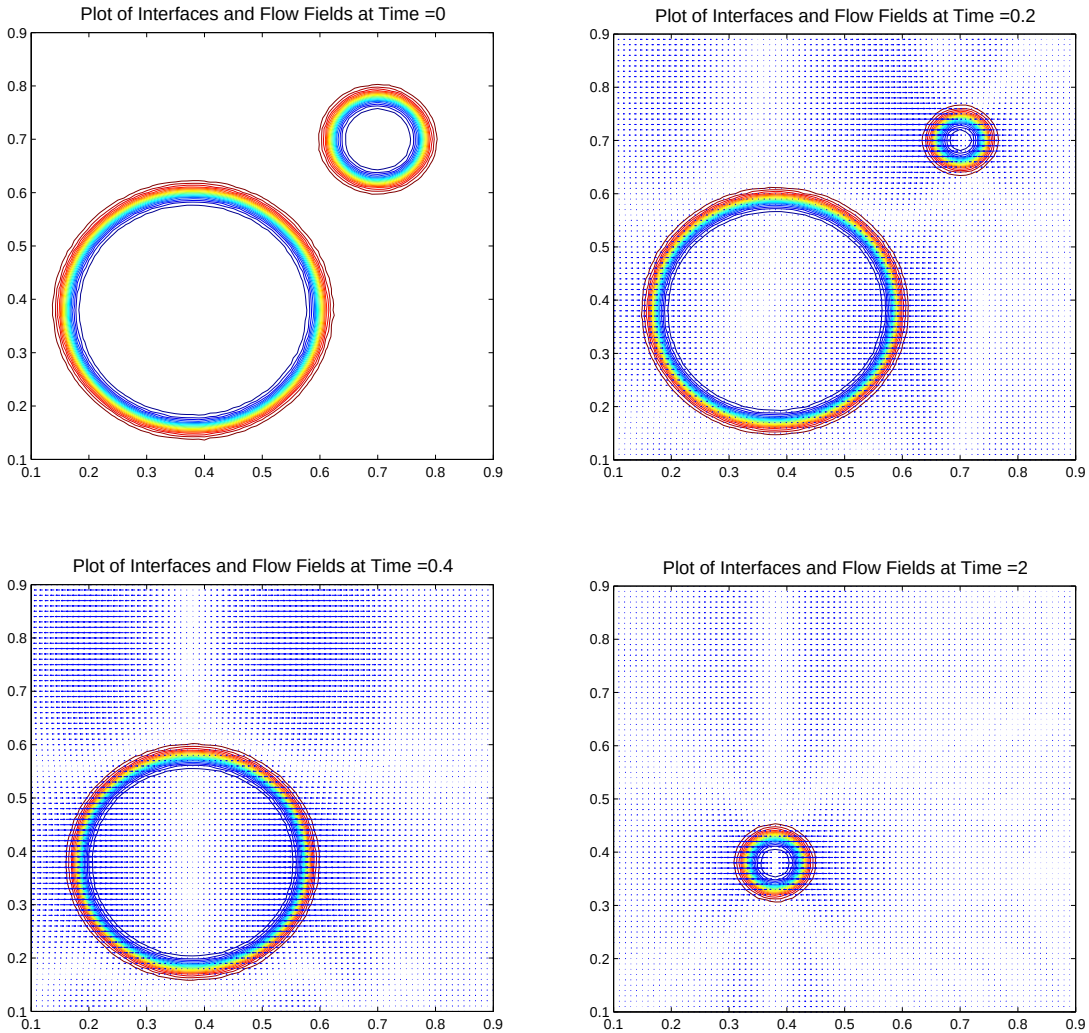

Figure: The time evolution results of phase field and velocity field from left to right and top to bottom ($t = 0.0, 0.2, 0.4, 2.0$ and $\Delta t = 0.001$).

Numerical Simulations

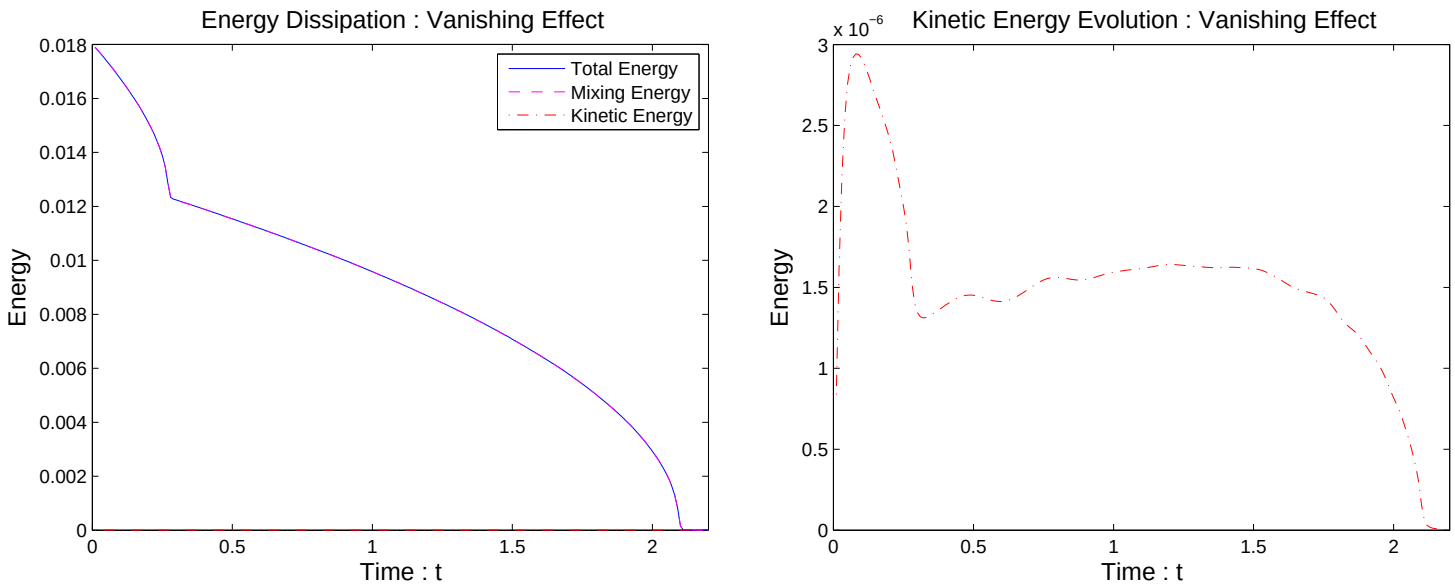

Figure: The total energy dissipation (left) and the kinetic energy (right) of vanishing phenomena in two-phase interface model.

Remarks

- LAP : Conservative Force
- MDP : Dissipative Force
- MDP : Whether the dissipation functional includes the "rate" functions in time t of all "variables"
- The conservative force in hydrodynamic system is consistent with the dissipative force
- MDP plays an important role in designing numerical algorithms to solve the hydrodynamic complex fluid problem
- Difficulty : The optimal order of convergence for the finite element space

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Thank You!