Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

An Introduction of Elastic Complex Fluids: An Energetic Variational Approach

Chun Liu, Penn State University

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Contents

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

1 PREFACE

Complex fluids such as polymeric solutions, liquid crystal solutions, pulmonary surfactant solutions, electro-rheological fluids, magneto-rheological fluids and blood suspensions exhibit many intricate rheological and hydrodynamic features that are very important to biological and industrial processes. Applications include the treatment of airway closure disease by surfactant injection; polymer additive to jets in inkjet printers, fuel injection, fire extinguishers; magneto-rheological damping of structural vibrations etc. The segregation, migration and aggregation of the particles and the stretching, coiling and entanglement of the molecules in the complex fluids that endows them with the unique rheological and hydrodynamic properties required for specific biological, physiological and industrial needs. One good example is the migration of blood cells in arteries towards the center axis (the Fahreus-Lynquist effect). This segregation leaves a low viscosity plasma marginal layer that helps reduces the overall resistance to blood flow. This complex physiological rheology has important implications in blood pressure, clotting, plaque formation and other cardiovascular diseases. An important goal of the large and multi-disciplinary field of fluid mechanics is to derive continuum partial differential equations (field equations) to describe the rheology of these various fluids and to solve these equations to explain and predict their macroscopic behavior.

The most common origin and manifestation of anomalous phenomena in complex fluids are different "elastic" effects. They can be the elasticity of deformable particles, elastic repulsion between charged liquid crystals, polarized colloids or multi-component phases, elasticity due to microstructures, or bulk elasticity endowed by polymer molecules in viscoelastic complex fluids. The physical properties are purely determined by the interplay of entropic and structural intermolecular elastic forces and interfacial interactions. These elastic effects can be represented in terms of certain internal variables, for example, the orientational order parameter in liquid crystals (related to their microstructures), the distribution density function in the dumb-bell model for polymeric materials, the magnetic field in magneto-hydrodynamic fluids, the volume fraction in mixture of different materials etc. The different rheological and hydrodynamic properties can be attributed to the special coupling between the transport of the internal variable and the induced elastic stress. In our energetic formulation, this represents a competition between the kinetic energy and the elastic energy. We look at the following system (a simplified Ericksen-Leslie system modeling the flow of nematic liquid crystals) as an example for such complex

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

fluids:

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla p - \nu\Delta u + \lambda\nabla\cdot(\nabla d\odot\nabla d) &= 0, \qquad (1.1)\\ d_t + (u\cdot\nabla)d - \gamma(\Delta d - f(d)) &= 0, \qquad (1.2)\end{aligned}$$

with $\nabla\cdot u = 0$, where u represents the flow velocity, p the pressure, d represents the *normed* director, $f(d) = F'(d)$ where $F(d)$ is the bulk part of the elastic energy. It is the coupling between the transport of d (material derivative here) and the induced elastic stress $(\nabla d\odot\nabla d)_{ij} = \sum_{k=1}^{n}(\nabla_i d_k)(\nabla_j d_k)$ that yields the following energy law, which presents the dissipative nature of the system:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|u|^2 + \lambda|\nabla d|^2 + 2\lambda F(d))dx = -\int_\Omega (\nu|\nabla u|^2 + \lambda\gamma|\Delta d - f(d)|^2)dx. \qquad (1.3)$$

On the other hand, the force balance (momentum) equation can be derived by the Least Action Principle, using the total energy functional and the way the internal variable d is transported. The competition between kinetic and elastic energy also produces the specific properties of the system, such as the stability and regularity of the hydrostatic configurations. When applied to micro-particles or molecules, the elastic energy determines the microstructures formation and how they interact with the fluid. The understanding of such underlying structures is also crucial in designing the accurate numerical algorithms in order to simulate the system, especially when the solutions involve singularities.

Most complex fluid behavior results from the multi-scale properties of the fluid material at the micro-structure scales. Hence, understanding complex fluid rheology and hydrodynamics must necessarily begin at the molecular and particulate level. The Fokker-Planck, Ginzburg-Landau or Liouville type statistical equations describing the nanoscale molecular dynamics or the microscale particulate dynamics are used to obtain rheological constitutive equations through least action principles, as have been done for viscoelastic polymeric fluids and liquid crystal solutions. The systems will satisfy the energy law (Second Law of Thermodynamics). The resulting partial differential equation system will involve multiple scales. In order to obtain the effective continuum equations at the macroscopic scale, mean field theories are often invoked to obtain closure in such field theoretic approaches. When these constitutive equations are inserted back into the Cauchy equation for force balance, the desired partial differential equation results.

The Navier-Stokes equation is the simplest of these, and fortunately, it does obey an energy law. On the other hand, the dumbbell model equa-

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

tion for polymeric materials loses the energy law after closure, even for the simplest Hookean case. Recently, more and more studies show that this classical approach is inadequate due to several deficiencies. Pertinent physics at the particulate and molecular level remains elusive for many complex fluids. For example, blood cell segregation shown in figure 1 has been attributed to particle deformation, inertia, asymmetry and a host of other origins. Even when the physics are known, some microscale phenomena remain unexplored due to mathematical and/or numerical difficulties. For example, defects in the liquid crystal have been shown to produce bulk flow due to the elastic pressure gradient they generate. The resulting flow can also destroy the defects and hence change the bulk rheology.

In this lecture note, we intend to introduce some of the mathematical tools, modeling, analysis and numerics, that are useful in studying these important and complicated materials. The brief description of the contents will suffice to show that this note is in no sense a systematic study of the broad area of complex fluids. Many important topics are not touched at all here. We hope this will just serve as an introduction and some reference for the students who will become interested in these fascinating subjects.

2 Calculus of Variations

We begin the short course by reviewing some basic mathematical tools in the theory of calculus of variations. All the materials can be found in the following references [33, 34, 90].

- L. C. Evans. *Partial Differential Equations*. AMS, 1998.
- L. C. Evans. *Weak Convergence Methods for Nonlinear Partial Differential Equations*. AMS, 1990.
- M. Struwe. *Variational Methods*. Spring-Verleg, 1990.

2.1 Euler-Lagrange equations.

For a given a Banach space A and a functional

$$E : A \longrightarrow D, \tag{2.1}$$

the Euler-Lagrange equation is defined as:

$$DE(u) = 0, \tag{2.2}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

where $DE : A \to A^*$ is the Frechet derivative defined by

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(u + \epsilon v) = (DE(u), v) = 0. \tag{2.3}$$

Example. For a functional $W(u, \nabla u)$, the corresponding Euler-Lagrange equation will be

$$-\nabla \cdot \frac{\partial W}{\partial \nabla u} + \frac{\partial W}{\partial u} = 0, \tag{2.4}$$

which in the weak form will be

$$(\frac{\partial W}{\partial \nabla u}, nablav) + (\frac{\partial W}{\partial u}, v) = 0, \tag{2.5}$$

for any test function v.

2.2 Direct methods

The following basic concepts are crucial for the direct method of calculus of variation:

- Lower semicontinuous: $\{u \in A | W(u) > a\}$ is open in A.
- Sequentially weak lower semicontinuous: If $u_n \to u$ weakly in A, then

$$W(u) \leq \lim \inf_{n\to\infty} W(u_n). \tag{2.6}$$

- Coercivity: If $|u_n|_A \to \infty$, then $W(u) \to \infty$.

With these concepts, we can state the following theorem.

Theorem 1 *If A is a reflxive Banach space. W is a nonnegative functional and is both coercive and lower semiconinuous, then W attains its infinimun in A.*

Proof. See Evan's or Struwe's book.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

2.3 Convexity

Given a functional $W(x, u, \nabla u)$, it is convex if $W_{\nabla_i u \nabla_j u}(x, u, \nabla u)$ is non-negative when u is a minimizer.

Theorem 2 *If W is bounded below, convex in ∇u. Then W is weakly lower semicontinuous.*

Proposition 1 *If W is coercive and convex, then there exists at least one minimizer.*

Theorem 3 *(Uniqueness) If W is strictly convex, then the minimizer is unique.*

Example. Given $u \in W^{1,q}(\Omega)$ and

$$|W(x, z, p) \leq c(|p|^q + |z|^q + 1) \tag{2.7}$$

$$|\nabla_p W|, \, |\nabla_z W| \leq c(|p|^{q-1} + |z|^{q-1} + 1), \tag{2.8}$$

the weak solution of Euler-Lagrange equation is

- The minimizer satisfies the Euler-Lagrange equation.
- Coercivity: $W(x, z, p) \geq \alpha |p|^q - \beta$.
- Convexity in p gives the existence of minimizer.

2.4 Dynamics

The gradient flow (fastest decent): In the case when only the long time behavior of the solution are important, the gradient flow will determine the properties of the solution. Moreover, the gradient flow also gives a method to achieve the stationary solution of the Euler-Lagrange equations.

$$u_t = -\gamma \frac{\delta W}{\delta u}, \tag{2.9}$$

where γ represents the relaxation time. The solution of the above equation (with either Dirichlet or natural boundary conditions) satisfies the following dissipative law:

$$\frac{d}{dt} \int_\Omega W \, dx = -\frac{1}{\gamma} \int_\Omega |u_t|^2 \, dx. \tag{2.10}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Remark 1 *(Long time behavior) For time $t \to \infty$. From Fubini's Theorem, there exists a subsequence t_i such that $u_t(\cdot, t_i) \to 0$ and*

$$\frac{\delta W}{\delta u} \to 0, \tag{2.11}$$

hence $u(\cdot, t_i)$ approaches to a stationary solution.

Damped wave equation.

$$\epsilon u_{tt} + u_t = -\gamma \frac{\delta W}{\delta u}, \tag{2.12}$$

The energy law becomes

$$\frac{d}{dt} \int_\Omega \gamma W + \frac{\epsilon}{2} |u_t|^2 \, dx = - \int_\Omega |u_t|^2 \, dx. \tag{2.13}$$

It is from this energy law that we can see the long time behavior of the solution are determined by the gradient flow.

2.5 Hamilton's Principle

Hamilton's Principle, which is also referred to as Principle of virtual work or the Least Action Principle, are the most fundamental principle in mechanics. In fact, it gives the momentum equations — the force balance equations.

The material presented here can be founded in the following classical references [3, 1, 74]:

- V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1978.
- R. Abraham and J. E. Marsden. *Fundations of Mechanics.* Springer-Verlag, 1978.
- J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry.* Springer-Verlag, 1999.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

2.5.1 Flow map and deformation tensor

The evolution of all materials involves the following basic mechanical concepts:

- Lagrangian coordinate (original labelling): X.

  Eulerian coordinate (observer's coordinate): x.

- Flow map (trajectory): $x(X,t)$ such that

$$x_t = u(x(X,t),t),\ x(X,0) = X, \tag{2.14}$$

  where $u(x,t$ is the velocity field.

  Remark 2 *If u is Lip in x then the flow map is uniquely determined.*

- Deformation: $F_{ij}(X,t) = \frac{\partial x_i}{\partial X_j}$.

  Without ambiguity, we can define $F(x(X,t),t) = F(X,t)$. The simple application of the Chain Rule gives the following important transport equation of F:

$$F_t + u \cdot \nabla F = \nabla u F. \tag{2.15}$$

- Each of the following equivalent statements will represent the incompressibility of the material.

  1. $\det F = 1$;
  2. $\operatorname{div} u = 0$ (from the identity $\delta \det F = \det F \operatorname{tr}(F^{-1}\delta F)$.
  3. $\nabla \cdot F_t + (u \cdot \nabla)(\nabla \cdot F) = 0$.

2.5.2 Variation of the domain v.s. variation of the function

Given an energy functional $W(\phi, \nabla\phi)$, depending on some variable ϕ, in order to find the critical point, we can employ each of the following two methods:

- Euler-Lagrange equation (variation with respect to ϕ): $\frac{\delta W}{\delta \phi} = 0$, which is expressed in the weak form as

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

$$(\frac{\partial W}{\partial \nabla \phi}, \nabla \psi) + (\frac{\partial W}{\partial \phi}, \psi) = 0, \tag{2.16}$$

for any test function ψ.

We note that the usual energy estimates are derived by setting $\psi = \phi$.

- Variation with respect to domain: $\frac{\delta W}{\delta x} = 0$, and the result can be expressed in the weak form as

$$(\frac{\partial W}{\partial \nabla \phi} \otimes \nabla \phi - WI, \nabla y) = 0, \tag{2.17}$$

for any test function y.

The formal equivalency of the two procedure is reflected in the following theorem.

Theorem 4 *Given an energy functional $W(\phi, \nabla \phi)$, all solutions of the Euler-Lagrangian equation:*

$$-\nabla \cdot \frac{\partial W}{\partial \nabla \phi} + \frac{\partial W}{\nabla \phi} = 0$$

also satisfy the equation

$$\nabla \cdot (\frac{\partial W}{\partial \nabla \phi} \otimes \nabla \phi - WI) = 0.$$

The proof of the theorem is the consequences of direct computations. From this theorem, we can immediately make the following remarks:

- Pohozaev inequality: set $y = x$ (same as multiply Euler-Lagrange equation by $x \cdot \nabla \phi$). This extra inequality is very important in the study of semilinear elliptic equations [90, 34].

- The variation of the domain require more regularity than that of the normal weak solutions of the Euler-Lagrange equations. This is in connection of the stationary weak solution for harmonic maps [86].

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

2.5.3 Least action principle

The force balance equation (momentum conservation law) is the state such that the flow map minimizes the action functional:

$$\mathcal{A}(x) = \int_0^T \int_{\Omega_0} (\frac{1}{2}\rho|u|^2 - W(\phi(x))\det F \, dX dt. \tag{2.18}$$

Here $W(\phi$ is the elastic internal energy, $\frac{q}{2}\rho|u|^2$ is the kinetic energy.

2.6 Constraint problems

Most physical problems involve finding the minimizers (critical points) in a constraint class of functions. The method of Lagrange multiplier is the basic tool for such a problem. However, this brings some extra difficulties and we will illustrate this using the following examples.

2.6.1 Harmonic maps

This is a simpl, but most classical example that can illustrate the role of constraint in the calculus of variations and the difficulties associated with it [86, 85, 5, 9, 44, 61].

For any function $u : \Omega \to B_1(0)$ with target space the unit sphere, we want to find the minimize the following Dirichlet energy:

$$W(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \, dx, \tag{2.19}$$

with Dirichlet boundary condition:

$$u|_{\partial\Omega} = u_0. \tag{2.20}$$

The Euler-Lagrange equation will be:

$$-\Delta u = \lambda(x)u, \tag{2.21}$$

where the Lagrange multiplier $\lambda(x) = |\nabla u|^2$ (with the help of the identity $\Delta uu = \Delta\frac{|u|^2}{2} - |\nabla u|^2$). Notice the difficulty of high nonlinearity on the right hand side of the equation. Moreover, the right hand side does posses the property of being a form of total derivation (like a Jacobian). Using this, Helein obtained the regularity in 2-dimensional cases [90].

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

2.6.2 Liquid crystals

For the uniaxial nematic liquid crystal materials, the bulk energy density in Ericksen's model are assumed to depend on the orientation vector (optical director) n, with $|n| = 1$, and the orientational order $s \in [-\frac{1}{2}, 1]$. Again we do not consider the effects due to surface energies, applied fields etc. The energy is given by

$$0 \leq w(s, n, \nabla s, \nabla n) \equiv w_0(s) + w_2(s, n, \nabla s, \nabla n). \tag{2.22}$$

Here the behavior of $w_0(s)$ is the bulk part of the energy.

The case $s = 1$ corresponding to the property that each molecules is perfectly aligned, and the case $s = -\frac{1}{2}$ means all molecules are lie in a plane perpendicular to the optical axis. Both situations are physically unrealistic and therefore we can have

$$w_0(-\frac{1}{2}) = w_0(1) = +\infty. \tag{2.23}$$

The term w_2 takes the form:

$$\begin{aligned} 2w_2 &= k_1|\text{ div } n|^2 + k_2|n \cdot \text{ curl } n|^2 + k_3|n \wedge \text{ curl } n|^2 && (2.24)\\ &+(k_2 + k_4)[\text{tr } (\nabla n)^2 - (\text{div } n)^2] + L_1|\nabla s|^2 + L_2(\nabla s \cdot n)^2 && (2.25)\\ &+L_3(\nabla s \cdot n)\text{div } n + L_4 \nabla s \cdot (\nabla n)n. && (2.26) \end{aligned}$$

where $k's$ and $L's$ are functions of s as well as the temperature θ.

With the help of the following identity

$$\begin{aligned} \text{div } (f[(\nabla n)n - (\text{div } n)n]) &= f[\text{tr } (\nabla n)^2 - (\text{div } n)^2] \\ &\quad +\nabla f \cdot [(\nabla n)n - (\text{ div } n)n] \end{aligned} \tag{2.27}$$

$$\begin{aligned} 2W_2 &= \overline{K}_1(\text{ div } n)^2 + K_2|n \cdot \text{ curl } n|^2 + \overline{K}_3|n \wedge \text{ curl } n|^2 \\ &\quad +(K_2 + K_4)[\text{tr } (\nabla n)^2 - (\text{ div } n)^2] \\ &\quad +K_5|\nabla s - (\nabla s \cdot n)n - \nu(\nabla n)n|^2 \\ &\quad +K_6(\nabla s \cdot n - \sigma \text{ div } n)^2 \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} \overline{K}_1 &= K_1 - \sigma^2 K_6 = K_1 - \tfrac{L_3^2}{4(L_1+L_2)} \\ \overline{K}_3 &= K_3 - \nu^2 K_5 = K_3 - \tfrac{L_4^2}{4L_1} \\ K_5 &= L_1, \quad K_6 = L_1 + L_2 \\ \nu &= -\tfrac{L_4}{2L_1}, \quad \sigma = -\tfrac{L_3}{2(L_1+L_2)}. \end{aligned} \tag{2.29}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

The size of various constants are characterized below.

$$\begin{array}{l} \overline{K}_1 > 0, \quad K_2 > |K_4|, \quad \overline{K}_3 > 0, \\ K_5 > 0, \quad K_6 > 0, \quad \text{for } s \neq 0. \\ \sigma, \nu \cong 0(s), \quad \overline{K}_1, K_2, \overline{K}_3, K_4 \cong 0(s^2) \\ K_6 \cong 0(s) \end{array} \tag{2.30}$$

The simplest form of the bulk energy density is

$$w_0(s) + k|\nabla s|^2 + s^2|\nabla n|^2 \tag{2.31}$$

Here one has $K_1 = K_3 = s^2$, $K_4 = L_2 = L_3 = L_4 = 0$, and $L_1 = k$. We shall see later this form of energy functional is closely related to the energy functional of maps from a domain in R^3 to a circular cone in $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$ the Minkowski space (cf. also [L3]).

The classical Oseen-Frank model can be derived from the Ericksen's model by imposing the additional constraint on the orientational order, $s = s^*$.

We note that the simplest form of such energy densities is $2w = |\nabla n|^2$. This corresponds to the case $k_1 = k_2 = k_3 = 1$ and $k_4 = q = 0$. The corresponding mathematical problem is to study harmonic maps from a domain to S^2 or RP^2.

Strong Anchoring Condition. When the surface of a container of liquid crystals is specially treated, the orientation of the liquid crystals molecules near the surface of container will aligned with the treatment and hence can be specified. This is usually referred to as the *strong anchoring condition*. Mathematically we can describe it as following Dirichlet boundary value problem.

2.6.3 Methods of penalty

We will just look at the harmonic problem. In order to avoid the nonlinearity in the problem, we will introduce the following approximate problem [18, 91]:

$$\min_{u \in H^1(R^3)} \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(|u|^2 - 1)^2 \, dx. \tag{2.32}$$

The above functional is also called the Ginzburg-Landau functional, which arises from the theory of superconductivity [23, 6, 26].

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

We can see, as $\epsilon \to 0$, u will convergent to a unit vector. The Euler-Lagrange equation of the approximate problem is:

$$-\Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0. \tag{2.33}$$

For each fixed ϵ, the solution is smooth. As ϵ approaches zero, the solution of the Ginzburg-Landau equation will convergence (weakly) to a solution of the harmonic map [91].

Finally, we will discuss a new type of relaxation that was discussed with M. Chipot and D. Kinderlehrer [19].

We will study the following minimization problem under relaxed constraint:

$$\min_{u \in A_\epsilon} E(u), \tag{2.34}$$

where $A_\epsilon = \{v \in H^1(\Omega), v|_{\partial\Omega} = g(x), \int_\Omega (|u|^2 - 1)^2\, dx \le \epsilon^2\}$. Notice, the relaxation is in the constraint, rather in the energy functional itself.

Lemma 1 *If u_ϵ is a minimizer of the problem (2.34), then*

$$\int_\Omega (|u_\epsilon|^2 - 1)^2\, dx = \epsilon^2. \tag{2.35}$$

Proof We will prove this lemma is by contradiction. If the statement is false, that is,

$$\int_\Omega (|u_\epsilon|^2 - 1)^2\, dx < \epsilon^2, \tag{2.36}$$

Then for variations $\delta\phi$ of small δ, we have $u_\epsilon + \delta\phi \in A_\epsilon$. Hence we have that u_ϵ will satisfies the Euler-Lagrange equation

$$-\Delta u_\epsilon = 0, \tag{2.37}$$

with boundary condition $g(x)$. So we get $u_\epsilon = \hat{u}$ which is independent to ϵ. $\int_\Omega (|u_\epsilon|^2 - 1)^2\, dx = \int_\Omega (|\hat{u}|^2 - 1)^2\, dx > \epsilon$ for ϵ sufficiently small, we get contradiction.

The following lemma is obvious from the definitions.

Lemma 2

$$\min_{u \in A_\epsilon} E(u) \le \min_{|u|=1} E(u) = M \tag{2.38}$$

where M is a constant independent to ϵ.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Proof Notice here we have $\min_{|u|=1} E(u) = \min_{u\in A_0} E(u)$ and lemma follows immediately.

From the lemma we see that if u_ϵ is the minimizer of the problem (2.34) for each ϵ, the they all satisfy the following equations:

$$-\Delta u_\epsilon = \lambda_\epsilon(|u_\epsilon|^2 - 1)u_\epsilon, \tag{2.39}$$

with boundary condition

$$u_\epsilon|_{\partial\Omega} = g(x), \tag{2.40}$$

and the uniform bound

$$\int_\Omega |\nabla u_\epsilon|^2\, dx \le M. \tag{2.41}$$

Pass to the limit of $\epsilon \to 0$, we have (up to a subsequence) that $u_\epsilon \to u^*$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere in Ω.

One takes the cross product of the equation (2.39) by u_ϵ to get

$$\nabla \cdot (\nabla u_\epsilon \times u_\epsilon) = 0, \tag{2.42}$$

Pass to the limit in the weak formulation, we can get that

$$\nabla \cdot (\nabla u^* \times u^*) = 0. \tag{2.43}$$

On the other hand, since we have

$$\int_\Omega (|u_\epsilon|^2 - 1)^2\, dx \to 0, \tag{2.44}$$

By Fatou's lemma,

$$|u^*| = 1. \tag{2.45}$$

We will have the following main theorem:

Theorem 5 *u^* satisfies the harmonic map equation:*

$$-\Delta u^* = |\nabla u^*|^2 u^*, \tag{2.46}$$

Proof To prove the theorem, we use (2.43) and get that:

$$\Delta u^* \times u^* = \nabla \cdot (\nabla u^* \times u^*) = 0. \tag{2.47}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

This mean that Δu^* is parallel to u^*. In a weak form, we see that, for any $v \in Hq_0(\Omega)$, $v - (v \cdot u^*)u^*$ is perpendicular to u^*. Then (2.47) is equivalent to

$$\int_\Omega \nabla u^* \nabla (v - (v \cdot u^*)u^*)\, dx = 0.$$

The left hand side is equal to

$$\int_\Omega \nabla u^* \nabla v - |\nabla u^*|^2 v \cdot u^* - \nabla (v \cdot u^*) u^* \cdot \nabla u^*\, dx.$$

The last term is equal to 0 since u^* is unit length.

$$\int_\Omega \nabla u^* \nabla v - |\nabla u^*|^2 v \cdot u^*\, dx = 0.$$

which is exactly the weak form of (2.46).

The cross product method in proving the convergence of the sequence was well know in the studying of the harmonic maps with target space being a sphere [91].

Finally, the following lemma gives more detailed information of the Lagrange multiplier λ_ϵ in the equation (2.39).

Lemma 3 *Suppose that Ω is strictly star-shaped with respect to 0 and the boundary $\partial\Omega$ is C^1. If λ_ϵ is the Lagrange multiplier as in (2.39), then*

$$M_1 \le -\lambda_\epsilon \epsilon^2 \le M_2, \tag{2.48}$$

where M_i are the constants independent of ϵ.

Proof We use the Pohozaev type of argument. Let Ω_h, $0 \le h \le h_0$, be the star-shaped domain that are closed to the original domain. h is the distance between the boundaries. The existence of these neighbouring domains can be adjustified by the smoothness of the domain. We multiplier the equation (2.39) by $(x \cdot \nabla)u_\epsilon$ and integrate over the domain Ω_h:

$$-\int_{\Omega_h} (\Delta u_\epsilon) \cdot (x \cdot \nabla) u_\epsilon\, dx = \int_{\Omega_h} \lambda_\epsilon (|u_\epsilon|^2 - 1) u_\epsilon \cdot (x \cdot \nabla) u_\epsilon\, dx. \tag{2.49}$$

The right hand side is equal to:

$$\begin{aligned} \int_{\Omega_h} \lambda_\epsilon (|u_\epsilon|^2 - 1) u_\epsilon \cdot (x \cdot \nabla) u_\epsilon\, dx &= \frac{\lambda_\epsilon}{4} \int_{\Omega_h} (x \cdot \nabla)(|u_\epsilon|^2 - 1)^2\, dx \quad (2.50) \\ &= \frac{d\lambda_\epsilon}{4} \int_{\Omega_h} (|u_\epsilon|^2 - 1)^2\, dx = \frac{d\lambda_\epsilon}{4} \epsilon^2. \end{aligned}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

The last equality uses the integration by parts and the constraint (2.35).

On the other hand, the left hand side is equal to:

$$\begin{aligned} & - \int_{\Omega_h} (\Delta u_\epsilon) \cdot (x \cdot \nabla) u_\epsilon \, dx \qquad (2.51) \\ = & \frac{(2-d)}{2} \int_{\Omega_h} |\nabla u_\epsilon|^2 \, dx - \frac{1}{2} \int_{\partial\Omega_h} |\nabla u_{\epsilon\nu}|^2 (x \cdot \nu) \, dx \\ + & \int_{\partial\Omega_h} (x \cdot \tau) u_{\epsilon\tau} (x \cdot \nu) u_{\epsilon\nu} \, dx \\ > & \frac{(2-d)}{2} M - \frac{1}{2} \int_{\partial\Omega_h} |\nabla u_{\epsilon\nu}|^2 (x \cdot \nu) \, dx, \\ + & \int_{\partial\Omega_h} (x \cdot \tau) u_{\epsilon\tau} (x \cdot \nu) u_{\epsilon\nu} \, dx \end{aligned}$$

for $n \geq 2$.

Now, we can integrate in the normal direction. We get that

$$M_1(h_0) \leq -\lambda_\epsilon \epsilon^2 \leq M_2(h_0), \qquad (2.52)$$

Notice that we have used the Cauchy's inequality and the property of strict star-shapeness of the domain.

The last theorem show that the Euler-Lagrange multiplier λ_ϵ is of order $O(\frac{1}{\epsilon^2})$. The constant h_0, hence the size of M_i, is determined by the smoothness of the boundary $\partial\Omega$.

3 Navier-Stokes equation

There are many references on the theory of Navier-Stokes equations [20, 94, 53, 72]. We will just list out some of them here:

- P. Constantin and C.Foias, *Navier-Stokes equation.* University of Chicago Press, 1988.
- R. Temam, *Navier-Stokes equation, theory and application.* AMS Chelsea Publishing, 1984.
- A. Majda and A. Bertozzi, *Vorticity and incompressible flow.* Cambridge University Press, 2002.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

- L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*. Pergamon Press, 1987.
- G. K. Bachelor *An inroduction to Fluid Mechanics*. Cambridge University Press, 1967.

3.1 Newtonian Fluids

The hydrodynamical systems for Newtonian fluids include the following equations:

Balance of mass:

$$\rho_t + \nabla \cdot (\rho u) = 0. \tag{3.1}$$

If $\rho = \rho_0$ is a constant, then $\nabla \cdot u = 0$. Notice, however that the reverse is not true.

Momentum equation (force balance equation).

$$\rho(u_t + u \cdot \nabla u) + \nabla p = \mu \Delta u. \tag{3.2}$$

This equation can be derived from the least action principle. Introduction of the viscosity through: postulating the dissipative term in energy law; or introduce random perturbation in the variation process (Peskin's work [79]).

Finally, the energy law:

$$\frac{d}{dt} \int_\Omega \frac{1}{2} \rho |u|^2 \, dx = - \int_\Omega \mu |\nabla u|^2 \, dx. \tag{3.3}$$

Notice in the incompressible fluids, with $nabla \cdot u = 0$, the energy equation is not an independent equation. It can be derived from the conservation of mass and the conservation of momentum equations.

In the case that the initial term can be neglected, the system will be the Stokes equation, which is a linear equation in velocity. For the inviscid fluids, the system becomes the Euler equation.

All the system will be equipped with proper Initial and Boundary conditions.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

3.1.1 Existence of global weak solution

The way to show the global existence of weak solutions (consistent with the energy laws), which is called the Larey-Hopf solution, is through the usual Galerkin scheme.

The goal of the Galerkin scheme is using the separation of variables method to approximate the original the problem (an infinite dimensional evolution problem) by finite dimensional ODE systems.

We first define the functional space:

$$\begin{aligned} V &= \{u \in H^1(\Omega) : \operatorname{div} u = 0\} \\ H &= \{u \in L^2(\Omega) : \operatorname{div} u = 0\}. \end{aligned} \tag{3.4}$$

Also, V' is the dual space of V. We have V is a subset of H which is a subset of V'.

The Stokes operator A is defined as a map from H onto the space

$$D(A) = \{u \in H,\, \Delta u \in H\}, \tag{3.5}$$

such that for any given f, $u = A^{-1}(f)$ satisfies the following Stokes problem:

$$-\Delta u + \nabla p = 0,\, \nabla \cdot u = 0. \tag{3.6}$$

The operator A is positive, selfadjoint. Since the inverse of A is a linear continuous map from H to $D(A)$ and it is compact. A can be viewed as a selfadjoint operator in H and its eigenfunction ϕ_i form a basis of H.

Theorem 6 *For any $f \in L^2(0,T,V')$ and the initial condition $u_0 \in H$ given, there exists a weak solution u to the Navier-Stokes equation such that*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H). \tag{3.7}$$

Moreover, the solution is unique when the dimension is 2.

Sketch of the proof. We look at the approximation of the solution u in the finite dimensional subspace spanned by the eigenfunction of the Stokes operator A. For any given integer n,

$$u_n = \sum_{i=1}^{n} g_{in}(t)\phi_i. \tag{3.8}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

The function u_m satisfies the system:

$$(u_n, v)_t + \mu((u_n, v)) - (u_m u_m, \nabla v) = (f, v), \qquad (3.9)$$
$$u_n(0) = P_n u_0,$$

where P_n is the orthogonal projection in H onto span$\{\phi_i\}$ and the test function v is any function in this space.

The above system is equivalent to a ODE system of the coefficients g_{in}. The ODE system always has a local solution. Moreover, we still have the following a priori estimate:

$$\frac{d}{dt}\frac{1}{2}|u_n|^2 + \mu|\nabla u_n|^2 = (f, u_n). \qquad (3.10)$$

Hence,

$$\sup |u_n|^2 + \int_0^T \mu|\nabla u_n|^2 \, dt \le M, \qquad (3.11)$$

where M is a constant depending on the initial condition and f. From this, frist, we can see that the ODE solution exists all the time. Secondly we can extract a subsequence (still denote as u_n) which convergence weakly in the space $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Finally, Aubin-Lions's compactness theorem shows that the weak limit is a solution of the original Navier-Stokes equation.

3.1.2 Existence of classical solution

Theorem 7 *For give $f \in L^\infty(0,T;H)$ and $u_0 \in V$. In 2 dimensional case, there exists a unique global solution*

$$u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V). \qquad (3.12)$$

However, such a solution exists when μ is large or u_0 is small.

Sketch of the proof. We will prove the theorem using the higher order energy estimates. For this we multiply the equation by Au and integrate by parts, we have

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \mu|Au|^2 + (uu, Au) = (f, Au). \qquad (3.13)$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

The right hand side can be bounded as

$$(f, AU) \leq \frac{\mu}{4}|Au|^2 + \frac{1}{\mu}|f|^2.$$

The trilinear term can be estimated by

$$(uu, Au) \leq |u|_4^2|Au|.$$

Using Ladyzhenskaya's inequalities to interpolate the L^4 norm by the L^2 and H^1 norms.

In 2-D, we have

$$|u|_4^2 \leq |u|_2|\nabla u|_2$$

and in 3-D

$$|u|_4^2 \leq |u|_2^{1/2}|\nabla u|_2^{3/2}$$

Hence we can show that in 2-D, we have the global classical solution. In 3-D, we can use the situation that either μ is large or u_0 is small to get that $|\nabla u|_2$ is in fact monotone in time.

3.1.3 Regularity

Theorem 8 *If a solution $u \in L^p(0, T; L^q(\Omega))$ is a solution of Navier-Stokes equation and $2/p + 3/q \leq 1$, then the solution is a unique classical solution.*

3.1.4 Partial regularity

Theorem 9 *There exists a weak solution of theNavier-Stokes equation such that the 1 dimensional Hausdorff measure of the singularity set is zero.*

4 Viscoelastic Materials

All complex fluids have distinguished viscoelastic properties.. The following references [7, 41, 54, 78, 82, 87] cover some of the most important area of the studies, both in mathematics and engineering/physics.

- R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics.* Weiley Interscience, New York, 1987.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

- M. E. Gurtin. *An Introduction to Continuum Mechanics*, volume 158 of *Mathematics in Science and Engineering*. Academic Press, 1981.
- R. G. Larson. *The Structure and Rheology of Complex Fluids*. Oxford, 1995.
- R. G. Owens and T. N. Phillips. *Computational Rheology*. Imperial College Press, London, 2002.
- M. Renardy, W. J. Hrusa, and J. A. Nohel. *Mathematical Problems in Viscoelasticity*, volume 35 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1987.
- W. R. Schowalter. *Mechanics of Non-Newtonian Fluids*. Pergamon Press, 1978.

4.1 Flow map and deformation tensor

In the context of hydrodynamics, the basic variable are the flow map (particle trajectory) $x(X,t)$. X is the original labeling (Lagrangian coordinate) of the particle. It is also referred to as material coordinate. x is the current (Eulerian) coordinate and referred to as reference coordinate. For a given velocity field $v(x,t)$ the flow map is defined by the following ordinary differential equation:

$$x_t(X,t) = v(x(X,t),t), \quad x(X,0) = X. \tag{4.1}$$

The deformation tensor $F(X,t)$ is defined as

$$F(X,t) = \frac{\partial x}{\partial X}. \tag{4.2}$$

When look in the Eulerian coordinate, we can define $\tilde{F}(x,t)$ such that $\tilde{F}(x(X,t),t) = F(X,t)$. With no ambiguity, we will not distinguish these two notations in this paper. Applying the chain rule, we see that $F(x,t)$ satisfies the following transport equation [70, 41, 54]:

$$F_t + v \cdot \nabla F = \nabla v F, \tag{4.3}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

which stands for $F_{ij_t} + v_k \nabla_k F_{ij} = \nabla_k v_i F_{kj}$. This is a direct consequence of the chain rule. Here we point out that in this paper, we use the notation $F_{ij} = \frac{\partial x_i}{\partial X_j}$ and $(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}$. This is different from notations in other papers by a transpose, for instance [54].

The incompressibility is represented as

$$\det F = 1. \tag{4.4}$$

By the identity of the variation of the determinant of a tensor

$$\delta \det F = \det F \mathrm{tr}\,(F^{-1}\delta F), \tag{4.5}$$

we see that $\nabla \cdot v = 0$. Moreover, we assume that the density $\rho = \rho_0$ to be a constant. This will replace the conservation of mass equation:

$$\rho_t + \nabla \cdot (\rho v) = 0. \tag{4.6}$$

Finally, in this case, if we denote $(\nabla \cdot F)_j = (\nabla_i F_{ij})$, we have [70, 41, 54]

$$(\nabla_i F_{ij})_t + v_k \nabla_k (\nabla_i F_{ij}) + \nabla_i v_k (\nabla_k F_{ij}) = \nabla_k v_i (\nabla_i F_{kj}) + \nabla_i \nabla_k v_i F_{kj}.$$

Using the incompressibility and switch the indices i and k of the first term on the right hand side, we have:

$$(\nabla \cdot F)_t + v \cdot \nabla (\nabla \cdot F) = 0. \tag{4.7}$$

4.2 Force Balance and Oldroyd-B systems

For general viscoelastic fluid, we start from the following conservation of momentum equation:

$$\rho(v_t + v \cdot \nabla v) = \nabla \cdot \tau, \tag{4.8}$$

where τ is total stress. In Newtonian flow, we have the constitutive equation $\tau = -pI + \mu D$, where p is the pressure, μ the viscosity and $D = \frac{\nabla v + \nabla^T v}{2}$ is the strain rate.

There have been many attempts to capture different non-Newtonian phenomena of the materials, such as those of Ericksen-Rivlin [88, 87] or high-grade fluid [48], Ladyzhenskaya where τ is nonlinear in the strain rate D [52]

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

and by Necas's group where viscosity depending on both D and p [45, 73]. All these models only involve instantaneous constitutive relation between the stress and strain.

For the nonlocal (in time) constitutive equations, there are the Maxwell model $\tau_t + \gamma\tau = \mu D$, the transport model $\tau_t + v \cdot \nabla\tau + \gamma\tau = \mu D$, and the Oldroyd (upper convective) models

$$\tau_t + v \cdot \nabla\tau - \nabla v\tau - \tau\nabla v^T + \gamma\tau = \mu D, \tag{4.9}$$

The constant γ in the above models represents the time scale for the elastic relaxation. It is associate to the Debra number $De = \frac{\mu}{\gamma}$, which indicates the relation between the characteristic flow time and the characteristic elastic time scales [7].

There are other types of Oldroyd models. Those are associated with the different ways the stress tensor is transported. For instance, the Johnson-Segaman model is just the linear combination of the upper convictive and the lower convective Oldroyd models.

We can also look at the following modified Oldroyd model:

$$\tau = -pI + \mu D + \tau_1, \tag{4.10}$$

and the elastic stress τ_1 satisfies the transport equation:

$$\tau_{1t} + v \cdot \nabla\tau_1 - \nabla v\tau_1 - \tau_1\nabla v^T + \gamma\tau_1 = \delta I, \tag{4.11}$$

The equation (4.11) can be related to the modified Oldroyd model (4.9) by simply change of variable as $\tau_1 = \tau - \eta I$, where $\eta = \mu/2$ [78].

The tensor $C = FF^T$ is usually called the Cauchy-Green strain tensor and $B = C^{-1}$ is the finger tensor [54, 41, 78]. In particular, the equation (4.11) is equivalent to

$$(F^{-1}\tau_1 F^{-T})_t + v \cdot \nabla(F^{-1}\tau_1 F^{-T}) = -\gamma(F^{-1}\tau_1 F^{-T}) + \delta F^{-1}F^{-T},$$

Hence, we can implicitly write the solution in the form :

$$\begin{aligned} \tau_1(x,t) \quad &= \quad \exp\{-\gamma t\}F(x,t)\tau_1(x,0)F^T(x,t) \\ &\quad +\delta\int_{-\infty}^{t}\exp\{-\gamma(t-s)\}F(x,t)F^{-1}(x,s)F^{-T}(x,s)F^T(x,t)\,ds. \end{aligned} \tag{4.12}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

From here, it is obvious that τ_1 is positive definite. In fact, in this case, we can define the *induced deformation* tensor $F_1 = \sqrt{\tau_1}$.

Lemma 4 *If a tensor τ satisfies the equation:*

$$\tau_t + v \cdot \nabla \tau - \nabla v \tau - \tau \nabla v^T = 0, \tag{4.13}$$

and the initial condition $\tau(x, 0) = \tau_0(x)$ is positive definite, then

$$\tau(x, t) = F \tau_0 F^T. \tag{4.14}$$

Moreover, the induced deformation tensor $F_1 = \sqrt{\tau}$ satisfies the same equation as (4.3):

$$(F)_t + v \cdot \nabla F = \nabla v F.$$

We remark that the above result, together with the results in [70] will allow us to obtain a global weak (Larey) solution for a small (induced) strain viscoelasticity. We notice that this type of results are different from the existence results of [39, 40, 82] and the more recent ones in [66, 16, 27] which will be discussed in the later sections.

Finally, we see that, for the Oldroyd model, the system satisfies the following energy law:

$$\frac{d}{dt} \int_\Omega \frac{1}{2} \rho |v|^2 + \frac{1}{2} \mathrm{tr}\, \tau_1 \, dx = - \int_\Omega \mu |D|^2. \tag{4.15}$$

4.3 Energetic Variational Formulation

In [70], in order to study the mixture of a fluid with a visco-elastic solid, we wrote the momentum equation for the viscoelastic materials in the Eulerian framework. Assuming the elastic energy of the solid is $W(F)$ where $F = [\partial x / \partial X]$ is the deformation tensor (strain). The following system (in weak form) gives the force balance equations (linear momentum equations):

$$\int_\Omega [\rho(v_t + (v \cdot \nabla) v) \cdot u - p \nabla \cdot u + \tau \cdot \nabla u \, dx = \int_\Omega \rho f \cdot u dx, \tag{4.16}$$

for any test function u, the elastic stress: $\tau = \mu D(v) + (1/J) S(F) F^T$, where $S(F) = [\partial W / \partial F]$ takes the Piola Kirchhoff form. Here we also adopt the

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

constraint $J = det(F) = 1$ for incompressibility. This momentum equation can be derived through the least action principle (Hamilton's principle). The action functional take the form:

$$A(x) = \int_0^T \int_{\Omega_0} \frac{1}{2}\rho |x_t(X,t)|^2 - W(F)\, dX dt, \tag{4.17}$$

where Ω_0 is the original domain occupied by the material. We use the fact that $J = \det F = 1$.

Now we take any one-parameter family of volume preserving flow map $x^\epsilon(X,t)$ with $\frac{dx^\epsilon}{d\epsilon}|_{\epsilon=0} = y$. From the fact that $J = \det F = 1$ and the identity (4.5), we have that $\nabla \cdot y = 0$. Now the equation (4.16) (without the viscosity dissipation term) can be seen just following the variation of A with respect to x:

$$\frac{d}{d\epsilon} A(x^\epsilon)|_{\epsilon=0} = 0. \tag{4.18}$$

We usually study the elasticity through the force balance equation, using the Lagrangian coordinate. Here we use the trajectory $x(X,t)$ as the unknown variable (or the displacement $x - X$). The equation reads as

$$\rho x_{tt} = -\frac{\delta W}{\delta x} = \nabla_X \cdot W_F + \nabla_X (F^{-T} p), \tag{4.19}$$

where p is the Lagrangian multiplier to the incompressibility condition. and it satisfies the energy law:

$$\frac{d}{dt} \int_{\Omega_0} \frac{1}{2}\rho |x_t|^2 + W(F)\, dX = 0. \tag{4.20}$$

In the case of Hookean (linear) elasticity, $W(F) = |F|^2 = \mathrm{tr}\,(FF^T)$, it becomes the usual wave equation:

$$\rho x_{tt} = \nabla_X \cdot W_F = \nabla_X \cdot F + \nabla_X (F^{-T} p) = \Delta_X x + \nabla_X (F^{-T} p). \tag{4.21}$$

We point out that it will be difficult to input the frame indifferent viscosity term in the above equations.

Again, the system (4.16) satisfies the energy estimate (second law of thermodynamics [41]):

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2}\rho |v|^2 + W(F)\, dx = -\int_{\Omega} \mu |D|^2\, dx. \tag{4.22}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

We notice that even in the linear elasticity case, the elastic stress term $\tau_2 = W_F F^T = FF^T$ is still nonlinear. In fact, it is always the same order as the energy. This is the main difficulty of the current setting. On the other hand, we can make the following observation. Using the fact that F satisfies the transport equation (4.3), we have

$$\tau_{2t} + v \cdot \nabla \tau_2 - \nabla v \tau_2 - \tau_2 \nabla v^T = 0 \tag{4.23}$$

and we recover the Oldroyd system (without the damping). Notice in this case that $W(F) = \operatorname{tr} \tau_2$. Hence the two energy laws are also consistent with each other.

In fact, we can also start with the above energy law and derive the linear momentum equations (hence the constitutive equations). This is also the approach that was used by Ericksen in the study of liquid crystal materials [30] and Gurtin for phase transitions [35].

The linear transport equation $F_t + (v \cdot \nabla)F = \nabla v\, F$ in tensor case can not be treated directly in the framework of [24] or [51]. We may apply the div-curl lemma [92] to obtain weak solutions [70]. However, this is not enough to achieve the convergence of the stress term. As an alternative, we used the polar decomposition (R be the rotation part and the symmetric U be small) and get the equations $R_t + u \cdot \nabla R = W(v)R$, $U_t + v \cdot \nabla U = R^T D(v) R$, where $F = R(I + U)$, $D(v)$, $W(v)$ are the symmetric skew components of ∇v. This was not the usual linear elastic formulation, rather, it was in the same sitting as the famous work by F. John [47] where he had applied the John-Nirenberg inequality [36] to study nonlinear elasticity for the static small strain cases. We linearized the elastic stress:

$$D\mathcal{W}(F)F^T = R(D\mathcal{W}(I) + D\mathcal{W}(I)U + \mathcal{C}(U) + O(U^2))R^T. \tag{4.24}$$

where we used the notation $\mathcal{C}(U)_{j\beta} = D^2\mathcal{W}(I)(U)_{j\beta} = \frac{\partial^2 \mathcal{W}}{\partial F_{i\alpha} \partial F_{j\beta}}(I) U_{i\alpha}$. The special form of the equation of R allowed us to get an approximate system for R and to generalize the tools for scalar transport equations [24] to this small strain case, and eventually leaded to the global existence of the approximate system [70].

Lin, Liu and Zhang study the existence of the original system. According to different situations, we let Ω be a bounded domain in $\mathbf{R}^2$ (or $\mathbf{R}^3$)

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

with smooth boundary, the whole domain or periodic boxes. The (linear) viscoelastic fluid system takes the following form:

$$\begin{aligned} F_t + v \cdot \nabla F &= \nabla v F, \\ v_t + v \cdot \nabla v + \nabla p &= \mu \Delta v + \nabla \cdot (F F^T), \\ \nabla \cdot v &= 0, \end{aligned} \tag{4.25}$$

where the $i-$th component of $\nabla \cdot (FF^T)$ on the right hand side of the momentum equation is $\nabla_j (F_{ik} F_{jk})$. The system has the initial condition:

$$F(x,0) = F_0(x),\ v(x,0) = v_0(x), \tag{4.26}$$

In cases of bounded domain, we chose the boundary condition: for any x on the boundary $\partial\Omega$,

$$F(x,t) = I,\ v(x,t) = 0, \tag{4.27}$$

The system satisfies the energy identity:

$$\frac{d}{dt} \int_\Omega \frac{1}{2}|v|^2 + \frac{1}{2}|F|^2 \, dx = -\int_\Omega \mu |\nabla v|^2 \, dx. \tag{4.28}$$

From the identity $(\nabla \cdot F)_t + v \cdot \nabla (\nabla \cdot F) = 0$ for the incompressible materials, if we assume that $\nabla \cdot F_0 = 0$, we have that $\nabla \cdot F = 0$ and $F = \nabla \times \phi$ where ϕ is a matrix. In 2-dimensional case, if we denote $\phi = (\phi_1, \phi_2)$, then the original system can be transformed (after adjusting the order and sign) into:

$$\begin{aligned} \phi_t + v \cdot \nabla \phi &= 0, \\ v_t + v \cdot \nabla v + \nabla p &= \mu \Delta v - \sum_{i=1}^{2} \Delta \phi_i \nabla \phi_i, \\ \nabla \cdot v &= 0. \end{aligned} \tag{4.29}$$

with initial condition:

$$\phi(x,0) = \phi_0,\ v(x,0) = v_0(x), \tag{4.30}$$

and in case of bounded domain, the boundary conditions: for any x on the boundary $\partial\Omega$,

$$\phi(x,t) = x,\ v(x,t) = 0, \tag{4.31}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

And the energy law becomes:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2+\frac{1}{2}|\nabla\phi|^2\,dx=-\int_\Omega \mu|\nabla v|^2\,dx. \tag{4.32}$$

The following theorems are proved in [65].

Theorem 10 *Let $k\geq 2$ be a positive integer, $\nabla\phi_0\in H^k(\Omega)$, $v_0\in H^k(\Omega)$, then there exists a positive time T, which depends only on $|\nabla\phi_0|_{H^2}$ and $|v_0|_{H^2}$, such that the system possesses a unique solution in the time interval $[0,T]$ with*

$$\begin{aligned}&\partial_t^j\nabla_x^\alpha v\in L^\infty([0,T];H^{k-2j-|\alpha|}(\Omega))\cap L^2([0,T];H^{k-2j-|\alpha|+1}(\Omega)),\\ &\partial_t^j\nabla_x^\alpha\nabla\phi\in L^\infty([0,T];H^{k-2j-|\alpha|}(\Omega)),\end{aligned} \tag{4.33}$$

for all j,α satisfying $2j+|\alpha|\leq k$. Moreover, if T^* is the maximal time of existence, then

$$\int_0^{T^*}|\nabla v|_{H^2}^2\,ds=+\infty. \tag{4.34}$$

Theorem 11 *Let Ω is a periodic box or the whole space R^2, $k\geq 2$ be a positive integer, $\nabla\phi_0\in H^k(\Omega)$ and $v_0\in\in H^k(\Omega)$. Furthermore, for some large enough constant C, we assume that,*

$$|\nabla v_0|_{H^2}+|\nabla\psi_0|_{H^2}\leq\frac{\mu}{C(1+\frac{1}{\mu})^3(1+\mu+\frac{1}{\mu})} \tag{4.35}$$

then the system (4.29) will have a unique global classical solution, such that,

$$|v|_{H^2}^2+|\nabla\psi|_{H^2}^2+\int_0^\infty(\mu|\nabla v|_{H^2}^2+\frac{1}{\mu}|\nabla\Delta\psi|_{L^2}^2)\,ds\leq\frac{\mu}{C(1+\mu+\frac{1}{\mu})}, \tag{4.36}$$

and (4.33) holds for $T=\infty$.

The results has been generalized to the general system in [56] and the small strain viscoelastic materials [55].

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

5 Liquid Crystal Flows

5.1 Ericksen-Leslie Theory

The hydrodynamical theory must describe not only orientation, as represented by the director field, $n(x,t)$ but macroscopic motion, represented by the velocity field $u(x,t)$.

We shall present the Ericksen-Leslie's set up (the corresponding static theory will be that of Oseen-Frank).

As usual, for liquids idealized as incompressible, we have the equation of continuity

$$\text{div } u = 0\,,\ u = (u_1, u_2, u_3), \tag{5.1}$$

representing conservation of mass. In general terms, equations of motion for u are of conventional form, i.e.,

$$\rho\left(\frac{\partial u_i}{\partial t} + u_{i,j}u_j\right) = t_{ij,j} + f_i\,, \tag{5.2}$$

where ρ is the (constant) mass density, f the body force, t the stress tensor. Then the stress tensor t can be written as

$$t = t^s + t^D\,, \tag{5.3}$$

the superscript s indicating a part covered by static theory, D indicating a dissipative part, vanishing when there is no motion.

Under various physical considerations, Leslie and Ericksen derived that

$$\begin{aligned} t^s_{ij} &= -p\delta_{ij} + W\delta_{ij} - \tau_{kj}n_{k,i}\,, \\ \tau_{ij} &= \tfrac{\partial W}{\partial n_{i,j}}\,, \end{aligned} \tag{5.4}$$

where W is the Oseen-Frank energy density (with $q = 0$ in nematics) and p is the pressure. Regarding the motion of n, as suggested by static theory,

$$n \wedge h = 0, \tag{5.5}$$

h being the total molecular field. There is an equivalent formulation, rephrasing this in terms of a balancing of moments. (Again one ignores the effect of the electromagnetic field).

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Similarly,

$$\begin{aligned} h &= h^s + h^D\,, \\ h_i^s &= \tfrac{\partial W}{\partial n_i} - \tau_{ij,j}. \end{aligned} \tag{5.6}$$

The terms t^D, h^D in dynamics have been treated from various viewpoints. In the purely dissipative model (parabolic system), the constitutive assumption presumes that t^D, h^D are linear functions of ∇u and

$$\dot{n} = \frac{\partial n}{\partial t} + u \cdot \nabla n, \tag{5.7}$$

with coefficients depending on n. Under further symmetry assumptions, as well as thermodynamics and mechanical arguments, the constitutive equations reduce to the form

$$t^D = \frac{\partial \Delta}{\partial \nabla u}\,, \tag{5.8}$$

$$h^D = \frac{\partial \Delta}{\partial \dot{n}}\,, \tag{5.9}$$

Δ being a dissipation function. In terms of the variables A and N given by

$$2A = \nabla u + \nabla u^t\,, \tag{5.10}$$

$$N = \dot{n} - \frac{1}{2}(\nabla u - \nabla u^{\mathbf{t}})n\,, \tag{5.11}$$

this function has the form

$$\begin{aligned} 2\Delta \quad &= \alpha_1(n \cdot An)^2 + \alpha_4(trA^2) + (\alpha_5 + \alpha_6)\|n \otimes An\|^2 \\ &+\gamma_1\|N\|^2 + 2\gamma_2 N \cdot An \geq 0. \end{aligned} \tag{5.12}$$

Here the scalar $\alpha's$ and $\gamma's$, the measure of viscosity, depend on the material and the temperature.

Based on a somewhat different argument, Leslie [Le2] obtained more general parabolic-hyperbolic systems. This system can be written in the following more concise form:

$$\rho\dot{u} = \text{ div } \left(-pI + \nabla n^T \cdot W_q + \frac{\partial \Delta}{\partial \nabla u}\right) + F. \tag{5.13}$$

$$\sigma\ddot{n} = -W_n + \text{div } W_q + \frac{\partial \Delta}{\partial \dot{n}} + \gamma n + G. \tag{5.14}$$

Here p is (as before) the pressure, γ is a Lagrange multiplies due to the constraint $|n| = 1$, and F, G are external forces.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

The system (5.14) is derived from the conservation law of the form (proposed by J. Ericksen)

$$\begin{aligned} \tfrac{d}{dt} \quad & \int_\Omega (\rho u^2 + W + \sigma|\dot{n}|^2)dx \\ & = -\int_\Omega \Delta dx + \text{ boundary terms and harmless terms.} \end{aligned} \tag{5.15}$$

The first system (with $\sigma = 0$) is parabolic, and can be thought of as nonlinear coupling between harmonic maps heat flow and Navier-Stokes equations.

The second system is a parablic-hyperbolic couples systems. Here one has a nonlinear coupling between wave maps with dumping effect and Navier Stokes equations.

Remark 3 *In Ericksen's equations, if we choose* $k_1 = k_2 = k_3 = 1, k_4 = q = 0, \alpha_1 = \alpha_4 = \alpha_5 + \alpha_6 = \gamma_2 = 0, \gamma_1 = 1$, *then the coupled systems can be written as*

$$\begin{cases} \frac{\partial}{\partial t}u^i + u \cdot \nabla u^i = \Delta u^i + \nabla_i p - (n_{x_i} \cdot n_{x_j})_{x_j} \\ i = 1, 2, 3. \\ \frac{\partial}{\partial t}n^i + u \cdot \nabla n^i - \Omega^i_j n^j = \Delta n^i + |\nabla n|^2 n^i \\ i = 1, 2, 3. \end{cases} \tag{5.16}$$

In addition, we have two constraint $divu = 0$ *and* $|n| = 1$. *Where* $\Omega_j = \frac{1}{2}[u^i_{x_j} - u^j_{x_i}]$. *There is also a similar version for Leslie's equations.*

If $u \equiv 0$, *then we have*

$$\frac{\partial n}{\partial t} = \Delta n + |\nabla n|^2 n \, , \, |n| = 1. \tag{5.17}$$

which is the equation of heat flow of harmonic maps from $\Omega \to \mathbf{S}^2$.

Remark 4 . *the first equation concerning balance of linear moments becomes*

$$(n_{x_i} \cdot n_{x_j})_{x_j} = \frac{\partial}{\partial x_i} P. \tag{5.18}$$

Not all weak solutions of (5.17) satisfy (5.18) those weak solutions of (5.17) and, in addition (5.18), have to satisfy so called energy-monotonicity inequality. In the static case, those solutions are exactly those called [SU] stationary solutions. They satisfy the energy monotonicity inequality.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

5.2 Existence and Regularity

In order to understand the Ericksen-Leslie theory, we will look at the following system. Although the system is simplified, it retained most mathematical and physical difficulties of the original system. Moreover, it emphasizes the special coupling between the director and the flow field.

$$u_t + (u \cdot \nabla)u + \nabla p - \nu \, div D(u) + \lambda \, div(\nabla d \odot \nabla d) = 0, \tag{5.19}$$

$$\nabla \cdot u = 0, \tag{5.20}$$

$$d_t + (u \cdot \nabla)d - \gamma(\Delta d - f(d)) = 0, \tag{5.21}$$

with initial conditions

$$u|_{t=0} = u_0, \quad d|_{t=0} = d_0, \tag{5.22}$$

satisfying either the Dirichlet boundary condition [LiLi95]

$$u = 0, \quad d = d_0, \tag{5.23}$$

or the *free surface* boundary conditions [LiSh00]

$$u \cdot n = 0, \quad ((\nabla \times u) \times n) \times n = 0, \quad \frac{\partial d}{\partial n} = 0, \tag{5.24}$$

on the boundary $\partial\Omega$ of the domain with n being the outward normal.

In the above system, u represents the velocity of the liquid crystal fluid, p the pressure, d represents the *normed* director of the molecule. The vectors $u, d : \Omega \times \Re^+ \to \Re^n$, and the function $p : \Omega \times \Re^+ \to \Re$, where $\Omega \subset \Re^n$ is a bounded smooth domain (or a polygonal domain) with boundary $\partial\Omega$. $D(u) = (1/2)(\nabla u + (\nabla u)^T)$ is the stretching tensor, $\sigma^v = pI + \nu D(u)$ is the fluid viscosity part of the stress tensor, $(\nabla d \odot \nabla d)_{ij} = \sum_{k=1}^n (\nabla_i d_k)(\nabla_j d_k)$, and finally, $f(d)$ is a polynomial of d which satisfies $f(d) = F'(d)$ where $F(d)$ is the bulk part of the elastic energy. The choice of $F(d)$ is such that the maximal principle for $|d|$ holds in the equation (5.21), that is, if $|d| \leq 1$ on the boundary and in the initial data, the $|d| \leq 1$ is true everywhere at any time. Usually, we chose $F(d)$ to be the Ginzburg-Landau penalization

$$F(d) = \frac{1}{4\epsilon^2}(|d|^2 - 1)^2. \tag{5.25}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Again, we see that (5.20) represents the fact that the fluid is incompressible. (5.21) is the evolution of the director, the left hand side shows that d is transported by the flow. We want to point out that the momentum equation (5.19) can be derived through the following least action principle.

The above highly simplified system, in fact, captures all the mathematical difficulties (as shown in the the later discussions) and the physical characteristics of the original model. To demonstrate the later one, we will derive the linear momentum equation (force balance) equation using the least action (Hamiltonian) principle.

Let us begin by computing the variations of the following "elastic" part of the "action" functional among all the volume preserving flow maps:

$$A(x) = \int_0^T \int_\Omega \frac{\lambda}{2}|\nabla d|^2 - h(d)\, dxdt. \tag{5.26}$$

We look at the volume preserving flow maps $x(X,t)$ such that

$$x_t(X,t) = v(x(X,t),t), \quad x(X,0) = X. \tag{5.27}$$

Here we can view X as the Lagrangian (initial) material coordinate and $x(X,t)$ the Eulerian (reference) coordinate.

In order to perform the variation, we look at the one parameter family of such maps x^ϵ such that:

$$x^0 = x, \quad \frac{dx^\epsilon}{d\epsilon} = y. \tag{5.28}$$

for any y such that $\nabla_x \cdot y = 0$.

Now we computer the variation of $A(x^\epsilon) = A(d(x^\epsilon,t))$ with respect to ϵ:

$$\begin{aligned}
0 = \frac{d}{d\epsilon}|_{\epsilon=0} A(x^\epsilon) &= \int_0^T \int_{\Omega_0} \lambda \nabla_x^i d \frac{d}{d\epsilon}|_{\epsilon=0} \nabla_{x^\epsilon}^i d(x^\epsilon,t)) + f'(d)\nabla_x^j dy^j\, dXdt \\
&= \int_0^T \int_{\Omega_0} \lambda \nabla_x^i d \frac{d}{d\epsilon}|_{\epsilon=0} \big(\nabla_x^j d(x^\epsilon,t))\nabla_{x^\epsilon}^i x^j\big) + f'(d)\nabla_x^j dy^j\, dXdt \\
&= \int_0^T \int_{\Omega_0} \lambda \nabla_x^i d \nabla_x^j \nabla_x^i d(x,t) + \lambda \nabla_x^i d(x,t) \nabla_x^j d(x,t) \nabla_x^i y^j \\
&\qquad + f'(d)\nabla_x^j dy^j\, dXdt
\end{aligned}$$

Here we have used the fact that $\nabla_{x^\epsilon} x$ is the inverse matrix of $\nabla_x x^\epsilon$.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Since y is an arbitrary divergence free vector field, integration by parts gives the following equation:

$$u_t + (u \cdot \nabla)u + \nabla p + \lambda \, div(\nabla d \odot \nabla d) = 0. \tag{5.29}$$

We point out that the viscosity and other types of dissipation are due to other term.

From this derivation, it is easy to see the energy law of the system. Moreover, the hydrodynamic equilibrium will be a special kind of stationary solution of d, as we remarked in the last section. This can be viewed in the following spatial Noether theorem.

In [62], we study the wellposedness of the system by establishing the following basic energy law:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\|u\|^2 + \lambda\|\nabla d\|^2 + 2\lambda F(d))dx = -\int_\Omega (\nu\|\nabla u\|^2 + \lambda\gamma\|\Delta d - f(d)\|^2)dx, \tag{5.30}$$

for all $t \in (0, T]$.

This energy law together with a modified Galerkin method enables us to prove the existence of a weak solution of the above system (1)-(2):

Theorem 12 *Under the assumptions that $u_0(x) \in L^2$, and that $d_0(x) \in H^1(\Omega)$ with $d_0|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$, the system has a global weak solution (v, d) satisfying:*

$$\begin{aligned} u &\in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\ d &\in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \end{aligned} \tag{5.31}$$

for all $T \in (0, \infty)$.

We also proved the existence of the classical solution.

Theorem 13 *The system has a unique global classical solution (u, d) provided that $u_0(x) \in H^1(\Omega)$, $d_0(x) \in H^2(\Omega)$ and, that either $dim\,\Omega = 2$ or $dim\,\Omega = 3$ and $\nu \geq \nu(\gamma, \lambda, u_0, d_0)$.*

The stability result is established in the following theorem:

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Theorem 14 *Suppose $d^* \in H^2(\Omega)$ is an absolute minimum of the functional*

$$E(d) \equiv \frac{1}{2}\int_\Omega (\|\nabla d\|^2 + F(d))dx \tag{5.32}$$

in the sense that $E(d^*) \leq E(d)$ whenever $d = d^*$ on $\partial\Omega$. Then there is an $\epsilon > 0$, possibly depending on the system such that: whenever $\|d_0 - d^*\|_{H^2(\Omega)} + \|u_0\|_{H^1(\Omega)} < \epsilon$, then the original system has a unique global solution (u, d) with $u(x,t) \longrightarrow 0$ in $H^1(\Omega)$, as $t \longrightarrow +\infty$. Moreover, for any sequence $t_i \longrightarrow +\infty$, $d(x, t_i') \longrightarrow \tilde{d}$ in $H^2(\Omega)$, for a subsequence $\{t_i'\}$. Here $\tilde{d}$ is a critical point of E.

Theorem 15 *If the domain and the initial-boundary conditions in system are smooth enough, then there exists a suitable weak solution whose singular set has one-dimension Hausdorff measure zero in space-time.*

6 Free Interface Motion in Mixtures

The interfacial dynamics in the mixture of different fluids, solids or gas have attracted attentions for more than two centuries. Many surface properties, such as capillarity, are associated with the surface tension through special boundary conditions on the interfaces [80, 50, 29, 14].

In classical approaches, the interface is usually considered to be a free surface that evolves in time with the fluid (the kinematic boundary condition). The dynamics of the interface at each time is determined by the following stress (force) balance condition:

$$[T] \cdot n = mHn, \tag{6.1}$$

where $[T] = [\nu D(u) - pI]$ is the jump of the stress across the interface Γ_t, n is its normal, $D(u) = \frac{\nabla u + (\nabla u)^T}{2}$ is the stretching tensor, H is the mean curvature of the surface and m is the surface tension constant. This is the usual Young-Laplace junction condition (see, for instance, [4, 80, 50, 29]). The hydrodynamic system describing the mixture of two Newtonian fluids with a free interface will be the usual Navier-Stokes equations in each of the fluid domains (possibly with different density and viscosity) together with the kinematic and force balance (traction free) boundary conditions on the

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

interface. The weak form of such a system when the density ρ and viscosity ν may vary in the mixture can be represented exactly in the following form [68]:

$$\begin{aligned} \int_0^T \int_\Omega [\quad - \quad & \rho u v_t - \rho u u \cdot \nabla v + \nu \nabla u \nabla v - p \nabla \cdot v] \, dx dt \\ = \quad & \int_0^T \int_{\Gamma_t} m H n \cdot v \, ds dt \end{aligned} \tag{6.2}$$

for any test function v.

One classical method to study the moving interfaces is to employ a mesh that has grid points on the interfaces, and deforms according to the motion of the boundary, such as the boundary integral and boundary element methods (cf. [21, 95, 49] and their references). Keeping track of the moving mesh may entail computational difficulties and large displacement in internal domains may cause mesh entanglement. Typically, sophisticated remeshing schemes have to be used in these cases.

As an alternative, fixed-grid methods that *regularize* the interface have been highly successful in treating deforming interfaces. These include the volume-of-fluid (VOF) method [58, 59], the front-tracking method [38, 37] and the level-set method [15, 77]. Instead of formulating the flow of two domains separated by an interface, these methods represent the interfacial tension as a body-force or bulk-stress spreading over a narrow region covering the interface. Then a single set of governing equations can be written over the entire domain, and solved on a fixed grid in a purely Eulerian framework.

The energetic phase field model can be viewed as a physically motivated level-set method. Instead of choosing an artificial smoothing function for the interface, the diffuse-interface model describes the interface by a mixing energy. This idea can be traced to van der Waals [96], and is the foundation for the phase-field theory for phase transition and critical phenomena (see [25, 13, 12, 75, 76, 93] and the references therein). The phase field models allow topological changes of the interface [71] and over the years, they have attracted a lot of interests in the field of nonlinear analysis (cf. [2, 11, 17, 84, 89]). Similar to the popular level set formulations (see [77] for an

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

extensive discussion), they have many advantages in numerical simulations of the interfacial motion (cf.[15]). When the transition width approaches zero, the phase field model with diffuse-interface becomes identical to a sharp-interface level-set formulation and it can also be reduced properly to the classical sharp-interface model.

In this paper, we will illustrate some basic features and general approaches of this method.

6.1 An energetic variational approach with phase field method

Here we will present the simpliest case of the phase field method to study the mixture of two incompressible Newtonian fluids.

Introduce a "phase" function $\phi(x,t)$ to identify the two fluids ($\{x : \phi(x,t) = 1\}$ is occupied by fluid 1 and $\{x : \phi(x,t) = -1\}$ by fluid 2). Looking at the following Ginzburg-Landau type of mixing energy:

$$\tilde{W}(\phi, \nabla\phi) = \int_\Omega [\frac{\eta}{2}|\nabla\phi|^2 + \frac{1}{4\eta}(\phi^2 - 1)^2]\, dx.$$

We can view ϕ as volume fraction. The mixing density and viscosity will be functions of ϕ. The part of bulk energy represents the interaction of different volume fractions of individual species (like Flory-Huggins free energy [54, 28]). The gradient part plays the role of regularization (relaxation). The combination represents the competition between the (hydro)phobic and (hydro)philic effects between different species. The interface is represented by $\{x : \phi(x,t) = 0\}$, with the fixed transition layer of thickness η. The dynamics of ϕ can be driven by either Allen-Cahn or Cahn-Hillard types of gradient flow, depending on the choice of different dissipative mechanism. The later one preserves the overall volume fraction of two fluids. For Chan-Hillard case (where the volume is preserved):

$$\phi_t + u \cdot \nabla\phi = -\gamma \frac{\delta W}{\delta \phi} = -\gamma\Delta(\Delta\phi - f(\phi)), \tag{6.3}$$

where $f(\phi) = F'(\phi) = \frac{1}{\eta}(\phi^2 - 1)\phi$. u is the velocity field. The right hand side can be viewed as the variation with respect to ϕ is the regular L^2 space. The left hand side indicates that the variable ϕ is transported by the flow, on

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

top of the energy decent dynamics. As $\gamma \to 0$, where γ represents the elastic relaxation time, the limiting ϕ satisfies the transport equation, which is equivalent to the mass transport equation (for incompressible fluids). Hence this formulation can also be viewed as the link (relaxation) between the mass average (in the kinetic energy) and the volume average (in the elastic energy).

In case the variational space is take to be H^{-1}, then ϕ will take the dynamics as the Allen-Cahn equation:

$$\phi_t + u \cdot \nabla \phi = \gamma(\Delta \phi - f(\phi)), \tag{6.4}$$

Combining this elastic energy with the kinetic energy, we have the total energy $E = \int_\Omega [\frac{\rho}{2}|u|^2 + \frac{\lambda}{2}|\nabla \phi|^2 + \lambda F(\phi)]\, dx$. Using the least action principle (the principle of virtual work), we can derive the following linear momentum equation (balance of force equation) [68, 67, 42, 81]:

$$\rho(u_t + (u \cdot \nabla)u) + \nabla p - \nu \Delta u + \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi) = g(x). \tag{6.5}$$

The density will satisfy the forllowing transport equation:

$$\rho_t + u \cdot \rho = 0, \tag{6.6}$$

under the incompressibility condition for the velocity field:

$$\nabla \cdot u = 0. \tag{6.7}$$

The final system (6.3)(6.6)(6.7)(6.5) (together with the suitable boundary and initial conditions) will then possess the following energy law:

$$\frac{d}{dt} \int_\Omega [\frac{\rho}{2}|u|^2 + \frac{\lambda}{2}|\nabla \phi|^2 + \lambda F(\phi)]\, dx = - \int_\Omega [\nu |\nabla u|^2 + \gamma \lambda |\nabla(\Delta \phi - f(\phi))|^2]\, dx. \tag{6.8}$$

We can see that as $\eta \to 0$, the elastic force $\lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi)$ converges to a measure supported only on the interface, with magnitude proportional to the mean curvature [70]. Hence we recover the traction-free boundary condition with surface tension. Moreover, we can also derive the relation of our parameters into the sharp interface ones as the following:

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Consider a one-dimensional interface. We require that the diffuse mixing energy in the region be equal to the traditional surface energy:

$$\sigma = \lambda \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + f_0(\phi) \right\} dx. \tag{6.9}$$

Let us further assume that the diffuse interface is at equilibrium, and thus has zero chemical potential,

$$\frac{\delta F_{mix}}{\delta \phi} = \lambda \{ -\frac{d^2\phi}{dx^2} + f_0'(\phi) \} = 0. \tag{6.10}$$

Since $f_0(\pm\infty) = 0$ and $\frac{d\phi}{dx}\Big|_{x=\pm\infty} = 0$, this equation can be integrated once to give

$$\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 = f_0(\phi), \tag{6.11}$$

which implies equal partition of the free energy between the two terms at equilibrium.

Eq. (6.11) can be solved together with the boundary condition $\phi(0) = 0$, and we obtain the equilibrium profile for $\phi(x)$:

$$\phi(x) = \tanh \left(\frac{x}{\sqrt{2}\epsilon} \right) \tag{6.12}$$

Thus, the capillary width ϵ is a measure of the thickness of the diffuse interface. More specifically, 90% of variation in ϕ occurs over a thickness of 4.1641ϵ, while 99% of the variation corresponds to a thickness of 7.4859ϵ.

Substituting Eq. (6.12) into Eq. (6.9), we arrive at the following matching condition for the interfacial tension σ:

$$\sigma = \frac{2\sqrt{2}}{3} \frac{\lambda}{\epsilon} \tag{6.13}$$

As the interfacial thickness ϵ shrinks toward zero, so should the energy density parameter λ; their ratio gives the interfacial tension in the sharp interface limit.

Obviously, the correspondence between the diffuse- and sharp-interface models is meaningful only when the former is at equilibrium. During the

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

relaxation of the diffuse interface (cf. Eq. 6.3), one cannot speak of a constant interfacial tension. Although one may view this as a deficiency of the diffuse-interface model, it in fact reflects the reality that the interface has its own dynamics which cannot be summarized by a constant σ except under limiting conditions. To anticipate the results in Section 3.3, we note that f_{anch} may also contribute to the surface energy, thus giving rise to an anisotropic "interfacial tension" that is not encompassed by the traditional version of the concept.

These models allow for topological changes of the interface ([71, 8, 46, 43]) and have many other advantages in numerical simulations of the interfacial motion, and have seen many applications in the physics and engineering literature [81, 97].

The conservative dynamics of the above diffuse-interface model can be formulated in the classical procedures of Lagrangian mechanics [1, 3]. The starting point is the Lagrangian $L = T - F$, where T and F are the kinetic and potential energies of the system. The least action principle requires that the action integral $I = \int L\, dt$ be stationary under variations of "paths". This will lead to a momemtum equation, with elastic stresses arising from the microstructural changes embodied in the free energy F, and evolution equations for the field variables whose momenta are included in T.

The least action principle (variation on the flow maps), which gives the momentum equation, and the fastest decent dynamics or other types of gradient flows (variation on the phase variables) are due to different physical principles. However, they are related in the static case: the first one is equivalent to the variation with respect to the domain and the second one is the variation of the same functional with respect to the function. It is clear that if the solutions are smooth (or regular enough), they are equivalent. The discrepancy between these two equations requires the presence of the singularities and defects.

The existence of the hydrodynamic equilibrium states for the coupled systems (the static solution with the velocity $u = 0$) can be viewed as a diret consequence of the special relation between the solution of the Euler-Lagrange equation of the elastic energy and the solution of the equation from variation of the domain to such an energy. Formally, it can be summerized

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

into the following simple theorem (see, for example, [64]):

Theorem 6.1 *Given an energy functional $W(\phi, \nabla\phi)$, all solutions of the Euler-Lagrangian equation:*

$$-\nabla \cdot \frac{\partial W}{\partial \nabla \phi} + \frac{\partial W}{\nabla \phi} = 0 \tag{6.14}$$

also satisfy the equation

$$\nabla \cdot (\frac{\partial W}{\partial \nabla \phi} \otimes \nabla \phi - WI) = 0. \tag{6.15}$$

This theorem guarantees the existence of the hydrodynamic equilibrium states for most systems. It also gives the stability results [62] and shows that all solutions of the system (6.24-6.28) will approach to an equilibrium state as $t \to +\infty$. One can also derive from Theorem 6.1 the usual Pohozaev identity [90] by writing the equations (6.14) and (6.15) in weak forms.

It is the generality of this energetic variational procedure, especially in accommodating microstructured fluids via the free energy F, that has made the diffuse-interface (phase field) method our choice for tackling interfacial problems of complex fluids. Conceivably, any complex fluid with a properly defined free energy can be included in this formulation. In this paper, we will be dealing with two kinds: The thermo-induced Marangoni-Benard convection, and the mixture involving nematic liquid crystals, which are described by a regularized Leslie-Ericksen model [22, 30, 31, 57]. The latter also introduces the issue of surface anchoring.

- Level set formulation: work of Hamilton, Evans-Spruck, Evans-Soner-Souganidis, Xinfu Chen
- Giga and Solonnikov: Classical results.
- General solutions;

6.2 Marangoni-Benard convection

The conventional Marangoni-Benard convection is described by the following two phase fluids with a sharp interface, involving the Boussinsq approxima-

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

tion.

$$\begin{aligned} \nabla \cdot u &= 0, && (6.16)\\ \rho_0(u_t + (u\cdot\nabla)u) + \nabla p - \nu \,\mathrm{div} D(u) &= -\rho g j, && (6.17)\\ \theta_t + u\cdot\nabla\theta &= k\Delta\theta. && (6.18) \end{aligned}$$

Here ρ is the temperature dependent density, g is the gravitational acceleration, j is the upward direction. u and p stands for the fluid velocity and the pressure, θ is the temperature. k is the thermal diffusion. Moreover, we assume that

$$\rho = \rho_0[1 - \alpha(\theta - \theta_0)]. \tag{6.19}$$

With the usual initial and boundary conditions, the interface conditions take the form:

$$\begin{aligned} &\eta_t + u\cdot\eta = 0, && (6.20)\\ &[T]\cdot n = -\sigma K n + (t\cdot\nabla\sigma)t. && (6.21) \end{aligned}$$

Equation (6.20) is the kinematic condition, representing the surface ($\eta = 0$) evolve with the fluid. (6.21) is the traction (T) free boundary (balance of forces) condition. The surface tension depends on the temperature $\sigma = \sigma_0 - \sigma_1\theta$.

In order to incoporate this effect in the phase field model and still mantain the energy law, we consider the action function:

$$\begin{aligned} A(x) \quad = \quad & \int_0^T \int_{\Omega_0} \frac{1}{2}\rho_0 |x_t(X,t)|^2 \qquad (6.22)\\ & -\frac{\lambda(x(X,t))}{2}(|\nabla_x \phi(x(X,t),t)|^2 + F(\phi(x(X,t),t)))\, dX dt. \end{aligned}$$

Here we can view X as the Lagrangian (initial) material coordinate and $x(X,t)$ the Eulerian (reference) coordinate. Ω_0 is the initial domain occupied by the fluid. The notion that $\phi(x(X,t),t)$ indicated that ϕ is transported by the flow field. The special feature in this case is the spatial dependence of λ. In fact, it can be a function of temperature that is transported by the flow.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

For incompressible materials, we look at the volume preserving flow maps $x(X,t)$ such that

$$x_t(X,t) = v(x(X,t),t), \quad x(X,0) = X. \tag{6.23}$$

We arrive at the following system:

$$\begin{aligned}
&\rho_0(u_t + (u\cdot\nabla)u) + \nabla p - \nu \operatorname{div} D(u) \qquad (6.24)\\
&= -\nabla\cdot(\lambda\nabla\phi\otimes\nabla\phi - \frac{\lambda}{2}|\nabla\phi|^2 - \frac{\lambda}{4\epsilon^2}(\phi^2-1)^2)\\
&-(1+\phi)g(\rho_1-\rho_0)j - (1-\phi)g(\rho_2-\rho_0)j,\\
&\phi_t + (u\cdot\nabla)\phi + \gamma\Delta(\Delta\phi - f(\phi)) = 0, \qquad (6.25)\\
&\nabla\cdot u = 0, \qquad (6.26)\\
&\theta_t + u\cdot\nabla\theta = k\Delta\theta. \qquad (6.27)
\end{aligned}$$

with initial conditions

$$u|_{t=0} = u_0, \quad d|_{t=0} = d_0, \tag{6.28}$$

and appropriate boundary conditions. In our simulations, we choose the period boundary conditions. The parameter λ is a linear function of the tempreature θ.

Here we used the classical Boussinesq approximation, which is the linear version of all different types of average approaches. The "background" density can be treated as a constant ρ_0 and the difference between the actual density and ρ_0 will contribute only to the buoyancy force [60].

Moreover, we see that

$$\begin{aligned}
&-\nabla\cdot(\lambda\nabla\phi\otimes\nabla\phi - \frac{\lambda}{2}|\nabla\phi|^2 - \frac{\lambda}{4\epsilon^2}(\phi^2-1)^2)\\
&= -\lambda\Delta\phi\nabla\phi - \frac{\lambda}{2}\nabla|\nabla\phi|^2 - (\nabla\lambda\cdot\nabla\phi)\nabla\phi\\
&+\frac{\nabla\lambda}{2}|\nabla\phi|^2 + \frac{\lambda}{2}\nabla|\nabla\phi|^2 + \frac{1}{4\epsilon^2}\nabla\lambda(\phi^2-1)^2 + \frac{1}{4\epsilon^2}\lambda\nabla(\phi^2-1)^2.
\end{aligned}$$

And the right hand side convergence to $-\sigma Hn + \nabla\sigma - (\nabla\sigma\cdot n)n = -\sigma Hn + (\nabla\sigma\cdot t)t$ where t is the tangential direction of the interface. This recovers the traction free boundary condition (6.21).

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

In order to avoid the using of the Boussinsq approximation, which is only valid when the density of the mixture does not vary much, we can solve the transport eqaution (6.6) instead. The momentum equation will becomes:

$$\begin{aligned} &\rho(u_t + (u\cdot\nabla)u) + \nabla p - \nu \,\mathrm{div} D(u) \\ &= -\nabla\cdot(\lambda\nabla\phi\otimes\nabla\phi - \frac{\lambda}{2}|\nabla\phi|^2 - \frac{\lambda}{4\epsilon^2}(\phi^2-1)^2) - \rho g j, \end{aligned} \tag{6.29}$$

More generally, the viscosity in different components can also be different.

Since we can view ϕ as the approxiamtion of the volume fraction, an alternative is to use the following "average" density and viscosity as follows:

$$\begin{aligned} \frac{1}{\rho(\phi)} &= \frac{1+\phi}{2\rho_1} + \frac{1-\phi}{2\rho_2}, \\ \frac{1}{\nu(\phi)} &= \frac{1+\phi}{2\nu_1} + \frac{1-\phi}{2\nu_2}, \end{aligned} \tag{6.30}$$

where ρ_1, ρ_2 are the corresponding density and ν_1, ν_2 are the viscosity constants. The reason to choose the harmonic average as in (6.30) is that the solution of the Cahn-Hilliard equation (6.3) does not satisfy the maximal principle. Hence, the linear average can not be guaranteed to be bounded away from zero. However, due to the L^∞-bound of the solution [10], the harmonic averages lead to desired properties. This approach can be replaced using the normal linear averages in the case when (6.3) is replaced by the Allen-Cahn equation (6.4) for which the solution satisfies the maximal principle.

The modified momentum equation with variable density and viscosity takes the form

$$\begin{aligned} (\rho(\phi)u)_t &+ (u\cdot\nabla)(\rho(\phi)u) + \nabla p - \mathrm{div}(\nu(\phi)D(u)) \\ &+ \lambda\nabla\cdot(\nabla\phi\otimes\nabla\phi) = -\rho(\phi) g j, \end{aligned} \tag{6.31}$$

where $g(x)$ is the external body force. As the equation (6.25) converges to the pure transport equation, together with the incompressibility condition (6.25), the density ρ will satisfy the continuity equation:

$$\rho_t + \nabla\cdot(\rho u) = 0. \tag{6.32}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

6.3 Mixtures involving liquid crystals

In an immiscible blend of a nematic liquid crystal and a Newtonian fluid, there are three types of elastic energies: mixing energy of the interface, bulk distortion energy of the nematic, and the anchoring energy of the liquid crystal molecules on the interface.

We again use the previously discussed Ginzburg-Laudan energy for the mixing energy.

$$f_{mix}(\phi, \nabla\phi) = \frac{1}{2}\lambda|\nabla\phi|^2 + f_0(\phi), \tag{6.33}$$

with a double-well potential for F

$$f_0 = \frac{\lambda}{4\epsilon^2}(\phi^2 - 1)^2, \tag{6.34}$$

The nematic has rod-like molecules whose orientation can be represented by a unit vector $\boldsymbol{n}(\boldsymbol{x})$ known as the director. When the director field is not uniform, the nematic has a Oseen-Frank distortion energy [22]:

$$f_{bulk} = \frac{1}{2}K_1(\nabla \cdot \boldsymbol{n})^2 + \frac{1}{2}K_2(\boldsymbol{n} \cdot \nabla \times \boldsymbol{n})^2 + \frac{1}{2}K_3(\boldsymbol{n} \times \nabla \times \boldsymbol{n})^2, \tag{6.35}$$

where K_1, K_2, K_3 are elastic constants for the three canonical types of orientational distortion: splay, twist and bend. We will adopt the customary one-constant approximation: $K = K_1 = K_2 = K_3$, so that the Frank energy simplifies to $f_{bulk} = \frac{K}{2}\nabla\boldsymbol{n} : (\nabla\boldsymbol{n})^{\mathrm{T}}$. Liu & Walkington [69] used a modified model by allowing a non-unity director whose length indicates the order parameter. Thus, the regularized Frank elastic energy becomes:

$$f_{bulk} = K\left[\frac{1}{2}\nabla\boldsymbol{n} : (\nabla\boldsymbol{n})^{\mathrm{T}} + \frac{(|\boldsymbol{n}|^2 - 1)^2}{4\delta^2}\right], \tag{6.36}$$

The second term on the right hand side serves as a penalty whose minimization is simply the Ginzburg-Landau approximation of the constraint $|\boldsymbol{n}| = 1$ for small δ. The advantage of this regularized formulation is that the energy is now bounded for orientational defects, which are non-singular points where $|\boldsymbol{n}| = 0$. This makes the numerical treatment much easier. Note that the regularization is based on the same idea as in Cahn-Hilliard's mixing energy. It is also related to Ericksen's theory of uniaxial nematics with a variable order parameter [32].

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

Depending on the chemistry of the two components, the rod-like molecules of the nematic phase prefer to orient on the interface in a certain direction known as the easy direction. The two most common types of anchoring are planar anchoring, where all directions in the plane of the interface are easy directions, and homeotropic anchoring, where the easy direction is the normal to the interface.

In the classical sharp interface picture, the anchoring energy is a surface energy. In our diffuse-interface model, however, we write it as a volumetric energy density in the same vein as the mixing energy:

$$f_{anch} = \frac{A}{2}(\boldsymbol{n} \cdot \nabla\phi)^2 \tag{6.37}$$

for planar anchoring, and

$$f_{anch} = \frac{A}{2}[|\boldsymbol{n}|^2|\nabla\phi|^2 - (\boldsymbol{n} \cdot \nabla\phi)^2] \tag{6.38}$$

for homeotropic anchoring. In these two equations, the positive parameter A indicates the strength of the anchoring.

Finally, the total free energy density for the two-phase material is written as:

$$f(\phi, \boldsymbol{n}, \nabla\phi, \nabla\boldsymbol{n}) = f_{mix} + \frac{1+\phi}{2} f_{bulk} + f_{anch} \tag{6.39}$$

where $\frac{1+\phi}{2}$ is the volume fraction of the nematic component, and $\phi = 1$ in the purely nematic phase. This energy is equivalent to that of Rey [83], and contains all the physics discussed there.

The induced elastic energy will be:

$$\boldsymbol{\sigma}^e = -\lambda(\nabla\phi \otimes \nabla\phi) - K\frac{1+\phi}{2}(\nabla\boldsymbol{n}) \cdot (\nabla\boldsymbol{n})^{\mathrm{T}} - \boldsymbol{G}, \tag{6.40}$$

where $\boldsymbol{G} = A(\boldsymbol{n} \cdot \nabla\phi)\boldsymbol{n} \otimes \nabla\phi$ for planar anchoring and $\boldsymbol{G} = A[(\boldsymbol{n} \cdot \boldsymbol{n})\nabla\phi - (\boldsymbol{n} \cdot \nabla\phi)\boldsymbol{n}] \otimes \nabla\phi$ for homeotropic anchoring. Note that the asymmetry of $\boldsymbol{G}$ reflects the fact that surface anchoring exerts a net torque on the fluid. Bulk distortion will give rise to an asymmetric stress as well if the elastic constants are unequal [22]. Morever, from the derivation of the previous section, we see that the anchoring energy f_{anch}, hence the term $\boldsymbol{G}$, induces a Marangoni force along isotropic-nematic interfaces

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

For our model system of a blend of a nematic and a Newtonian fluid, the field variables are velocity $\boldsymbol{v}$, pressure p, phase function ϕ and director $\boldsymbol{n}$. We write the continuity and momentum equations in the usual form:

$$\nabla \cdot \boldsymbol{v} = 0, \tag{6.41}$$

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla p + \nabla \cdot \boldsymbol{\sigma}, \tag{6.42}$$

where $\boldsymbol{\sigma}$ is the deviatoric stress tensor.

Based on the free energy in equation (6.39), a generalized chemical potential can be defined as $\delta F / \delta \phi$. If one assumes a generalized Fick's law that the mass flux be proportional to the gradient of the chemical potential, the Cahn-Hilliard equation is obtained as an evolution equation for ϕ [13]:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \nabla \phi = \nabla \cdot \left[\gamma_1 \, \nabla \left(\frac{\delta F}{\delta \phi} \right) \right] \tag{6.43}$$

where γ_1 is the mobility, taken to be a constant in this paper. The diffusion term on the right hand side has contributions from all three forms of free energy.

The rotation of $\boldsymbol{n}$ is determined by the balance between a viscous torque and an elastic torque. The latter, also known as the molecular field [22], arises from the free energies of the system:

$$\boldsymbol{h} = -\frac{\delta F}{\delta \boldsymbol{n}} = K \left[-\nabla \cdot \left(\frac{1+\phi}{2} \nabla \boldsymbol{n} \right) + \frac{1+\phi}{2} \frac{(\boldsymbol{n}^2 - 1)\boldsymbol{n}}{\delta^2} \right] + \boldsymbol{g}, \tag{6.44}$$

where $\boldsymbol{g} = A(\boldsymbol{n} \cdot \nabla \phi) \nabla \phi$ for planar anchoring, and $\boldsymbol{g} = A[(\nabla \phi \cdot \nabla \phi)\boldsymbol{n} - (\boldsymbol{n} \cdot \nabla \phi) \nabla \phi]$ for homeotropic anchoring. Now the evolution equation of $\boldsymbol{n}$ is written as:

$$\frac{\partial \boldsymbol{n}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{n} = \gamma_2 \boldsymbol{h}, \tag{6.45}$$

where the constant γ_2 determines the relaxation time of the director field. Equation (6.45) is a simplified version of the Leslie-Ericksen equation [22].

Equations (6.41), (6.42), (6.43) and (6.45) form the complete set of equations governing the evolution of the nematic-Newtonian two-phase system.

In this paper, we assume that the two phases have the same constant density, with negligible volume change upon mixing. Thus, the mixture is

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

incompressible with a solenoidal velocity. In general, however, the diffuse-interface method is not restricted to equal-density components. When the two phases have differing densities, one approach is to view the mixture as a compressible fluid with $\nabla \cdot \boldsymbol{v} \neq 0$ in the mixing layer, where $\boldsymbol{v}$ is a mass-averaged velocity [71]. As an alternative, [67] have proposed a picture in which the components mix by advection only without diffusion. Thus, the velocity at a spatial point is defined as that of the component occupying that point; it is spatially continuous and remains solenoidal. An inhomogeneous average density is established from the initial condition, which is later transported by the velocity field. Finally, if the density difference is small, the Bousinesq approximation can be employed [67]. .

A solution to the above governing equations obeys an energy law.

$$\frac{d}{dt}\int_\Omega \left(\frac{\rho}{2}|\boldsymbol{v}|^2 + f\right) d\Omega = -\int_\Omega \left(\mu \nabla \boldsymbol{v} : \nabla \boldsymbol{v}^{\mathrm{T}} + \gamma_1 \left|\nabla \frac{\delta F}{\delta \phi}\right|^2 + \gamma_2 \left|\frac{\delta F}{\delta \boldsymbol{n}}\right|^2\right) d\Omega, \tag{6.46}$$

where f is the system's potential energy density. Physically, the law states that the total energy of the system (excluding thermal energy) will decrease from internal dissipation. The work of Lin and Liu [62, 63] can be used to rigorously prove the well-posedness of such a system. Under such an energy law, a finite-dimensional approximation to the governing equations, such as a finite-element or spectral scheme, can be shown to be guaranteed to converge [69]. This constitutes one of the advantages of our method over previous methods that do not maintain the system's total energy budget.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

7 Magneto-hydrodynamics (MHD)

7.1 Introduction

We are interested in the following unsteady incompressible Magneto Hydrodynamics (MHD) system:

$$\begin{aligned}
\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p &= \mu\Delta\boldsymbol{u} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{b}, && (7.1)\\
\nabla\cdot\boldsymbol{u} &= 0, && (7.2)\\
\frac{1}{c}\partial_t\boldsymbol{b} + \nabla\times\boldsymbol{e} &= \boldsymbol{0}, && (7.3)\\
\nabla\times\boldsymbol{b} &= \frac{4\pi}{c}\boldsymbol{j}, && (7.4)\\
\sigma(\boldsymbol{e} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{b}) &= \boldsymbol{j}, && (7.5)
\end{aligned}$$

in a smooth bounded domain in either $\mathbf{R}^2$ or $\mathbf{R}^3$. We also equip the system with no-slip and perfectly conducting wall conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma = \partial\Omega, \tag{7.6}$$

where ρ is the fluid density, $\boldsymbol{u}$ the fluid velocity, $\boldsymbol{b}$, $\boldsymbol{j}$ and $\boldsymbol{e}$ are the magnetic field, the electric current density and the electric field respectively. μ, σ and c are physical constants representing the viscosity coefficient, the electric conductivity and the speed of light.

We see that (7.1) consists of the Navier-Stokes equation governing the motion of the solenoidal fluid motion, coupled with the Faraday equation describing the evolution of the magnetic field, through the Lorentz force $\boldsymbol{j}\times\boldsymbol{b}$ and the electro transport $\boldsymbol{u}\times\boldsymbol{b}$.

7.2 The evolution of the magnetic field

First, let us look at the transport equation of $\boldsymbol{b}$, and take divergence of the equation with respect to the spatial variable. We have that

$$(\nabla\cdot\boldsymbol{b})_t = 0. \tag{7.7}$$

Hence we see that $\boldsymbol{b}$ is a divergence free vector field (assume this is true for the initial datum). This, combined with the fact that $\boldsymbol{u}$ is also divergence

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

free, we have

$$\begin{aligned}\nabla \times (\boldsymbol{u} \times \boldsymbol{b}) &= (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} + \boldsymbol{u}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{u}) \\ &= (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{b}.\end{aligned}$$

We can now rewrite the evolution equation (7.4) of the magnetic field $\boldsymbol{b}$ as:

$$\boldsymbol{b}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} = -\frac{c}{\sigma}\nabla \times \boldsymbol{j}. \tag{7.5}$$

The right hand side (7.7) is the dissipation term, while the left hand side can be written as $\boldsymbol{b}_t + \mathcal{L}_{\boldsymbol{u}}\boldsymbol{b}$, where $\mathcal{L}_{\boldsymbol{u}}\boldsymbol{b}$ is the Lie derivative of $\boldsymbol{b}$ with respect to the flow field $\boldsymbol{u}$.

Again we look at F, the deformation tensor $\frac{\partial x}{\partial X}$, where x is the Eulerian (reference) coordinate and X is the Lagrangian (material) coordinate. The flow trajectory of a partial will be $x(X,t)$, where

$$\frac{\partial}{\partial t}x(X,t) = \boldsymbol{u}(x(X,t),t), \quad x(X,0) = X. \tag{7.5}$$

In case of incompressible materials, x is a volume preserving diffeomorphism and $\det F = 1$. In this case, we have the following identity [41, 70]:

$$\frac{d}{dt}F(x(X,t),t) = \frac{\partial}{\partial t}F(x,t) + \boldsymbol{u}(x,t) \cdot \nabla F(x,t) = \nabla \boldsymbol{u}(x,t)F(x,t). \tag{7.5}$$

The transport $\boldsymbol{b}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u}$ represents exactly the relation $\boldsymbol{b}(x(X,t),t) = F^{-1}\boldsymbol{b}_0(X)$, where $\boldsymbol{b}_0$ is the initial magnetic fields. From here, we see that F carries all the transport information of $\boldsymbol{b}$. This makes the MHD system very much related to the viscoelastic system discussed in the earlier sections.

7.3 The energy law

The system (7.1) admits the following energy law:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{8\pi}|\boldsymbol{b}|^2\,dx = -\int_\Omega \mu|\nabla \boldsymbol{u}|^2 + \frac{4\pi}{c\sigma}|\boldsymbol{j}|^2\,dx. \tag{7.5}$$

This energy law can be derived by multiply (7.2) by $\boldsymbol{u}$ and (7.4) by $\boldsymbol{b}$, add the results together, and integration by parts. The special cancellation of the term from the Lorentz force and the term from the transport of the magnetic field shows the special coupling that will be discussed in the later sections.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

7.4 The linear momentum equation

The linear momentum equation (without the dissipation term $\mu \boldsymbol{u}$) in (7.1) can be derived through the variations of the following "action" functional in the space of volume preserving flow maps:

$$\begin{aligned} A(x) &= \int_0^T \int_\Omega \frac{1}{2}\rho|\boldsymbol{u}(x,t)|^2 + \frac{1}{8\pi}|\boldsymbol{b}(x,t)|^2\, dxdt \\ &= \int_0^T \int_{\Omega_0} \frac{1}{2}\rho|x_t(X,t)|^2 + \frac{1}{8\pi}|F(X,t)\boldsymbol{b}(x(X,t)|^2\, dXdt \\ &= A_1(x) + \frac{1}{4\pi}A_2(x). \end{aligned} \tag{7.6}$$

Here we use the fact that the Jacobian of x with respect to X is 1, due to the incompressibility.

In order to perform the variation, we look at the one parameter family of such maps x^ϵ such that:

$$x^0 = x, \quad \frac{dx^\epsilon}{d\epsilon} = y. \tag{7.4}$$

for any y such that $\nabla_x \cdot y = 0$.

We computer the variation of $A(x^\epsilon) = A(\phi(x^\epsilon, t))$ with respect to ϵ. The contribution of the kinetic part $\frac{1}{2}\rho|\boldsymbol{u}(x,t)|^2$ will give the Euler part of the equation (7.2). Suppose that $F^\epsilon = \frac{x^\epsilon}{X}$, the contribution of the magnetic field will be:

$$\begin{aligned} \frac{d}{d\epsilon}|_{\epsilon=0} A_2(x^\epsilon) &= \int_0^T \int_{\Omega_0} \big(F(X,t)\boldsymbol{b}(x(X,t), \frac{d}{d\epsilon}|_{\epsilon=0}(F^\epsilon(X,t)\boldsymbol{b}(x^\epsilon(X,t)))\big)\, dXdt \\ &= \int_0^T \int_{\Omega_0} \big(F(X,t)\boldsymbol{b}(x(X,t), F(X,t)(y\cdot\nabla_x)\boldsymbol{b}\big) \\ &+ \big(F(X,t)\boldsymbol{b}(x(X,t), \nabla_X y\boldsymbol{b}(x(X,t))\big)\, dXdt. \end{aligned}$$

Since y is an arbitrary divergence free vector field, integration by parts gives the following equation:

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p = \frac{1}{4\pi}\nabla\cdot(\boldsymbol{b}\otimes\boldsymbol{b}) + \frac{1}{2\pi}\nabla|\boldsymbol{b}|^2, \tag{7.1}$$

Since we have the identity:

$$(\nabla\times\boldsymbol{b})\times\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \frac{1}{2}\nabla|\boldsymbol{b}|^2 = \nabla\cdot(\boldsymbol{b}\otimes\boldsymbol{b}) - \frac{1}{2}\nabla|\boldsymbol{b}|^2, \tag{7.1}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

we have that

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p_1 = \frac{1}{4\pi}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} = \frac{1}{c}\boldsymbol{j}\times\boldsymbol{b}. \tag{7.1}$$

From the above derivation, we see that the Lorentz force comes from the fact that the magnetic field is transport by the flow map as a covariant 1-form and the inclusion of the magnetic energy of $\boldsymbol{b}$ in the total energy.

7.5 The dynamics of magnetic field lines

In the case that $\sigma = \infty$, then we are in the case that

$$\boldsymbol{e} = -\frac{1}{c}\boldsymbol{u}\times\boldsymbol{b}. \tag{7.1}$$

Taking the $\nabla\times$ on the both sides of the equation yields that

$$\begin{aligned}(\nabla\times\boldsymbol{b})_t \quad &+ \quad (\boldsymbol{u}\cdot\nabla)(\nabla\times\boldsymbol{b}) - ((\nabla\times\boldsymbol{b})\cdot\nabla)\boldsymbol{u} \\ &- \quad (\boldsymbol{b}\cdot\nabla)(\nabla\times\boldsymbol{u}) + ((\nabla\times\boldsymbol{u})\cdot\nabla)\boldsymbol{b} = 0.\end{aligned} \tag{7.2}$$

In the 2-dimensional case, that is, both $\boldsymbol{u}$ and $\boldsymbol{b}$ depend only on the first two coordinate x_1, x_2 and the third component $\boldsymbol{u}^3 = \boldsymbol{b}^3 = 0$, then we have

$$(\nabla\times\boldsymbol{b})_t + (\boldsymbol{u}\cdot\nabla)(\nabla\times\boldsymbol{b}) - (\boldsymbol{b}\cdot\nabla)(\nabla\times\boldsymbol{u}) = 0. \tag{7.1}$$

If we are in a irrotational flow field, we have that

$$(\nabla\times\boldsymbol{b})_t + (\boldsymbol{u}\cdot\nabla)(\nabla\times\boldsymbol{b}) = 0. \tag{7.1}$$

Hence we can see that if $(\nabla\times\boldsymbol{b})$ concentrate on a curve (interface) at the initial time, it will also be concentrated a curve.

Finally, since $\nabla\cdot\boldsymbol{b} = 0$, we have that

$$\boldsymbol{b} = \nabla\times\mathbf{A}, \tag{7.1}$$

where $\mathbf{A}$ is the electric potential with the Columb gauge $\nabla\cdot\mathbf{A} = 0$. The electric current $\boldsymbol{j} = -\Delta\mathbf{A}$. Moreover, it satisfies the transport equation:

$$\mathbf{A}_t + (\boldsymbol{u}\cdot\nabla)\mathbf{A} = 0. \tag{7.1}$$

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

In the 2-dimensional case, we have the simpler form

$$\boldsymbol{b} = \nabla^{\perp}\phi = (\phi_y, -\phi_x), \tag{7.1}$$

for a scalar function $\phi(x, y)$. The current $\boldsymbol{j} = -\Delta\phi \mathbf{e}_3$ and it satisfies the transport equation:

$$\phi_t + (\boldsymbol{u} \cdot \nabla)\phi = 0. \tag{7.1}$$

Moreover, the Lorentz force

$$\boldsymbol{j} \times \boldsymbol{b} = \Delta\phi\nabla\phi. \tag{7.1}$$

The level set of ϕ, $\{\phi = c\}$ is the magnetic field lines. The dynamics of such curves is very important in understanding the MHD equations.

References

[1] R. Abraham and J. E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Ratiu and Richard Cushman.

[2] N. D. Alikakos, P. W. Bates, and X. F. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rational Mech. Anal.*, 128(2):165–205, 1994.

[3] V. I. Arnold. *Mathematical methods of classical mechanics*. Springer-Verlag, New York, 1989. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[4] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, paperback edition, 1999.

[5] F. Bethuel and H. Brezis. Regularity of minimizers of relaxed problems for harmonic maps. *J. Funct. Anal.*, 101(1):145–161, 1991.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids: Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

[6] F. Bethuel, H. Brezis, and F. Helein. *Ginzburg–Landau Vorticies*. Klumer, 1995.

[7] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics*. Weiley Interscience, New York, 1987.

[8] F. Boyer. A theoretical and numerical model for the study of incompressible mixture flows. *Computers and Fluids*, 31:41–68, 2002.

[9] H. Brezis, J. Coron, and E. Lieb. Harmonic maps with defects. *Comm. Math. Phys.*, 107(4):649–705, 1986.

[10] L. A. Caffarelli and N. E. Muler. An L^∞ bound for solutions of the Cahn-Hilliard equation. *Arch. Rational Mech. Anal.*, 133(2):129–144, 1995.

[11] G. Caginalp and X. F. Chen. Phase field equations in the singular limit of sharp interface problems. In *On the evolution of phase boundaries (Minneapolis, MN, 1990–91)*, pages 1–27. Springer, New York, 1992.

[12] J. W. Cahn and S. M. Allen. A microscopic theory for domain wall motion and its experimental varification in fe-al alloy domain growth kinetics. *J. Phys. Colloque*, C7:C7–51, 19778.

[13] J. W. Cahn and J. E. Hillard. Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Phys.*, 28:258–267, 1958.

[14] P. M. Chaikin and T. C. Lubensky. *Principles of Condensed Matter Physics*. Cambridge, 1995.

[15] Y. C. Chang, T. Y. Hou, B. Merriman, and S. Osher. A level set formulation of eulerian interface capturing methods for incompressible fluid flows. *J. Comput. Phys.*, 124(2):449–464, 1996.

[16] J. Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. MAth. Anal.*, 33(1):84–112, 2001.

[17] X. F. Chen. Generation and propagation of interfaces in reaction-diffusion systems. *Trans. Amer. Math. Soc.*, 334, 1992.

[18] Y. M. Chen and M. Struwe. Regularity for heat flow for harmonic maps. *Math. Z.*, 201:83–103, 1989.

[19] M. Chipot, D. Kinderlehrer, and C. Liu. Some variational problems with relaxed constraints. *unpublished*, 1997.

[20] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[21] V. Cristini, J. Blawzdziewicz, and M. Loewenberg. Drop breakup in three-dimensional viscous flows. *Phys. Fluids*, 10:1781–1783, 1998.

[22] P. G. de Gennes and J. Prost. *The Physics of Liquid Crystals*. Oxford University Press, 1993.

[23] J. Deang, Q. Du, M. Gunzburger, and J. Peterson. Vortices in superconductors: modelling and computer simulations. *Philos. Trans. Roy. Soc. London*, 355(1731):1957–1968, 1997.

[24] R. J. DiPerna and P. L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[25] J. E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.*, 88(2):95–133, 1985.

[26] W. E. Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity. *Phys. D*, 77(4):383–404, 1994.

[27] W. E, T. Li, and P. Zhang. Well-posedness for the dumbbell model of polymeric fluids. *Comm. in Math. Phy.*, 248 (2):409–427, 2004.

[28] W. E and P. Palffy-Muhoray. Phase separation in incompressible systems. *Phys. Rev. E (3)*, 55(4):R3844–R3846, 1997.

[29] D. A. Edwards, H. Brenner, and D. T. Wasan. *Interfacial Transport Process and Rheology*. Butterworth-Heinemann, 1991.

[30] J. Ericksen. Conservation laws for liquid crystals. *Trans. Soc. Rheol.*, 5:22—34, 1961.

[31] J. Ericksen. Continuum theory of nematic liquid crystals. *Res. Mechanica*, 21:381–392, 1987.

[32] J. Ericksen. Liquid crystals with variable degree of orientation. *Arch Rath. Mech. Anal.*, 113:97–120, 1991.

[33] L. C. Evans. *Weak Convergence Methods for Nonlinear Partial Differential Equations*, volume 74. A.M.S. Regional Conference Series in Mathematics, 1989.

[34] L. C. Evans. *Partial Differential Equations*. A.M.S., 1998.

[35] E. Freid and M. E. Gurtin. Continuum theory of thermally induced phase transitions based on an order parameter. *Physica D*, 68:326–343, 1993.

[36] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer–Verlag, Berlin, New York, 1983.

[37] J. Glimm, J. W. Grove, X. L. Li, and D. C. Tan. Robust computational algorithms for dynamic interface tracking in three dimensions. *SIAM J. Sci. Comput.*, 21(6):2240–2256 (electronic), 2000.

[38] J. Glimm, X. L. Li, Y. Liu, and N. Zhao. Conservative front tracking and level set algorithms. *Proc. Natl. Acad. Sci. USA*, 98(25):14198–14201 (electronic), 2001.

[39] C. Guillopé and J. C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[40] C. Guillopé and J. C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO Modél. Math. Anal. Numér.*, 24(3):369–401, 1990.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

[41] M. E. Gurtin. *An Introduction to Continuum Mechanics*, volume 158 of *Mathematics in Science and Engineering*. Academic Press, 1981.

[42] M. E. Gurtin. Multiphase thermodynamics with interfacial structure, 1. Heat conduction and the capillary balance law. *Archive for Rational Mechanics and Analysis*, 104:195–221, 1988.

[43] M. E. Gurtin, D. Polignone, and J. Viñals. Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.*, 6(6):815–831, 1996.

[44] R. Hardt and F. H. Lin. Stability of singularities of minimizing harmonic maps. *J. Differential Geom.*, 29(1):113–123, 1989.

[45] J. Hron, J. Málek, J. Ne$\boldsymbol{v}$cas, and K. R. Rajagopal. Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear-dependent viscosities. *Math. Comput. Simulation*, 61(3-6):297–315, 2003. MODELLING 2001 (Pilsen).

[46] D. Jacqmin. Calculation of two-phase Navier-Stokes flows using phase-field modeling. *J. Comput. Phys.*, 155(1):96–127, 1999.

[47] F. John. Rotation and strain. *Comm. Pure Appl. Math.*, 14:391–413, 1961.

[48] D. D. Joseph. Instability of the rest state of fluids of arbitrary grade greater than one. *Arch. Rational Mech. Anal.*, 75(3):251–256, 1980/81.

[49] R. Khayat. Three-dimensional boundary-element analysis of drop deformation for newtonian and viscoelastic systems. *Int. J. Num. Meth. Fluids*, 34:241–275, 2000.

[50] V. V. Krotov and A. I. Rusanov. *Physicochemical Hydrodynamics of Capillary Systems*. Inperial College Press, London, 1999.

[51] S. N. Kruzkov. First order quasilinear equations in several independent variables. *Math. USSR Sbornik*, 10(2):217–243, 1970.

[52] O. A. Ladyzhenskaya and G. A. Seregin. On the regularity of solutions of two-dimensional equations of the dynamics of fluids with nonlinear viscosity. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 259(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 30):145–166, 298, 1999.

[53] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics*. Pergamon Press, 2 edition, 1987.

[54] R. G. Larson. *The Structure and Rheology of Complex Fluids*. Oxford, 1995.

[55] Z. Lei, C. Liu, and Y. Zhou. Existence of small strain incompressible viscoelasticity. *preprint*, 2005.

[56] Z. Lei, C. Liu, and Y. Zhou. Incompressible viscoelasticity fluids: Existence of classical solutions with near equilibrium initial datum. *Submitted to Comm. Math. Phys.*, 2005.

[57] F. Leslie. Some constitutive equations for liquid crystals. *Archive for Rational Mechanics and Analysis*, 28:265–283, 1968.

[58] J. Li and Y. Renardy. Numerical study of flows of two immiscible liquids at low reynolds number. *SIAM Review*, 42:417–439, 2000.

[59] J. Li and Y. Renardy. Shear-induced rupturing of a viscous drop in a bingham liquid. *J. Non-Newtonian Fluid Mech.*, 95:235–251, 2000.

[60] J. Lighthill. *Waves in Fluids*. Cambridge, 1978.

[61] F. H. Lin. Mapping problems, fundamental groups and defect measures. *Acta Math. Sin. (Engl. Ser.)*, 15(1):25–52, 1999.

[62] F. H. Lin and C. Liu. Nonparabolic dissipative systems, modeling the flow of liquid crystals. *Comm. Pure Appl. Math.*, XLVIII(5):501–537, 1995.

[63] F. H. Lin and C. Liu. Global existence of solutions for the Ericksen Leslie–system. *Arch. Rat. Mech. Ana.*, 154(2):135–156, 2001.

[64] F. H. Lin and C. Liu. Static and dynamic theories of liquid crystals. *Journal of Partial Differential Equations*, 14(4):289–330, 2001.

[65] F. H. Lin, C. Liu, and P. Zhang. On a micro-macro model for polymeric fluids near equilibrium. *Comm. Pure Appl. Math.*, LIX:1–29, 2005.

[66] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[67] C. Liu and J. Shen. A phase field model for the mixture of two incompressible fluids and its approximation by a fourier-spectral method. *Physica D*, 179:211–228, 2003.

[68] C. Liu and S. Shkoller. Variational phase field model for the mixture of two fluids. *preprint*, 2001.

[69] C. Liu and N. J. Walkington. Approximation of liquid crystal flows. *SIAM Journal on Numerical Analysis*, 37(3):725–741, 2000.

[70] C. Liu and N. J. Walkington. An eulerian description of fluids containing visco-hyperelastic particles. *Arch. Rat. Mech. Ana.*, 159:229–252, 2001.

[71] J. Lowengrub and L. Truskinovsky. Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 454(1978):2617–2654, 1998.

[72] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[73] J. Málek, J. Ne$\boldsymbol{v}$cas, and K. R. Rajagopal. Global analysis of the flows of fluids with pressure-dependent viscosities. *Arch. Ration. Mech. Anal.*, 165(3):243–269, 2002.

[74] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New

York, second edition, 1999. A basic exposition of classical mechanical systems.

[75] G. B. McFadden, A. A. Wheeler, R. J. Braun, S. R. Coriell, and R. F. Sekerka. Phase-field models for anisotropic interfaces. *Phys. Rev. E (3)*, 48(3):2016–2024, 1993.

[76] W. W. Mullins and R. F. Sekerka. On the thermodynamics of crystalline solids. *J. Chem. Phys.*, 82, 1985.

[77] S. Osher and J. Sethian. Fronts propagating with curvature dependent speed: Algorithms based on Hamilton Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[78] R. G. Owens and T. N. Phillips. *Computational Rheology.* Imperial College Press, London, 2002.

[79] C. S. Peskin. A random-walk interpretation of the incompressible Navier-Stokes equations. *Comm. Pure Appl. Math.*, 38(6):845–852, 1985.

[80] R. F. Probstein. *Physicochemical Hydrodynamics, An Introduction.* John Wiley and Sons, INC., 1994.

[81] T. Qian, X. P. Wang, and P. Sheng. Molecular scale contact line hydrodynamics of immiscible flows. *preprint*, 2002.

[82] M. Renardy, W. J. Hrusa, and J. A. Nohel. *Mathematical Problems in Viscoelasticity*, volume 35 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific & Technical, Harlow, 1987.

[83] A. D. Rey. Vescoelastic theory for nematic interfaces. *Physical Review E*, 61(2):1540–1549, 2000.

[84] J. Rubinstein, P. Sternberg, and J. B. Keller. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.*, 49(1):116–133, 1989.

[85] R. Schoen and K. Uhlenbeck. Correction to: "A regularity theory for harmonic maps". *J. Differential Geom.*, 18(2):329, 1983.

[86] R. Schoen and K. Uhlenbeck. Regularity of minimizing harmonic maps into the sphere. *Invent. Math.*, 78(1):89–100, 1984.

[87] W. R. Schowalter. *Mechanics of Non-Newtonian Fluids*. Pergamon Press, 1978.

[88] M. Slemrod. Constitutive relations for Rivlin-Ericksen fluids based on generalized rational approximation. *Arch. Ration. Mech. Anal.*, 146(1):73–93, 1999.

[89] H. M. Soner. Convergence of the phase-field equations to the Mullins-Sekerka problem with kinetic undercooling [97d:80007]. In *Fundamental contributions to the continuum theory of evolving phase interfaces in solids*, pages 413–471. Springer, Berlin, 1999.

[90] M. Struwe. *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer-Verlag, 1990.

[91] M. Struwe. *Geometric Evolution Problems*. AMS, 1992.

[92] L. Tartar. Compensated compactness and applications to partial differential equations. In R. J. Knops, editor, *Research Notes in Mathematics, Nonlinear Analysis, and Mechanics, Heriot Watt Symposium*, volume 4. Pitman Press, 1979.

[93] J. E. Taylor and J. W. Cahn. Linking anisotropic sharp and diffuse surface motion laws via gradient flows. *J. Statist. Phys.*, 77(1-2):183–197, 1994.

[94] R. Temam. *Navier-Stokes Equations*. North Holland, 1977.

[95] E. M. Toose, B. J. Geurts, and J. Kuerten. A boundary integral method for two-dimensional (non)-newtonian drops in slow viscous flow. *J. Non-Newtonian Fluid Mech.*, 60:129–154, 1995.

Chun Liu Intro to EnVarA p. 286-337 of
Hou, T. Y., C. Liu and J.-g. Liu, Eds. (2009). Multi-scale Phenomena in Complex Fluids:
Modeling, Analysis and Numerical Simulations
Singapore, World Scientific Publishing Company.

[96] J. van der Waals. The thermodynamic theory of capillarity under the hypothesis of a continuous density variation. *J. Stat. Phys.*, 20:197–244, 1893.

[97] P. Yue, J. Feng, C. Liu, and J. Shen. A diffuse-interface method for simulating two-phase flows of complex fluids. *Journal of Fluid Mechanics*, 515:293–317, 2004.