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A GENERALIZED POISSON–NERNST–PLANCK–NAVIER–STOKES MODEL ON THE FLUID WITH THE CROWDED CHARGED PARTICLES: DERIVATION AND ITS WELL-POSEDNESS*

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Abstract. We derive a hydrodynamic model of the compressible conductive fluid by using an energetic variational approach, which could be called a generalized Poisson–Nernst–Planck–Navier–Stokes system. This system characterizes the micro-macro interactions of the charged fluid and the mutual friction between the crowded charged particles. In particular, it reveals the cross-diffusion phenomenon which does not happen in the fluid with the dilute charged particles. The cross-diffusion is tricky; however, we develop a general method to show that the system is globally asymptotically stable under small perturbations around a constant equilibrium state. Under some conditions, we also obtain the optimal decay rates of the solution and its derivatives of any order. Our method will apply equally well to a class of cross-diffusion systems if their linearized diffusion matrices are diagonally dominant.

Key words. Poisson–Nernst–Planck–Navier–Stokes equations, energetic variational approach, well-posedness, cross-diffusion

AMS subject classifications. 35Q35, 35Q92, 76W05

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1. Introduction.

Throughout this paper, for the sake of the unified presentation, we give a list:

ρ	mass density of fluid	ϕ	electrostatic potential
u	macroscopic velocity of fluid	ϵ	dielectric constant
v	negative charge distribution	z_v	valence of negative ions
u_v	effective velocity of negative charge	z_w	valence of positive ions
w	positive charge distribution	e	charge of one electron
u_w	effective velocity of positive charge	p	pressure function
μ, μ'	viscosity coefficients	D_v	diffusivity of negative ions
$D_{v,w}$	mobility coefficients	D_w	diffusivity of positive ions
k_B	Boltzmann constant	T	absolute temperature

The study of the transport of charged particles has been a very hot topic since it plays an important role in real life. Many phenomena and processes, including electrophoresis and electroosmosis [45, 57], for instance, in physics, chemistry, biology, and engineering, could be attributed to these models in light of many classical

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fluid models, such as the inviscid Euler system [14], the viscous Navier–Stokes system [47], magnetohydrodynamics system [15, 16], Maxwell system [48, 56], and so forth. In this paper, our starting point is the Poisson–Nernst–Planck (PNP) system in [9]. We first review some known results about the PNP system. In general, the PNP system models the dynamics of the microscopic charged particles (cations, anions) in compressible or incompressible fluids by neglecting the macroscopic fluid effect. The original PNP system reads as

$$\begin{cases} v_t = \operatorname{div}\left[D_v\left(\nabla v + \dfrac{z_v e}{k_B T} v\nabla\phi\right)\right], \\ w_t = \operatorname{div}\left[D_w\left(\nabla w + \dfrac{z_w e}{k_B T} w\nabla\phi\right)\right], \\ -\epsilon\Delta\phi = z_v e v + z_w e w, \end{cases} \tag{1.1}$$

which is usually used to describe the dynamics of the *dilute* charged particles [24, 25, 56]. There is the energy dissipation law of (1.1),

$$\begin{aligned} \frac{d}{dt}E^{total} &:= \frac{d}{dt}\int k_B T\,(v\ln v + w\ln w) + \frac{1}{2}\phi(z_v e v + z_w e w) \\ &= -\int \frac{k_B T}{D_v} v\,|u_v|^2 + \frac{k_B T}{D_w} w\,|u_w|^2 := -\triangle, \end{aligned} \tag{1.2}$$

where E^{total} is the total energy including thermo-fluctuations (Gibbs entropy) of the ion species and the electric potential energy, $\triangle$ is the entropy production (energy dissipation rate), $u_v (u_w)$ satisfying (2.11) ((2.12)) is the effective velocity of the negative (positive) charge, and ϕ is the solution to the Poisson equation $(1.1)_3$, respectively. Hsieh et al. [33] combined the energy dissipation law (1.2) and (2.11)–(2.12) to derive the above equations $(1.1)_{1,2}$ by using an energetic variational approach (EVA). For the existence, uniqueness, and long time behavior of solutions to the Cauchy problem or initial-boundary problem of (1.1), we could refer to [5, 28, 46, 59, 61, 67, 77] and the references therein. For the study of the PNP system with n ion species, we could refer to [8, 39, 51]. For the other topics of the PNP system, such as the steady-state problem, the quasi-neutral limit, etc., we could refer to [4, 23, 27, 42, 52, 65, 68, 71, 75, 76]. In addition, the PNP system (1.1) coupled with an incompressible Navier–Stokes system has been studied; cf. [6, 36, 37, 38, 50, 69, 79].

However, the model might need to be modified when one deals with the solution with crowded charged particles. Such a case can be found in ion channels and electrodes of batteries [19, 20, 21]. In this case, the mutual friction between different ion species is inevitable since it has an important impact on the dynamics of the species themselves. To analyze such an impact, Hsieh et al. [33] modified the above dissipation as the following; however, the total energy is unchanging,

$$\triangle^* = \int \frac{k_B T}{D_v} v\,|u_v|^2 + \frac{k_B T}{D_w} w\,|u_w|^2 + \frac{k_B T}{D_{v,w}} vw\,|u_v - u_w|^2,$$

where the added third term (the relative velocity differences) is responsible for the dissipation arising from the friction between particles. Then, their modified PNP model is

(1.3)
$$\begin{cases} v_t = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+v)\left(\nabla v + \frac{z_v e}{k_B T} v\nabla\phi\right) + v\left(\nabla w + \frac{z_w e}{k_B T} w\nabla\phi\right)\right)\right], \\ w_t = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+w)\left(\nabla w + \frac{z_w e}{k_B T} w\nabla\phi\right) + w\left(\nabla v + \frac{z_v e}{k_B T} v\nabla\phi\right)\right)\right], \\ -\epsilon\Delta\phi = z_v e v + z_w e w, \end{cases}$$

where we have set $D_v = D_w = D_{v,w} = 1$ for brevity.

Motivated by [33], we modify the energy dissipation law by considering the combination of the microscopic (atomic) energy law and macroscopic (hydrodynamic) energy law, which could be reasonable since the microscopic charged particle and the macroscopic flow interact with each other. More precisely, we explore the connection between the PNP system and Navier–Stokes system by using an EVA stated as in section 2. Then, we derive a generalized coupled Poisson–Nernst–Planck-Navier–Stokes (PNP-NS) system:

(1.4)
$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + \mu'\nabla \operatorname{div} u - \nabla v - \nabla w + \epsilon\Delta\phi\nabla\phi, \\ v_t + \operatorname{div}(vu) = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+v)\left(\nabla v + z_v e v\nabla\phi\right) + v\left(\nabla w + z_w e w\nabla\phi\right)\right)\right], \\ w_t + \operatorname{div}(wu) = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+w)\left(\nabla w + z_w e w\nabla\phi\right) + w\left(\nabla v + z_v e v\nabla\phi\right)\right)\right], \\ -\epsilon\Delta\phi = z_v e v + z_w e w. \end{cases}$$

The above system (1.4) could describe the dynamics of the compressible viscous conductive fluid with the *crowded* charged particles in $\mathbb{R}^3$. Comparing the model (1.3) and (1.4) with (1.1), we find that the cross-diffusion terms appear in the equations $(1.3)_{1,2}$ and $(1.4)_{3,4}$. Then, we can conclude that the mutual friction between different particles will lead to the cross-diffusion phenomenon. For the cross-diffusion phenomenon, readers could refer to [53, 54]. As we know, it is hard to deal with the cross-diffusion problems since generally there is no maximum principle. In addition, the cross-diffusion induced instability happens; see [35, 74]. About the cross-diffusion models, there are two typical examples. One is the well-known Shigesada–Kawasaki–Teramoto (SKT) system, which was first proposed to model the spatial segregation of two competing species in [70]. Since then, many researchers have been devoted to studying the SKT system. For more details, readers could refer to [10, 11, 17, 32, 53, 54, 80]. The other model is called (Patlak–)Keller–Segel model, which was derived to model the chemotaxis; cf. [31, 43, 44, 66, 78]. Besides, we could also refer to [7, 12, 18, 40, 41, 49] for other cross-diffusion models. However, our derived cross-diffusion system (1.4) is new and more complicated since it consists of coupled hyperbolic-parabolic-elliptic equations. To some extent, it can be used to model and simulate some important biophysical processes. So, it will be very interesting to solve (1.4) in the following.

In this paper, without loss of generality, we set $\mu = \mu' = \epsilon = e = z_w = 1$ and $z_v = -1$ in (1.4). Then, the system (1.4) is reduced to

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0, \\
(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p - \Delta u - \nabla \operatorname{div} u = -\nabla v - \nabla w + \Delta\phi\nabla\phi, \\
v_t + \operatorname{div}(vu) = \operatorname{div}\left[\dfrac{1}{1+v+w}\left((1+v)\left(\nabla v - v\nabla\phi\right) + v\left(\nabla w + w\nabla\phi\right)\right)\right], \\
w_t + \operatorname{div}(wu) = \operatorname{div}\left[\dfrac{1}{1+v+w}\left((1+w)\left(\nabla w + w\nabla\phi\right) + w\left(\nabla v - v\nabla\phi\right)\right)\right], \\
\Delta\phi = v - w.
\end{cases} \tag{1.5}
$$

Here we assume that the pressure $p = p(\rho)$ is a smooth function such that

$$p'(\rho) > 0 \text{ for } \rho > 0. \tag{1.6}$$

We also assume $p'(1) = 1$ without loss of generality. We will look for the solutions $(\rho, u, v, w)(x,t)$ to the Cauchy problem for (1.5) with the initial data

$$(\rho, u, v, w)(x,t)\mid_{t=0} = (\rho_0, u_0, v_0, w_0)(x), \;\; x \in \mathbb{R}^3, \tag{1.7}$$

and the far-field behavior

$$\lim_{|x|\to+\infty} (\rho, u, v, w)(x,t) = (1, 0, 1, 1). \tag{1.8}$$

In addition, the Poisson equation $(1.5)_5$ implies the electrical neutrality of the far-field, i.e.,

$$\lim_{|x|\to+\infty} \phi(x,t) = 0. \tag{1.9}$$

We define the perturbation by

$$\varrho = \rho - 1, \;\; u = u, \;\; V = v - 1, \;\; W = w - 1, \;\; \phi = \phi.$$

Then, the Cauchy problem (1.5)–(1.9) becomes

$$
\begin{cases}
\varrho_t + \operatorname{div}((\varrho+1)u) = 0, \\
(\varrho+1)\left(u_t + u\cdot\nabla u\right) + p'(\varrho+1)\nabla\varrho - \Delta u - \nabla \operatorname{div} u = -\nabla V - \nabla W + \Delta\phi\nabla\phi, \\
V_t + \operatorname{div}((V+1)u) \\
\qquad = \operatorname{div}\left(\dfrac{(V+2)\nabla V + (V+1)\nabla W}{V+W+3}\right) - \operatorname{div}\left(\dfrac{(V+1)(V-W+1)\nabla\phi}{V+W+3}\right), \\
W_t + \operatorname{div}((W+1)u) \\
\qquad = \operatorname{div}\left(\dfrac{(W+2)\nabla W + (W+1)\nabla V}{V+W+3}\right) + \operatorname{div}\left(\dfrac{(W+1)(W-V+1)\nabla\phi}{V+W+3}\right), \\
\Delta\phi = V - W, \\
(\varrho, u, V, W)\mid_{t=0} = (\varrho_0, u_0, V_0, W_0).
\end{cases} \tag{1.10}
$$

Set $\widetilde{U} = (V, W)^T$. The perturbed cross-diffusion equations $(1.10)_3$–$(1.10)_4$ could be rewritten as

$$\widetilde{U}_t - \operatorname{div}(A(\widetilde{U})\nabla\widetilde{U}) = \mathcal{N}(\widetilde{U}).$$

Here the diffusion matrix $A(\widetilde{U})$ is given as

$$A(\widetilde{U}) := \begin{pmatrix} A_{11}(V,W) & A_{12}(V,W) \\ A_{21}(V,W) & A_{22}(V,W) \end{pmatrix},$$

where

$$A_{11}(V,W) := \frac{V+2}{V+W+3},\ A_{12}(V,W) := \frac{V+1}{V+W+3},$$
$$A_{21}(V,W) := \frac{W+1}{V+W+3},\ A_{22}(V,W) := \frac{W+2}{V+W+3}.$$

We claim that the linearized matrix for $A(\widetilde{U})$ is diagonally dominant, i.e., the linear part of $A(\widetilde{U})$, which by (4.1) is equal to

$$\mathcal{L}(A) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

is diagonally dominant. This is a key point to derive the effective energy estimates in the later sections. In fact, we develop a simple approximation scheme to prove the local solution, where the cross-diffusion could be controlled by using some detailed energy estimates. In particular, the unique local solution holds for large initial data. Since the linearized diffusion matrix is diagonally dominant, we manage to derive the refined a priori estimates. Then, the a priori estimates ensure that the local solution can be extended to the global one by using a continuous argument. To prove the time-decay rates, a crucial step is to find a better energy estimate for $\nabla\phi$ given by Lemma 5.1 on the premise that the global small solution has been obtained. Further, we could establish the refined differential inequality given by Lemma 5.2, which helps us to derive the time-decay rates of the solution and its derivatives of any order. Our main results, which include the global small solution and the time-decay rates, could be applied to a class of cross-diffusion systems whose linearized diffusion matrices are diagonally dominant. For example, it is suitable for a more general PNP-NS system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + \mu'\nabla \operatorname{div} u - \nabla\varphi(v) - \nabla\psi(w) + \epsilon\Delta\phi\nabla\phi, \\ v_t + \operatorname{div}(vu) \\ \qquad = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+v)\left(\nabla\varphi(v) + z_v ev\nabla\phi\right) + v\left(\nabla\psi(w) + z_w ew\nabla\phi\right)\right)\right], \\ w_t + \operatorname{div}(wu) \\ \qquad = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+w)\left(\nabla\psi(w) + z_w ew\nabla\phi\right) + w\left(\nabla\varphi(v) + z_v ev\nabla\phi\right)\right)\right], \\ -\epsilon\Delta\phi = z_v ev + z_w ew, \end{cases}$$

where the functions $\varphi(v), \psi(w)$ could be considered as being smooth.

Notation. In this paper, we use $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1 \leqslant p \leqslant \infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. The symbol ∇^ℓ with an integer $\ell \geqslant 0$ stands for the usual any spatial derivatives of order ℓ. When $\ell < 0$ or ℓ is not a positive integer, ∇^ℓ stands for Λ^ℓ defined by $\Lambda^\ell f := \mathscr{F}^{-1}(|\xi|^\ell \mathscr{F} f)$, where $\mathscr{F}$ is the usual Fourier transform operator and $\mathscr{F}^{-1}$ is its inverse.

Throughout this paper we let C denote some positive universal constants. We will use $a \lesssim b$ if $a \leqslant Cb$. We use C_0 to denote the constants depending on the initial data. For simplicity, we write $\|(A,B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f\,dx$.

The notation $\mathcal{C}^k(0,T;B)(k \geqslant 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, $\mathcal{L}^k_\infty(0,T;B)(k \geqslant 0)$ denotes the space of B-valued k-times boundedly differentiable functions on $[0,T]$, and $\mathcal{L}_2(0,T;B)$ denotes the space of B-valued L^2-functions on $(0,T)$.

Our main result about the asymptotic stability of the Cauchy problem (1.10) under small perturbations around a constant equilibrium state is stated in the following theorem.

Theorem 1.1. *Denote $U(t) = (\varrho, u, V, W)(t)$. Assume that $U_0 := (\varrho_0, u_0, V_0, W_0) \in H^k$ for an integer $k \geqslant 3$. If the initial H^3 norm $\|U_0\|_{H^3}$ is sufficiently small, then the Cauchy problem* (1.10) *admits a unique global solution $(U, \nabla\phi)(t)$ satisfying for all $t \geqslant 0$ and $3 \leqslant \ell \leqslant k$,*

$$\|U(t)\|_{H^\ell} + \left(\int_0^t \|\nabla\varrho(\tau)\|^2_{H^{\ell-1}} + \|\nabla(u,V,W)(\tau)\|^2_{H^\ell} + \|\Delta\phi(\tau)\|^2_{H^\ell}\,d\tau\right)^{1/2} \leqslant C\|U_0\|_{H^\ell}. \tag{1.11}$$

If we additionally assume that $\|\nabla^{-1}(V_0 - W_0)\|_{L^2}$ is sufficiently small and $U_0 \in \dot{B}^{-s}_{2,\infty}$ with $0 < s \leqslant 3/2$, then for all $t \geqslant 0$,

$$\left\|\nabla^\ell U(t)\right\|_{H^{k-\ell}} \leqslant C_0(1+t)^{-\frac{\ell+s}{2}} \quad \text{for} \quad 0 \leqslant \ell \leqslant k-1 \tag{1.12}$$

and

$$\left\|\nabla^\ell \nabla\phi(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{\ell+1+s}{2}} \quad \text{for} \quad 0 \leqslant \ell \leqslant k-2. \tag{1.13}$$

We give some remarks in the following.

Remark 1.2. Here we could remove the smallness of $\|\nabla^{-1}(V_0 - W_0)\|_{L^2}$ by assuming only $\nabla^{-1}(V_0 - W_0) \in L^2$. Then we could prove that the time-decay rates (1.12) hold still for $\ell = 0, 1$. However, it is hard to obtain the higher-order decay since we cannot prove the inequality (5.23) for $\ell > 1$. In addition, the restriction $\nabla^{-1}(V_0 - W_0) \in L^2$ may be natural since it could be looked as the restriction for the initial electric field $\nabla\phi_0$ by

$$\left\|\nabla^{-1}(V_0 - W_0)\right\|_{L^2} = \left\|\nabla^{-1}\Delta\phi_0\right\|_{L^2} = \|\nabla\phi_0\|_{L^2}.$$

Remark 1.3. If $s = 3/2$, then the time-decay rates (1.12) could be regarded as being optimal in the sense that it is consistent with the decay of the heat kernel.

Remark 1.4. The Besov spaces $\dot{B}^{-s}_{2,\infty}$ are given by Definition A.5. By Lemma A.6, we have $L^p \subset \dot{B}^{-s}_{2,\infty}$ for $1 \leqslant p < 2$. So, our time-decay rates (1.12)–(1.13) with $s = 3(1/p - 1/2)$ also follow if we replace $U_0 \in \dot{B}^{-s}_{2,\infty}$ $(0 < s \leqslant 3/2)$ by $U_0 \in L^p$ $(1 \leqslant p < 2)$.

Remark 1.5. This paper focuses on the whole space. In fact, the PNP-NS system in some bounded domain could also be derived by the EVA, and then the corresponding initial-boundary value problems could be considered by imposing some appropriate boundary conditions. This is the forthcoming work by using different methods.

This paper is structured as follows. In section 2, we derive the generalized PNP-NS system (1.4) by using the EVA. Then, we construct the local solution and establish the a priori estimates in sections 3–4, respectively. In section 5, we prove the global solution and obtain the time-decay rates of solutions. In Appendix A, we list out some technical ingredients used often in this paper, such as various interpolation inequalities and definition of Besov spaces.

2. Derivation of models. In this section, we will use an EVA, together with a prescribed energy dissipation law, to derive a generalized PNP-NS model. Such a model could simulate the dynamics of the fluid with the crowded charged particles (ions, etc.).

First, by the law of conservation of mass, we have

$$\rho_t + \operatorname{div}(\rho u) = 0. \tag{2.1}$$

And Gauss's law gives the Poisson equation

$$-\epsilon\Delta\phi = z_v e v + z_w e w. \tag{2.2}$$

Solving the Poisson equation (2.2), we obtain

$$\phi(x) = \frac{e}{\epsilon}\int G(x-y)(z_v v + z_w w)(y)\,dy, \tag{2.3}$$

where the kernel $G(\cdot) = \frac{1}{4\pi|\cdot|}$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^3$.

The energy dissipation law could be read as the following form:

$$\boxed{\frac{d}{dt}E^{total} = -\triangle.} \tag{2.4}$$

Here the total energy is given by

$$\begin{aligned} E^{total} &= \underbrace{\int k_B T(v\ln v + w\ln w) + \frac{\epsilon}{2}|\nabla\phi|^2\,dx}_{\text{microscopic}} + \underbrace{\int \frac{\rho}{2}|u|^2 + \omega(\rho)\,dx}_{\text{macroscopic}} \\ &= \int k_B T(v\ln v + w\ln w)\,dx \\ &\quad + \frac{e^2}{2\epsilon}\iint G(x-y)(z_v v + z_w w)(x)(z_v v + z_w w)(y)\,dydx, \end{aligned} \tag{2.5}$$

where we have used integration by parts and (2.3). And the dissipation is given by

$$\begin{aligned} \triangle = \int \Bigg(& k_B T\left(\frac{v}{D_v}|u_v - u|^2 + \frac{w}{D_w}|u_w - u|^2 + \frac{vw}{D_{v,w}}|u_v - u_w|^2\right) \\ & + \mu|\nabla u|^2 + \mu'|\operatorname{div} u|^2\Bigg)\,dx. \end{aligned} \tag{2.6}$$

In the following, we set $k_B = T = 1$ for brevity. Then, we begin to use the EVA to derive the equations of motion, as in [26, 33, 79]. In fact, the EVA is to find out the conservative force and the dissipative force by computing some appropriate variations for the action functionals set by the total energy and dissipation with the help of the least action principle (LAP) (or Hamilton's principle) [1, 2, 26, 29, 34] and the maximum dissipation principle (MDP) (or Onsager's principle) [26, 34, 62, 63, 64]. Then, the total force balance gives rise to the motion equations. So, the first step is to set the action functionals as

$$\mathcal{A}_1 = -\int_0^{t^*}\int v\ln v + w\ln w + \frac{e^2}{2\epsilon}\int G(x-y)(z_v v + z_w w)(x)(z_v v + z_w w)(y)\,dydxdt,$$
$$\mathcal{A}_2 = \int_0^{t^*}\int \frac{\rho}{2}|u|^2 - \omega(\rho)\,dxdt,$$

which correspond to the microscopic part and the macroscopic part of the total energy, respectively.

Second, by the LAP, taking the variation of $\mathcal{A}_1$ (for any smooth $\tilde{v}$ with compact support) with respect to v yields

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{A}_1(v+\varepsilon\tilde{v}) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(-\int_0^{t^*}\int (v+\varepsilon\tilde{v})\ln(v+\varepsilon\tilde{v})\,dxdt\right)\\
&\quad + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[-\frac{e^2}{\epsilon}\int_0^{t^*}\iint G(x-y)(z_v v + z_v\varepsilon\tilde{v} + z_w w)(x)(z_v v + z_w w)(y)dydxdt\right]\\
&= \int_0^{t^*}\int (-\ln v - 1 - z_v e\phi)\,\tilde{v}\,dxdt.
\end{aligned}$$

Since $\tilde{v}$ is arbitrary, we obtain

$$\frac{\delta\mathcal{A}_1}{\delta v} = -\ln v - 1 - z_v e\phi \Rightarrow F_{\text{conservative},v} = v\nabla\frac{\delta\mathcal{A}_1}{\delta v} = -\nabla v - z_v e v\nabla\phi.$$

By the MDP, taking the variation of $\frac{1}{2}\triangle$ (for any smooth $\tilde{u}$ with compact support) with respect to u_v yields

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{2}\triangle(u_v+\varepsilon\tilde{u})\\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{2}\int\frac{v}{D_v}\left|u_v+\varepsilon\tilde{u}-u\right|^2 + \frac{vw}{D_{v,w}}\left|u_v+\varepsilon\tilde{u}-u_w\right|^2\,dx\\
&= \int\left(\frac{v}{D_v}(u_v-u) + \frac{vw}{D_{v,w}}(u_v-u_w)\right)\cdot\tilde{u}\,dx.
\end{aligned}$$

Since $\tilde{u}$ is arbitrary, we obtain

$$\frac{v}{D_v}(u_v-u) + \frac{vw}{D_{v,w}}(u_v-u_w) = 0.$$

Accordingly,

$$F_{\text{dissipative},v} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u_v} = \frac{v}{D_v}(u_v-u) + \frac{vw}{D_{v,w}}(u_v-u_w).$$

The total force balance for the negative charge yields

$$v\nabla\frac{\delta\mathcal{A}_1}{\delta v} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u_v},$$

i.e.,

$$-\nabla v - z_v ev\nabla\phi = \frac{v}{D_v}(u_v - u) + \frac{vw}{D_{v,w}}(u_v - u_w). \tag{2.7}$$

Similarly, for the positive charge, we have the total force balance:

$$w\nabla\frac{\delta\mathcal{A}_1}{\delta w} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u_w},$$

i.e.,

$$-\nabla w - z_w ew\nabla\phi = \frac{w}{D_w}(u_w - u) + \frac{vw}{D_{v,w}}(u_w - u_v). \tag{2.8}$$

For simplicity of derivation, we set $D_v = D_w = D_{v,w} = 1$ from now on. Then, by (2.7) and (2.8), we obtain

$$vu_v = vu - \frac{1}{1+v+w}\left((1+v)\left(\nabla v + z_v ev\nabla\phi\right) + v\left(\nabla w + z_w ew\nabla\phi\right)\right) \tag{2.9}$$

and

$$wu_w = wu - \frac{1}{1+v+w}\left((1+w)\left(\nabla w + z_w ew\nabla\phi\right) + w\left(\nabla v + z_v ev\nabla\phi\right)\right). \tag{2.10}$$

In the meantime, for the microscopic negative and positive charge, we have

$$v_t + \operatorname{div}(vu_v) = 0, \tag{2.11}$$

$$w_t + \operatorname{div}(wu_w) = 0. \tag{2.12}$$

Plugging (2.9), (2.10) into (2.11), (2.12), respectively, we obtain

$$v_t + \operatorname{div}(vu) = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+v)\left(\nabla v + z_v ev\nabla\phi\right) + v\left(\nabla w + z_w ew\nabla\phi\right)\right)\right] \tag{2.13}$$

and

$$w_t + \operatorname{div}(wu) = \operatorname{div}\left[\frac{1}{1+v+w}\left((1+w)\left(\nabla w + z_w ew\nabla\phi\right) + w\left(\nabla v + z_v ev\nabla\phi\right)\right)\right]. \tag{2.14}$$

For the macroscopic action functional $\mathcal{A}_2$, by referring to [26], we have

$$F_{\text{macro-conservative}} = \frac{\delta\mathcal{A}_2}{\delta x} = -\left((\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho)\right),$$

where

$$p(\rho) := \omega'(\rho)\rho - \omega(\rho). \tag{2.15}$$

On the other hand, by the MDP again, taking the variation of $\frac{1}{2}\triangle$ (for any smooth $\tilde{u}$

with compact support) with respect to u yields

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\triangle(u+\varepsilon\tilde{u}) \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\int \Big(v\,|u_v - u - \varepsilon\tilde{u}|^2 + w\,|u_w - u - \varepsilon\tilde{u}|^2 \\
&\qquad\qquad + \mu|\nabla u + \varepsilon\nabla\tilde{u}|^2 + \mu'|\operatorname{div} u + \varepsilon \operatorname{div}\tilde{u}|^2\Big)\,dx, \\
&= \int v(u_v - u)\cdot(-\tilde{u}) + w(u_w - u)\cdot(-\tilde{u}) + \mu\nabla u : \nabla\tilde{u} + \mu' \operatorname{div} u \operatorname{div}\tilde{u}\,dx, \\
&= \int (-\mu\Delta u - \mu'\nabla \operatorname{div} u + v(u-u_v) + w(u-u_w))\cdot\tilde{u}\,dx.
\end{aligned}$$

Since $\tilde{u}$ is arbitrary, we obtain

$$-\mu\Delta u - \mu'\nabla \operatorname{div} u + v(u-u_v) + w(u-u_w) = 0.$$

Accordingly,

$$F_{\text{macro-dissipative}} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u} = -\mu\Delta u - \mu'\nabla \operatorname{div} u + v(u-u_v) + w(u-u_w).$$

The macroscopic force balance yields

$$\frac{\delta\mathcal{A}_2}{\delta x} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u},$$

i.e.,

$$(\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) = \mu\Delta u + \mu'\nabla \operatorname{div} u + v(u_v - u) + w(u_w - u).$$

Plugging (2.9) and (2.10) into the above equation, by the Poisson equation (2.2), we obtain

$$(\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) = \mu\Delta u + \mu'\nabla \operatorname{div} u - \nabla v - \nabla w + \epsilon\Delta\phi\nabla\phi. \tag{2.16}$$

Finally, we collect (2.1), (2.2), (2.13), (2.14), and (2.16) to obtain the compressible PNP-NS system

$$\begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0, \\
(\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) = \mu\Delta u + \mu'\nabla \operatorname{div} u - \nabla v - \nabla w + \epsilon\Delta\phi\nabla\phi, \\
v_t + \operatorname{div}(vu) = \operatorname{div}\left[\dfrac{1}{1+v+w}\left((1+v)\left(\nabla v + z_v ev\nabla\phi\right) + v\left(\nabla w + z_w ew\nabla\phi\right)\right)\right], \\
w_t + \operatorname{div}(wu) = \operatorname{div}\left[\dfrac{1}{1+v+w}\left((1+w)\left(\nabla w + z_w ew\nabla\phi\right) + w\left(\nabla v + z_v ev\nabla\phi\right)\right)\right], \\
-\epsilon\Delta\phi = z_v ev + z_w ew.
\end{cases} \tag{2.17}$$

Now, we can conclude the following proposition.

PROPOSITION 2.1. *The PNP-NS system* (2.17) *satisfies the following energy dissipation law:*

$$\frac{d}{dt}\int \frac{\rho}{2}|u|^2+\omega(\rho)+v\ln v+w\ln w+\frac{\epsilon}{2}|\nabla\phi|^2\,dx = -\int\left(v\,|u_v-u|^2+w\,|u_w-u|^2+vw\,|u_v-u_w|^2+\mu|\nabla u|^2+\mu'|\operatorname{div}u|^2\right)dx. \tag{2.18}$$

Conversely, we could use an EVA to derive the PNP-NS system (2.17).

Proof. We have derived the PNP-NS system (2.17) in the above. Multiplying the first four equations in (2.17) by $\omega'(\rho)$, u, $\ln v+1+z_ve\phi$, $\ln w+1+z_we\phi$, respectively, summing them up, and then integrating over $\mathbb{R}^3$, we obtain (2.18). ∎

We give some remarks in the following.

Remark 2.2. Solving ODE (2.15) directly, we obtain

$$\omega(\rho)=\begin{cases}\rho\ln\rho & \text{if}\quad p(\rho)=\rho,\\ \rho^\gamma & \text{if}\quad p(\rho)=(\gamma-1)\rho^\gamma,\ \gamma>1.\end{cases}$$

Remark 2.3. The variations $\frac{\delta E^{total}}{\delta v}=\ln v+1+z_ve\phi$ and $\frac{\delta E^{total}}{\delta w}=\ln w+1+z_we\phi$ are called the (electro)chemical potential [22] of negative and positive ions, respectively.

3. Local solution.

In this section, we will prove the local existence and uniqueness of solutions to the Cauchy problem (1.10).

For brevity, we denote

$$U=(\varrho,u,V,W),\ U_n=(\varrho_n,u_n,V_n,W_n),\ n=0,1,2\dots$$

and

$$h_\epsilon=J_\epsilon*h,\ h_{0,\epsilon}=J_\epsilon*h_0,$$

where J_ϵ is the Friedrichs' mollifier for some $\epsilon>0$. We rewrite (1.10) as

$$\begin{cases}\varrho_t+u\cdot\nabla\varrho+(\varrho+1)\operatorname{div}u=0,\\ u_t-\dfrac{1}{\varrho+1}\Delta u-\dfrac{1}{\varrho+1}\nabla\operatorname{div}u=g^1,\\ V_t-A_{11}(V,W)\Delta V-A_{12}(V,W)\Delta W=g^2,\\ W_t-A_{22}(V,W)\Delta W-A_{21}(V,W)\Delta V=g^3,\\ \Delta\phi=V-W,\\ U\,|_{t=0}=U_0,\end{cases} \tag{3.1}$$

where

$$\begin{aligned}g^1&:=-u\cdot\nabla u-\frac{p'(\varrho+1)}{\varrho+1}\nabla\varrho+\frac{1}{\varrho+1}\Delta\phi\nabla\phi-\frac{1}{\varrho+1}\nabla V-\frac{1}{\varrho+1}\nabla W,\\ g^2&:=-\operatorname{div}u-\operatorname{div}(Vu)+\nabla A_{11}(V,W)\cdot\nabla V+\nabla A_{12}(V,W)\cdot\nabla W\\ &\quad+\operatorname{div}\left(\frac{(V+1)(W-V-1)}{V+W+3}\nabla\phi\right),\end{aligned}$$

$$g^3 := -\operatorname{div} u - \operatorname{div}(Wu) + \nabla A_{22}(V,W)\cdot\nabla W + \nabla A_{21}(V,W)\cdot\nabla V + \operatorname{div}\left(\frac{(W+1)(W-V+1)}{V+W+3}\nabla\phi\right).$$

In the following, we will prove the local solvability of the Cauchy problem (3.1). First, we construct the suitable approximate system corresponding to (3.1). Then, such an approximate system could be solved by solving the corresponding linearized system of (3.1). At this time, we could obtain a sequence of approximate solutions. Finally, we could prove that the approximate solutions converge to the solution to the original system (3.1). Hence, we complete the proof of local solvability of the system (3.1).

Let $n = 2, 3, 4, \ldots$. We construct the approximate system of (3.1):

$$\begin{cases} \partial_t \varrho_n + u_{n-1}\cdot\nabla\varrho_n + (\varrho_{n-1}+1)\operatorname{div} u_n = 0, \\ \partial_t u_n - \dfrac{1}{\varrho_{n-1}+1}\Delta u_n - \dfrac{1}{\varrho_{n-1}+1}\nabla\operatorname{div} u_n = g^1_{n-1}, \\ \partial_t V_n - A_{11}(V_{n-1},W_{n-1})\Delta V_n - A_{12}(V_{n-1},W_{n-1})\Delta W_n = g^2_{n-1}, \\ \partial_t W_n - A_{22}(V_{n-1},W_{n-1})\Delta W_n - A_{21}(V_{n-1},W_{n-1})\Delta V_n = g^3_{n-1}, \\ U_n\,|_{t=0} = U_0,\ U_1 \equiv U_0. \end{cases} \tag{3.2}$$

Here the nonlinear functions are defined by

$$\begin{aligned} g^1_{n-1} :=& -u_{n-1}\cdot\nabla u_{n-1} - \frac{p'(\varrho_{n-1}+1)}{\varrho_{n-1}+1}\nabla\varrho_{n-1} + \frac{1}{\varrho_{n-1}+1}\Delta\phi_{n-1}\nabla\phi_{n-1} \\ &- \frac{1}{\varrho_{n-1}+1}\nabla V_{n-1} - \frac{1}{\varrho_{n-1}+1}\nabla W_{n-1}, \\ g^2_{n-1} :=& -\operatorname{div} u_{n-1} - \operatorname{div}(V_{n-1}u_{n-1}) + \nabla A_{11}(V_{n-1},W_{n-1})\cdot\nabla V_{n-1} \\ &+ \nabla A_{12}(V_{n-1},W_{n-1})\cdot\nabla W_{n-1} \\ &+ \operatorname{div}\left(\frac{(V_{n-1}+1)(W_{n-1}-V_{n-1}-1)}{V_{n-1}+W_{n-1}+3}\nabla\phi_{n-1}\right), \\ g^3_{n-1} :=& -\operatorname{div} u_{n-1} - \operatorname{div}(W_{n-1}u_{n-1}) + \nabla A_{22}(V_{n-1},W_{n-1})\cdot\nabla W_{n-1} \\ &+ \nabla A_{21}(V_{n-1},W_{n-1})\cdot\nabla V_{n-1} \\ &+ \operatorname{div}\left(\frac{(W_{n-1}+1)(W_{n-1}-V_{n-1}+1)}{V_{n-1}+W_{n-1}+3}\nabla\phi_{n-1}\right). \end{aligned}$$

To solve the above approximate system, we turn to consider the linearized system of (3.1) at $(\eta, \varpi, \tilde{v}, \tilde{w})$:

$$\begin{cases} L^0_{\eta,\varpi}(\varrho,u) := \varrho_t + \varpi\cdot\nabla\varrho + (\eta+1)\operatorname{div} u = f, \\ L^1_\eta(u) := u_t - \dfrac{1}{\eta+1}\Delta u - \dfrac{1}{\eta+1}\nabla\operatorname{div} u = g^1, \\ L^2_{\tilde{v},\tilde{w}}(V,W) := V_t - A_{11}(\tilde{v},\tilde{w})\Delta V - A_{12}(\tilde{v},\tilde{w})\Delta W = g^2, \\ L^3_{\tilde{v},\tilde{w}}(V,W) := W_t - A_{22}(\tilde{v},\tilde{w})\Delta W - A_{21}(\tilde{v},\tilde{w})\Delta V = g^3, \\ U\,|_{t=0} = U_0, \end{cases} \tag{3.3}$$

where η, $\varpi = (\varpi^1, \varpi^2, \varpi^3)^T$, $\tilde{v}$, $\tilde{w}$, f, $g^1 = (g^{11}, g^{12}, g^{13})^T$, g^2, and g^3 are regarded as the given functions. We introduce the definitions of some function spaces.

DEFINITION 3.1. *For $l = 2, 3$, we denote the function spaces*

$$\begin{aligned}\mathcal{E}(0,T;H^l) := \{&(\varrho,u,V,W)(x,t):\\ &\varrho(x,t)\in\mathcal{C}^0(0,T;H^l)\cap\mathcal{C}^1(0,T;H^{l-1}),\\ &(u,V,W)(x,t)\in\mathcal{C}^0(0,T;H^l)\cap\mathcal{C}^1(0,T;H^{l-2})\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}(0,T;H^l) := \{&(\varrho,u,V,W)(x,t):\\ &\varrho(x,t)\in\mathcal{L}^0_\infty(0,T;H^l)\cap\mathcal{L}^1_\infty(0,T;H^{l-1}),\\ &(u,V,W)(x,t)\in\mathcal{L}^0_\infty(0,T;H^l)\cap\mathcal{L}^1_\infty(0,T;H^{l-2})\}.\end{aligned}$$

In the whole proof of local solution, it is important to establish the following energy estimates of the linearized system (3.3). So, we start from this point.

3.1. Energy estimates for linearized system. Throughout this subsection, we assume for some $T > 0$,

$$E := \sup_{0\leqslant t\leqslant T}\|(\eta,\varpi,\tilde{v},\tilde{w})(t)\|_{H^3}. \tag{3.4}$$

We also consider the simple hyperbolic equation:

$$L_\varpi(\varrho) := \varrho_t + \varpi\cdot\nabla\varrho = f^0, \tag{3.5}$$

where $f^0 := f - (\eta+1)\operatorname{div} u$ is viewed as a known function.

First, we give some estimates for the commutators of the operators $L^0_{\eta,\varpi}$, L^1_η, $L^2_{\tilde{v},\tilde{w}}$, $L^3_{\tilde{v},\tilde{w}}$, L_ϖ, ∇^m, $m = 1, 2, \ldots$, and mollifiers J_ϵ, which will be used later.

LEMMA 3.2. *Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases}(\eta,\varpi,\tilde{v},\tilde{w})(t)\in\mathcal{L}^0_\infty(0,T;H^3),\ \eta(t)\geqslant\chi>-1,\\ U(t)\in\mathfrak{L}(0,T;H^l)\ \text{for } l=2,3.\end{cases}$$

Then we have the following estimates (1)–(3) *for all $t\in[0,T]$.*

(1) *For any m, $0\leqslant m\leqslant l-1$,*

$$\begin{cases}\|\nabla^m L_\varpi(\varrho)-L_\varpi(\nabla^m\varrho)\|_{L^2}\leqslant CE\|\nabla\varrho\|_{H^m},\\ \left\|\nabla^m L^0_{\eta,\varpi}(\varrho,u)-L^0_{\eta,\varpi}(\nabla^m\varrho,\nabla^m u)\right\|_{L^2}\leqslant CE\|\nabla(\varrho,u)\|_{H^m},\\ \left\|\nabla^m L^1_\eta(u)-L^1_\eta(\nabla^m u)\right\|_{L^2}\leqslant CE\|\nabla u\|_{H^m},\\ \left\|\nabla^m L^2_{\tilde{v},\tilde{w}}(V,W)-L^2_{\tilde{v},\tilde{w}}(\nabla^m V,\nabla^m W)\right\|_{L^2}\leqslant CE\|\nabla(V,W)\|_{H^m},\\ \left\|\nabla^m L^3_{\tilde{v},\tilde{w}}(V,W)-L^3_{\tilde{v},\tilde{w}}(\nabla^m V,\nabla^m W)\right\|_{L^2}\leqslant CE\|\nabla(V,W)\|_{H^m}.\end{cases} \tag{3.6}$$

(2) *For any m, $0\leqslant m\leqslant l-1$, as $\epsilon\to 0$,*

$$\begin{cases}\|J_\epsilon * L_\varpi(\varrho)-L_\varpi(\varrho_\epsilon)\|_{H^{m+1}}\to 0,\\ \left\|J_\epsilon * L^0_{\eta,\varpi}(\varrho,u)-L^0_{\eta,\varpi}(\varrho_\epsilon,u_\epsilon)\right\|_{H^{m+1}}\to 0,\\ \left\|J_\epsilon * L^1_\eta(u)-L^1_\eta(u_\epsilon)\right\|_{H^m}\to 0,\\ \left\|J_\epsilon * L^2_{\tilde{v},\tilde{w}}(V,W)-L^2_{\tilde{v},\tilde{w}}(V_\epsilon,W_\epsilon)\right\|_{H^m}\to 0,\\ \left\|J_\epsilon * L^3_{\tilde{v},\tilde{w}}(V,W)-L^3_{\tilde{v},\tilde{w}}(V_\epsilon,W_\epsilon)\right\|_{H^m}\to 0.\end{cases} \tag{3.7}$$

(3) *Further we assume that* $U(t) \in \mathfrak{L}(0,T;H^3)$ *and* $(\eta', \varpi', \tilde{v}', \tilde{w}')(t) \in \mathcal{L}^0_\infty(0,T;H^3)$ *with* $\eta'(t) \geqslant \chi > -1$. *Then we have*

(3.8)
$$\begin{cases} \left\|L^0_{\eta,\varpi}(\varrho,u) - L^0_{\eta',\varpi'}(\varrho,u)\right\|_{H^2} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|(\varrho,u)\|_{H^3} \|(\eta-\eta', \varpi-\varpi')\|_{H^2}, \\ \left\|L^1_\eta(u) - L^1_{\eta'}(u)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|u\|_{H^3} \|\eta-\eta'\|_{H^2}, \\ \left\|L^2_{\tilde{v},\tilde{w}}(V,W) - L^2_{\tilde{v}',\tilde{w}'}(V,W)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|(V,W)\|_{H^3} \|(\tilde{v}-\tilde{v}', \tilde{w}-\tilde{w}')\|_{H^2}, \\ \left\|L^3_{\tilde{v},\tilde{w}}(V,W) - L^3_{\tilde{v}',\tilde{w}'}(V,W)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|(V,W)\|_{H^3} \|(\tilde{v}-\tilde{v}', \tilde{w}-\tilde{w}')\|_{H^2}. \end{cases}$$

Proof. We refer to [58, Lemma 3.1, p. 74]. ∎

Next, we will show some energy estimates for the solutions to the linearized system.

LEMMA 3.3. *Let* $l = 1$, 2, *or* 3. *Assume that for some* $T > 0$,

$$\varpi(t) \in \mathcal{L}^0_\infty(0,T;H^3), \ f^0(t) \in \mathcal{L}^0_\infty(0,T;H^l).$$

If $\varrho(t) \in \mathcal{L}^0_\infty(0,T;H^l) \cap \mathcal{L}^1_\infty(0,T;H^{l-1})$ *solves* (3.5), *then for any* $t \in [0,T]$,

$$\|\varrho(t)\|_{H^l} \leqslant e^{CEt}\left(\|\varrho_0\|_{H^l} + \int_0^t e^{-CE\tau}\left\|f^0(\tau)\right\|_{H^l} d\tau\right). \tag{3.9}$$

Proof. We can refer to [58, Lemma 3.2, p. 77]. ∎

LEMMA 3.4. *Let* $l = 2$ *or* 3. *Assume that for some* $T > 0$,

$$(\tilde{v}, \tilde{w})(t) \in \mathcal{L}^0_\infty(0,T;H^3), \ (g^2, g^3)(t) \in \mathcal{L}^0_\infty(0,T;H^{l-1}).$$

If $(V,W)(t) \in \mathcal{L}^0_\infty(0,T;H^l) \cap \mathcal{L}^1_\infty(0,T;H^{l-2})$ *solves* $(3.3)_3$–$(3.3)_4$, *then* $(V,W)(t) \in \mathcal{L}_2(0,T;H^{l+1})$ *and there exists a constant* $\nu > 0$ *such that for any* $t \in [0,T]$,

$$\begin{aligned} &\|(V,W)(t)\|^2_{L^2} + \nu\int_0^t \|\nabla(V,W)(\tau)\|^2_{L^2} d\tau \\ &\quad\leqslant e^{C(1+E^2)t}\left(\|(V_0,W_0)\|^2_{L^2} + C\int_0^t \left\|(g^2,g^3)(\tau)\right\|^2_{L^2} d\tau\right) \end{aligned} \tag{3.10}$$

and for $1 \leqslant k \leqslant l$,

$$\begin{aligned} &\left\|\nabla^k(V,W)(t)\right\|^2_{L^2} + \nu\int_0^t \left\|\nabla^{k+1}(V,W)(\tau)\right\|^2_{L^2} d\tau \\ &\quad\leqslant e^{CE^2t}\left(\|\nabla(V_0,W_0)\|^2_{H^{k-1}} + C\int_0^t \left\|(g^2,g^3)(\tau)\right\|^2_{H^{k-1}} d\tau\right). \end{aligned} \tag{3.11}$$

Proof. Let

$$J := V + W, \quad K := V - W. \tag{3.12}$$

By $(3.3)_3$–$(3.3)_4$, we obtain

$$\begin{cases} J_t - \Delta J = g^2 + g^3, \\ K_t - \dfrac{1}{\tilde{v}+\tilde{w}+3}\Delta K - \dfrac{\tilde{v}-\tilde{w}}{\tilde{v}+\tilde{w}+3}\Delta J = g^2 - g^3. \end{cases} \tag{3.13}$$

Multiplying $(3.13)_1$ by J and integrating over $\mathbb{R}^3$, by the integration by parts and Cauchy's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|J\|_{L^2}^2 + \|\nabla J\|_{L^2}^2 = \int \left(g^2+g^3\right) J \leqslant C\|J\|_{L^2}^2 + C\left\|(g^2,g^3)\right\|_{L^2}^2. \tag{3.14}$$

Multiplying $(3.13)_2$ by K and integrating over $\mathbb{R}^3$, by the integration by parts and Cauchy's inequality, we obtain for any $\varepsilon > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|K\|_{L^2}^2 + \frac{1}{\tilde{v}+\tilde{w}+3}\|\nabla K\|_{L^2}^2 \\ &\quad = -\int \nabla\left(\frac{1}{\tilde{v}+\tilde{w}+3}\right)\nabla K K + \int \frac{\tilde{v}-\tilde{w}}{\tilde{v}+\tilde{w}+3}\Delta J K + \int \left(g^2-g^3\right) K \\ &\quad \leqslant \varepsilon\|\nabla K\|_{L^2}^2 + C_\varepsilon\left(1+E^2\right)\|\nabla J\|_{L^2}^2 + C_\varepsilon\left(1+E^2\right)\|K\|_{L^2}^2 + C\left\|(g^2,g^3)\right\|_{L^2}^2. \end{aligned} \tag{3.15}$$

Taking $\varepsilon > 0$ properly small, we obtain for some $\nu' > 0$,

$$\frac{d}{dt}\|K\|_{L^2}^2 + \nu'\|\nabla K\|_{L^2}^2 \leqslant C\left(1+E^2\right)\|\nabla J\|_{L^2}^2 + C\left(1+E^2\right)\|K\|_{L^2}^2 + C\left\|(g^2,g^3)\right\|_{L^2}^2. \tag{3.16}$$

Multiplying (3.16) by some fixed small constant and then adding it to (3.14) so that the first term on the right-hand side (RHS) of (3.16) could be absorbed, we can obtain for some $\tilde{\nu} > 0$,

$$\frac{d}{dt}\|(J,K)\|_{L^2}^2 + \tilde{\nu}\|\nabla(J,K)\|_{L^2}^2 \leqslant C(1+E^2)\|(J,K)\|_{L^2}^2 + C\left\|(g^2,g^3)\right\|_{L^2}^2.$$

By Gronwall's inequality, we obtain for some $\tilde{\nu} > 0$,

$$\begin{aligned} &\|(J,K)(t)\|_{L^2}^2 + \tilde{\nu}\int_0^t \|\nabla(J,K)(\tau)\|_{L^2}^2\, d\tau \\ &\quad \leqslant e^{C(1+E^2)t}\left(\|(J_0,K_0)\|_{L^2}^2 + C\int_0^t \left\|(g^2,g^3)(\tau)\right\|_{L^2}^2\, d\tau\right). \end{aligned} \tag{3.17}$$

Thus, we deduce the estimate (3.10) from (3.12) and (3.17). Define

$$L^4_{\tilde{v},\tilde{w}}(J,K) := K_t - \frac{1}{\tilde{v}+\tilde{w}+3}\Delta K - \frac{\tilde{v}-\tilde{w}}{\tilde{v}+\tilde{w}+3}\Delta J.$$

Next, applying $J_\epsilon *$ to (3.13), we have

$$\begin{cases} \partial_t J_\epsilon - \Delta J_\epsilon = g^2_\epsilon + g^3_\epsilon, \\ L^4_{\tilde{v},\tilde{w}}(J_\epsilon, K_\epsilon) = g^2_\epsilon - g^3_\epsilon + N_1^\epsilon, \end{cases} \tag{3.18}$$

where

$$N_1^\epsilon := L^4_{\tilde{v},\tilde{w}}(J_\epsilon,K_\epsilon) - J_\epsilon * L^4_{\tilde{v},\tilde{w}}(J,K).$$

By Lemma 3.2, we easily obtain for $0 \leqslant m \leqslant l-1$,

$$\|N_1^\epsilon\|_{H^m} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.19}$$

Multiplying $(3.18)_1$ by $-\Delta J_\epsilon$ and integrating over $\mathbb{R}^3$, we easily obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla J_\epsilon\|_{L^2}^2 + \|\Delta J_\epsilon\|_{L^2}^2 \leqslant C\left\|(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2. \tag{3.20}$$

Multiplying $(3.18)_2$ by $-\Delta K_\epsilon$ and integrating over $\mathbb{R}^3$, by Cauchy's inequality, we obtain for any $\varepsilon > 0$,

$$\begin{aligned} -\int L^4_{\tilde{v},\tilde{w}}(J_\epsilon, K_\epsilon)\Delta K_\epsilon &= -\int \left(g_\epsilon^2 - g_\epsilon^3 + N_1^\epsilon\right)\Delta K_\epsilon \\ &\leqslant \varepsilon\|\Delta K_\epsilon\|_{L^2}^2 + C_\varepsilon\left(\left\|(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2 + \|N_1^\epsilon\|_{L^2}^2\right). \end{aligned} \tag{3.21}$$

For the left-hand side of the above inequality, we have for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned} &-\int L^4_{\tilde{v},\tilde{w}}(J_\epsilon, K_\epsilon)\Delta K_\epsilon \\ &= -\int \left(\partial_t K_\epsilon - \frac{1}{\tilde{v}+\tilde{w}+3}\Delta K_\epsilon - \frac{\tilde{v}-\tilde{w}}{\tilde{v}+\tilde{w}+3}\Delta J_\epsilon\right)\Delta K_\epsilon \\ &= \frac{1}{2}\frac{d}{dt}\|\nabla K_\epsilon\|_{L^2}^2 + \int \frac{1}{\tilde{v}+\tilde{w}+3}|\Delta K_\epsilon|^2 + \int \frac{\tilde{v}-\tilde{w}}{\tilde{v}+\tilde{w}+3}\Delta J_\epsilon \Delta K_\epsilon \\ &\geqslant \frac{1}{2}\frac{d}{dt}\|\nabla K_\epsilon\|_{L^2}^2 + \int \frac{1}{\tilde{v}+\tilde{w}+3}|\Delta K_\epsilon|^2 - \tilde{\varepsilon}\|\Delta K_\epsilon\|_{L^2}^2 - C_{\tilde{\varepsilon}}E^2\|\Delta J_\epsilon\|_{L^2}^2. \end{aligned} \tag{3.22}$$

Plugging (3.22) into (3.21) and taking ε, $\tilde{\varepsilon}$ properly small, we obtain for some $\nu' > 0$,

$$\frac{d}{dt}\|\nabla K_\epsilon\|_{L^2}^2 + \nu'\left\|\nabla^2 K_\epsilon\right\|_{L^2}^2 \leqslant CE^2\|\Delta J_\epsilon\|_{L^2}^2 + C\left(\left\|(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2 + \|N_1^\epsilon\|_{L^2}^2\right). \tag{3.23}$$

Multiplying (3.23) by some fixed small constant and then adding it to (3.20) so that the first term on the RHS of (3.23) could be absorbed, we can obtain for some $\tilde{\nu} > 0$,

$$\frac{d}{dt}\|\nabla(J_\epsilon, K_\epsilon)\|_{L^2}^2 + \tilde{\nu}\left\|\nabla^2(J_\epsilon, K_\epsilon)\right\|_{L^2}^2 \leqslant C\left(\left\|(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2 + \|N_1^\epsilon\|_{L^2}^2\right). \tag{3.24}$$

Integrating (3.24) in time, we obtain

$$\begin{aligned} &\|\nabla(J_\epsilon, K_\epsilon)(t)\|_{L^2}^2 + \tilde{\nu}\int_0^t \left\|\nabla^2(J_\epsilon, K_\epsilon)(\tau)\right\|_{L^2}^2 d\tau \\ &\quad \leqslant \|\nabla(J_{0,\epsilon}, K_{0,\epsilon})\|_{L^2}^2 + C\int_0^t \left(\left\|(g_\epsilon^2, g_\epsilon^3)(\tau)\right\|_{L^2}^2 + \|N_1^\epsilon(\tau)\|_{L^2}^2\right) d\tau. \end{aligned} \tag{3.25}$$

As $\epsilon \to 0$, by (3.19), we obtain

$$\begin{aligned} &\|\nabla(J, K)(t)\|_{L^2}^2 + \tilde{\nu}\int_0^t \left\|\nabla^2(J, K)(\tau)\right\|_{L^2}^2 d\tau \\ &\quad \leqslant \|\nabla(J_0, K_0)\|_{L^2}^2 + C\int_0^t \left\|(g^2, g^3)(\tau)\right\|_{L^2}^2 d\tau \\ &\quad \leqslant e^{CE^2 t}\left(\|\nabla(J_0, K_0)\|_{L^2}^2 + C\int_0^t \left\|(g^2, g^3)(\tau)\right\|_{L^2}^2 d\tau\right). \end{aligned} \tag{3.26}$$

Thus, we can deduce (3.11) with $k = 1$ from (3.12) and (3.26). Applying ∇ to equation (3.18), we obtain

$$\begin{cases} \partial_t \nabla J_\epsilon - \nabla \Delta J_\epsilon = \nabla g_\epsilon^2 + \nabla g_\epsilon^3, \\ L^4_{\tilde{v},\tilde{w}}(\nabla(J_\epsilon, K_\epsilon)) = \nabla g_\epsilon^2 - \nabla g_\epsilon^3 + N_2^\epsilon, \end{cases} \tag{3.27}$$

where

$$N_2^\epsilon := L^4_{\tilde{v},\tilde{w}}(\nabla(J_\epsilon, K_\epsilon)) - \nabla L^4_{\tilde{v},\tilde{w}}(J_\epsilon, K_\epsilon).$$

By Lemma 3.2, we easily obtain,

$$\|N_2^\epsilon\|_{L^2} \leqslant CE \, \|\nabla(J_\epsilon, K_\epsilon)\|_{H^1} . \tag{3.28}$$

Here we can apply the inequality (3.24) to $\nabla(J_\epsilon, K_\epsilon)$ in (3.27) to prove (3.11) for $k = 2$. In fact, for $k = 2$, by (3.28), we deduce from (3.24) and (3.27) that for some $\tilde{\nu} > 0$,

$$\begin{aligned} &\frac{d}{dt} \left\|\nabla^2(J_\epsilon, K_\epsilon)\right\|_{L^2}^2 + \tilde{\nu} \left\|\nabla^3(J_\epsilon, K_\epsilon)\right\|_{L^2}^2 \\ &\quad \leqslant C \left(\left\|\nabla(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2 + \|\nabla N_1^\epsilon\|_{L^2}^2 + \|N_2^\epsilon\|_{L^2}^2 \right) \\ &\quad \leqslant C \left(\left\|\nabla(g_\epsilon^2, g_\epsilon^3)\right\|_{L^2}^2 + \|\nabla N_1^\epsilon\|_{L^2}^2 \right) + CE^2 \, \|\nabla(J_\epsilon, K_\epsilon)\|_{H^1}^2 . \end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned} &\left\|\nabla^2(J_\epsilon, K_\epsilon)(t)\right\|_{L^2}^2 + \tilde{\nu} \int_0^t \left\|\nabla^3(J_\epsilon, K_\epsilon)(\tau)\right\|_{L^2}^2 \, d\tau \\ &\quad \leqslant e^{CE^2 t} \Bigg(\left\|\nabla^2(J_{0,\epsilon}, K_{0,\epsilon})\right\|_{L^2}^2 + CE^2 \int_0^t \|\nabla(J_\epsilon, K_\epsilon)(\tau)\|_{L^2}^2 \, d\tau \\ &\qquad + C \int_0^t \left(\left\|\nabla(g_\epsilon^2, g_\epsilon^3)(\tau)\right\|_{L^2}^2 + \|\nabla N_1^\epsilon(\tau)\|_{L^2}^2 \right) d\tau \Bigg). \end{aligned}$$

Thus, by (3.19) and (3.26), as $\epsilon \to 0$, we obtain

$$\begin{aligned} &\left\|\nabla^2(J, K)(t)\right\|_{L^2}^2 + \tilde{\nu} \int_0^t \left\|\nabla^3(J, K)(\tau)\right\|_{L^2}^2 \, d\tau \\ &\quad \leqslant CE^2 e^{CE^2 t} \int_0^t \|\nabla(J, K)(\tau)\|_{L^2}^2 \, d\tau \\ &\qquad + e^{CE^2 t} \left(\left\|\nabla^2(J_0, K_0)\right\|_{L^2}^2 + C \int_0^t \left\|\nabla(g^2, g^3)(\tau)\right\|_{L^2}^2 \, d\tau \right) \\ &\quad \leqslant CE^2 t e^{CE^2 t} \sup_{0 \leqslant \tau \leqslant t} \|\nabla(J, K)(\tau)\|_{L^2}^2 \\ &\qquad + e^{CE^2 t} \left(\left\|\nabla^2(J_0, K_0)\right\|_{L^2}^2 + C \int_0^t \left\|\nabla(g^2, g^3)(\tau)\right\|_{L^2}^2 \, d\tau \right) \\ &\quad \leqslant e^{CE^2 t} \left(\|\nabla(J_0, K_0)\|_{H^1}^2 + C \int_0^t \left\|(g^2, g^3)(\tau)\right\|_{H^1}^2 \, d\tau \right). \end{aligned} \tag{3.29}$$

Thus, we deduce (3.11) with $k = 2$ from (3.12) and (3.29). Similar to the case for $k = 2$, we can prove (3.11) with $k = 3$. Hence, the proof of Lemma 3.4 is completed. ∎

LEMMA 3.5. *Let $l = 2$ or 3. Assume that for some $T > 0$ and some constant χ,*

$$\eta(t) \in \mathcal{L}^0_\infty(0,T;H^3),\ \eta(t) \geqslant \chi > -1,\ g^1(t) \in \mathcal{L}^0_\infty(0,T;H^{l-1}).$$

If $u(t) \in \mathcal{L}^0_\infty(0,T;H^l) \cap \mathcal{L}^1_\infty(0,T;H^{l-2})$ solves $(3.3)_2$, then $u(t) \in \mathcal{L}_2(0,T;H^{l+1})$ and there exists a constant $\nu > 0$ such that for any $t \in [0,T]$,

$$\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u(\tau)\|^2_{L^2}\, d\tau \leqslant e^{C(1+E^2)t}\left(\|u_0\|^2_{L^2} + C\int_0^t \left\|g^1(\tau)\right\|^2_{L^2} d\tau\right) \tag{3.30}$$

and for $1 \leqslant k \leqslant l$,

$$\begin{aligned} \left\|\nabla^k u(t)\right\|^2_{L^2} &+ \nu \int_0^t \left\|\nabla^{k+1} u(\tau)\right\|^2_{L^2} d\tau \\ &\leqslant e^{CE^2 t}\left(\|\nabla u_0\|^2_{H^{k-1}} + C\int_0^t \left\|g^1(\tau)\right\|^2_{H^{k-1}} d\tau\right). \end{aligned} \tag{3.31}$$

Proof. The proof is similar to Lemma 3.4. ∎

LEMMA 3.6. *Let $l = 2$ or 3. Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases} (\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{L}^0_\infty(0,T;H^3), \\ \eta(t) \geqslant \chi > -1, \\ f(t) \in \mathcal{L}^0_\infty(0,T;H^l), \\ (g^1, g^2, g^3)(t) \in \mathcal{L}^0_\infty(0,T;H^{l-1}). \end{cases}$$

If $U(t) \in \mathfrak{L}(0,T;H^l)$ solves the system (3.3), then $(u, V, W)(t) \in \mathcal{L}_2(0,T;H^{l+1})$ and there exists a constant $\nu > 0$ such that for any $t \in [0,T]$,

$$\begin{aligned} &\|U(t)\|_{H^l},\ \left(\nu\int_0^t \|\nabla(u,V,W)(\tau)\|^2_{H^l}\, d\tau\right)^{1/2} \\ &\quad\leqslant e^{C(1+E)^2 t}\left(\|U_0\|_{H^l} + \int_0^t \|f(\tau)\|_{H^l}\, d\tau + \left(C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|^2_{H^{l-1}} d\tau\right)^{1/2}\right). \end{aligned}$$

Proof. First, we have

$$\begin{aligned} \int_0^t \left\|f^0(\tau)\right\|_{H^l} d\tau &= \int_0^t \|f(\tau)\|_{H^l} + \|(\eta+1)\operatorname{div} u\|_{H^l}\, d\tau \\ &\leqslant \int_0^t \|f(\tau)\|_{H^l} + C(1+E)\|\nabla u\|_{H^l}\, d\tau \\ &\leqslant \int_0^t \|f(\tau)\|_{H^l}\, d\tau + C(1+E)t^{1/2}\left(\int_0^t \|\nabla u\|^2_{H^l}\, d\tau\right)^{1/2}. \end{aligned}$$

By (3.9)–(3.11) and (3.30)–(3.31), we obtain

$$\begin{aligned}
\|U(t)\|_{H^l} &\leqslant e^{C(1+E)^2 t}\Bigg(\|\varrho_0\|_{H^l} + \int_0^t \left\|f^0(\tau)\right\|_{H^l}\, d\tau \\
&\qquad + \|(u_0,V_0,W_0)\|_{H^l} + \left(C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right)^{1/2}\Bigg) \\
&\leqslant e^{C(1+E)^2 t}\Bigg(\|\varrho_0\|_{H^l} + \int_0^t \|f(\tau)\|_{H^l}\, d\tau \\
&\qquad + C(1+E)t^{1/2}\left(\int_0^t \|\nabla u\|_{H^l}^2\, d\tau\right)^{1/2} \\
&\qquad + \|(u_0,V_0,W_0)\|_{H^l} + \left(C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right)^{1/2}\Bigg) \\
&\leqslant e^{C(1+E)^2 t}\Bigg(\|\varrho_0\|_{H^l} + \int_0^t \|f(\tau)\|_{H^l}\, d\tau \\
&\qquad + C(1+E)t^{1/2} e^{C(1+E^2)t}\Bigg(\|(u_0,V_0,W_0)\|_{H^l}^2 \\
&\qquad\qquad + \int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\Bigg)^{1/2} \\
&\qquad + \|(u_0,V_0,W_0)\|_{H^l} + \left(C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right)^{1/2}\Bigg) \\
&\leqslant e^{C(1+E)^2 t}\Bigg(\|U_0\|_{H^l} + \int_0^t \|f(\tau)\|_{H^l}\, d\tau \\
&\qquad + \left(C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right)^{1/2}\Bigg).
\end{aligned}$$

Here, we have used that

$$C(1+E)t^{1/2}e^{C(1+E^2)t} \leqslant C(1+E)^2 t + e^{C(1+E^2)t} \leqslant e^{C(1+E)^2 t}.$$

By (3.10)–(3.11) and (3.30)–(3.31), we also obtain

$$\nu\int_0^t \|\nabla(u,V,W)(\tau)\|_{H^l}^2\, d\tau \leqslant e^{C(1+E)^2 t}\left(\|U_0\|_{H^l}^2 + C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right).$$

The proof of Lemma 3.6 is completed. ∎

3.2. Solving the linearized system. Assume that for some $T > 0$ and some constant χ,

$$\begin{cases}
(\eta,\varpi,\tilde v,\tilde w)(t) \in \mathcal{C}^0(0,T;H^3), \\
\eta(t) \geqslant \chi > -1, \\
f(t) \in \mathcal{C}^0(0,T;H^3), \\
(g^1,g^2,g^3)(t) \in \mathcal{C}^0(0,T;H^2).
\end{cases}$$

In what follows, we will solve the Cauchy problem (3.3):

$$
\begin{cases}
L^0(\varrho,u) := \varrho_t + \varpi\cdot\nabla\varrho + (\eta+1)\operatorname{div} u = f,\\
L^1(u) := u_t - \dfrac{1}{\eta+1}\Delta u - \dfrac{1}{\eta+1}\nabla\operatorname{div} u = g^1,\\
L^2(V,W) := V_t - A_{11}(\tilde v,\tilde w)\Delta V - A_{12}(\tilde v,\tilde w)\Delta W = g^2,\\
L^3(V,W) := W_t - A_{22}(\tilde v,\tilde w)\Delta W - A_{21}(\tilde v,\tilde w)\Delta V = g^3,\\
U\mid_{t=0} = U_0.
\end{cases}
\tag{3.32}
$$

First, we consider the Cauchy problem

$$
\begin{cases}
L(\varrho) := \varrho_t + \varpi\cdot\nabla\varrho = f^0,\\
\varrho\mid_{t=0} = \varrho_0,
\end{cases}
\tag{3.33}
$$

where ϖ and $f^0 = f - (\eta+1)\operatorname{div} u$ are regarded as known.

LEMMA 3.7. *Let $l = 1$ or 2. Assume that for some $T > 0$,*

$$
\begin{cases}
\varpi(t) \in \mathcal{C}^0(0,T;H^3),\\
f^0(t) \in \mathcal{C}^0(0,T;H^l).
\end{cases}
$$

If $\varrho_0 \in H^l$, then the Cauchy problem (3.33) has a unique solution

$$
\varrho(t) \in \mathcal{C}^0(0,T;H^l) \cap \mathcal{C}^1(0,T;H^{l-1})
$$

such that for any $t \in [0,T]$,

$$
\|\varrho(t)\|_{H^l} \leqslant e^{CEt}\left(\|\varrho_0\|_{H^l} + \int_0^t e^{-CE\tau}\left\|f^0(\tau)\right\|_{H^l}\,d\tau\right).
\tag{3.34}
$$

Proof. We can refer to [58, Proposition 4.1, p. 83]. ∎

Next, we solve the Cauchy problem of $(3.32)_2$–$(3.32)_4$.

LEMMA 3.8. *Let $l = 2$ or 3. Assume that for some $T > 0$ and some constant χ,*

$$
\begin{cases}
(\eta,\tilde v,\tilde w)(t) \in \mathcal{C}^0(0,T;H^3),\\
\eta(t) \geqslant \chi > -1,\\
(g^1,g^2,g^3)(t) \in \mathcal{C}^0(0,T;H^{l-1}).
\end{cases}
$$

If $(u_0,V_0,W_0) \in H^l$, then the Cauchy problem of $(3.32)_2$–$(3.32)_4$ has a unique solution

$$
(u,V,W)(t) \in \mathcal{C}^0(0,T;H^l) \cap \mathcal{C}^1(0,T;H^{l-2})
$$

such that for any $t \in [0,T]$,

$$
\begin{aligned}
&\|(u,V,W)(t)\|_{H^l}^2 + \nu\int_0^t \|\nabla(u,V,W)(\tau)\|_{H^l}^2\,d\tau\\
&\quad\leqslant e^{C(1+E^2)t}\left(\|(u_0,V_0,W_0)\|_{H^l}^2 + C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right),
\end{aligned}
\tag{3.35}
$$

where $\nu > 0$ is some constant.

Proof. For the existence and uniqueness of the solution, we could refer to [13, Theorem 2.5.1, p. 108]. Then the energy estimates (3.35) follow from Lemmas 3.4–3.5. ∎

Now, we can use Lemmas 3.7–3.8 to obtain the solution of the Cauchy problem (3.32).

PROPOSITION 3.9. *Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases} (\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{C}^0(0, T; H^3), \\ \eta(t) \geqslant \chi > -1, \\ f(t) \in \mathcal{C}^0(0, T; H^2), \\ (g^1, g^2, g^3)(t) \in \mathcal{C}^0(0, T; H^1). \end{cases}$$

If $U_0 \in H^2$, then the Cauchy problem (3.32) *has a unique solution*

$$U(t) \in \mathcal{E}(0, T; H^2) \tag{3.36}$$

such that

$$(u, V, W)(t) \in \mathcal{L}_2(0, T; H^3) \tag{3.37}$$

and for any $t \in [0, T]$,

$$\begin{aligned} &\|U(t)\|_{H^2}, \ \left(\nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{H^2}^2 \, d\tau\right)^{1/2} \\ &\leqslant e^{C(1+E)^2 t} \left(\|U_0\|_{H^2} + \int_0^t \|f(\tau)\|_{H^2} \, d\tau + \left(C \int_0^t \left\|(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 \, d\tau\right)^{1/2}\right), \end{aligned} \tag{3.38}$$

where $\nu > 0$ is some constant.

Proof. For any $\epsilon > 0$, let $(\varrho_\epsilon, u_\epsilon, V_\epsilon, W_\epsilon)(t)$ be the solution of

$$\begin{cases} L^0(\varrho_\epsilon, u_\epsilon) = f, \\ L^1(u_\epsilon) = g_\epsilon^1, \\ L^2(V_\epsilon, W_\epsilon) = g_\epsilon^2, \\ L^3(V_\epsilon, W_\epsilon) = g_\epsilon^3 \end{cases} \tag{3.39}$$

with the initial data

$$\varrho_\epsilon \mid_{t=0} = \varrho_0, \ (u_\epsilon, V_\epsilon, W_\epsilon) \mid_{t=0} = (u_{0,\epsilon}, V_{0,\epsilon}, W_{0,\epsilon}).$$

Since $(g_\epsilon^1, g_\epsilon^2, g_\epsilon^3)(t) \in \mathcal{C}^0(0, T; H^2)$ and $(u_{0,\epsilon}, V_{0,\epsilon}, W_{0,\epsilon}) \in H^3$, Lemma 3.8 implies that

$$(u_\epsilon, V_\epsilon, W_\epsilon)(t) \in \mathcal{C}^0(0, T; H^3) \cap \mathcal{C}^1(0, T; H^1)$$

and

$$-(\eta + 1) \operatorname{div} u_\epsilon(t) \in \mathcal{C}^0(0, T; H^2).$$

Thus, Lemma 3.7 gives

$$\varrho_\epsilon(t) \in \mathcal{C}^0(0, T; H^2) \cap \mathcal{C}^1(0, T; H^1).$$

Further, by Lemma 3.6, for the difference of solutions for any $\epsilon, \epsilon' > 0$, we have the estimate for any $t \in [0, T]$,

$$
\begin{aligned}
&\|(\varrho_\epsilon - \varrho_{\epsilon'}, u_\epsilon - u_{\epsilon'}, V_\epsilon - V_{\epsilon'}, W_\epsilon - W_{\epsilon'})(t)\|_{H^2},\\
&\left(\nu \int_0^t \|\nabla(u_\epsilon - u_{\epsilon'}, V_\epsilon - V_{\epsilon'}, W_\epsilon - W_{\epsilon'})(\tau)\|_{H^2}^2 \, d\tau\right)^{1/2}\\
&\quad \leqslant e^{C(1+E)^2 t}\Bigg(\|(u_{0,\epsilon} - u_{0,\epsilon'}, V_{0,\epsilon} - V_{0,\epsilon'}, W_{0,\epsilon} - W_{0,\epsilon'})\|_{H^2}\\
&\qquad + \left(C \int_0^t \left\|(g_\epsilon^1 - g_{\epsilon'}^1, g_\epsilon^2 - g_{\epsilon'}^2, g_\epsilon^3 - g_{\epsilon'}^3)(\tau)\right\|_{H^1}^2 \, d\tau\right)^{1/2}\Bigg).
\end{aligned} \tag{3.40}
$$

Since

$$
\begin{cases}
\|(u_{0,\epsilon} - u_{0,\epsilon'}, V_{0,\epsilon} - V_{0,\epsilon'}, W_{0,\epsilon} - W_{0,\epsilon'})\|_{H^2} \to 0,\\
\left\|(g_\epsilon^1 - g_{\epsilon'}^1, g_\epsilon^2 - g_{\epsilon'}^2, g_\epsilon^3 - g_{\epsilon'}^3)(t)\right\|_{H^1} \to 0,
\end{cases}
$$

as $\epsilon, \epsilon' \to 0$ in (3.40), we obtain the solution $U(t)$ to the problem (3.32), which satisfies (3.36). The energy estimate (3.38) follows from Lemma 3.6. The uniqueness and (3.37) follow from (3.38). The proof of Proposition 3.9 is completed. ∎

Finally, we will prove the existence of the solution to the Cauchy problem (3.32) in $\mathcal{E}(0, T; H^3)$ for some $T > 0$.

PROPOSITION 3.10. *Assume that for some $T > 0$ and some constant χ,*

$$
\begin{cases}
(\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{C}^0(0, T; H^3),\\
\eta(t) \geqslant \chi > -1,\\
(g^1, g^2, g^3)(t) \in \mathcal{C}^0(0, T; H^2).
\end{cases}
$$

If $U_0 \in H^3$, then the Cauchy problem (3.32) *with $f \equiv 0$ has a unique solution*

$$U(t) \in \mathcal{E}(0, T; H^3) \tag{3.41}$$

such that

$$(u, V, W)(t) \in \mathcal{L}_2(0, T; H^4) \tag{3.42}$$

and for any $t \in [0, T]$,

$$
\begin{aligned}
&\|U(t)\|_{H^3}, \ \left(\nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2}\\
&\quad \leqslant e^{C(1+E)^2 t}\left(\|U_0\|_{H^3} + \left(C \int_0^t \left\|(g^1, g^2, g^3)(\tau)\right\|_{H^2}^2 \, d\tau\right)^{1/2}\right),
\end{aligned} \tag{3.43}
$$

where $\nu > 0$ is some constant.

Proof. Applying ∇ to (3.32) with $f \equiv 0$, we obtain

(3.44)
$$\begin{cases} L^0(\nabla\varrho, \nabla u) = -(\nabla\varpi)^T \cdot \nabla\varrho - \nabla\eta \operatorname{div} u := \tilde{f}, \\ L^1(\nabla u) = -\dfrac{1}{(\eta+1)^2}\Delta u \cdot (\nabla\eta)^T - \dfrac{1}{(\eta+1)^2}\nabla \operatorname{div} u \cdot (\nabla\eta)^T + \nabla g^1 := g^{1'}, \\ L^2(\nabla V, \nabla W) = \nabla A_{11}\Delta V + \nabla A_{12}\Delta W + \nabla g^2 := g^{2'}, \\ L^3(\nabla V, \nabla W) = \nabla A_{22}\Delta W + \nabla A_{21}\Delta V + \nabla g^3 := g^{3'}, \\ \nabla U \mid_{t=0} = \nabla U_0 \in H^2. \end{cases}$$

Next, we could combine an iteration method and Proposition 3.9 to solve the Cauchy problem (3.44) in $\mathcal{E}(0,T;H^2)$. Denote

$$\mathcal{U}^m := ([\nabla\varrho]^{(m)}, [\nabla u]^{(m)}, [\nabla V]^{(m)}, [\nabla W]^{(m)}) \text{ for } m = 0, 1, 2, \ldots.$$

Noting

$$\begin{aligned} \left\|\tilde{f}\right\|_{H^2} \leqslant CE\left\|\nabla(\varrho, u)\right\|_{H^2}, \ \left\|(g^{1'}, g^{2'}, g^{3'})\right\|_{H^1} \\ \leqslant \left\|\nabla(g^1, g^2, g^3)\right\|_{H^1} + CE\left\|\nabla^2(u, V, W)\right\|_{H^1}, \end{aligned}$$

the Cauchy problem (3.44) could be solved by the iteration

$$\mathcal{U}^0(t) := \nabla U_0,$$

and $\mathcal{U}^m(t)$, $m = 1, 2, 3, \ldots$, is the solution, belonging to $\mathcal{C}^0(0,T;H^2)$, of the problem

(3.45)
$$\begin{cases} L^0([\nabla\varrho]^{(m)}, [\nabla u]^{(m)}) = -(\nabla\varpi)^T \cdot [\nabla\varrho]^{(m-1)} - \nabla\eta \displaystyle\sum_{i=1}^{3}([u_{x_i}]^{(m-1)}), \\ L^1([\nabla u]^{(m)}) = -\dfrac{1}{(\eta+1)^2}\left(\operatorname{div}([\nabla u]^{(m-1)}) + \nabla \displaystyle\sum_{i=1}^{3}([u_{x_i}]^{(m-1)})\right) \cdot (\nabla\eta)^T + \nabla g^1, \\ L^2([\nabla V]^{(m)}, [\nabla W]^{(m)}) = \nabla A_{11} \operatorname{div}[\nabla V]^{(m-1)} + \nabla A_{12} \operatorname{div}[\nabla W]^{(m-1)} + \nabla g^2, \\ L^3([\nabla V]^{(m)}, [\nabla W]^{(m)}) = \nabla A_{22} \operatorname{div}[\nabla W]^{(m-1)} + \nabla A_{21} \operatorname{div}[\nabla V]^{(m-1)} + \nabla g^3, \\ \mathcal{U}^m \mid_{t=0} = \nabla U_0. \end{cases}$$

We next estimate the approximation $\{\mathcal{U}^m(t)\}$.

$$\begin{aligned} \left\|\mathcal{U}^0(t)\right\|_{H^2}^2 &\leqslant \left\|\nabla U_0\right\|_{H^2}^2, \\ \left\|\mathcal{U}^1(t)\right\|_{H^2}^2 &\leqslant e^{C(1+E^2)t}\Bigg(\left\|\nabla U_0\right\|_{H^2}^2 + \int_0^t CE^2\left\|\nabla(\varrho_0, u_0)\right\|_{H^2}^2 \, d\tau \\ &\quad + C\int_0^t \Big(\left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 \\ &\quad + CE^2\left\|\nabla^2(u_0, V_0, W_0)\right\|_{H^1}^2\Big) \, d\tau\Bigg) \\ &\leqslant e^{C(1+E^2)t}\left(\left\|\nabla U_0\right\|_{H^2}^2 + C\int_0^t \left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 \, d\tau\right). \end{aligned}$$

For $m = 1, 2, 3, \ldots$, we have the difference system

$$\begin{cases} L^0([\nabla\varrho]^{(m+1)} - [\nabla\varrho]^{(m)}, [\nabla u]^{(m+1)} - [\nabla u]^{(m)}) = G_0^m, \\ L^1([\nabla u]^{(m+1)} - [\nabla u]^{(m)}) = G_1^m, \\ L^2([\nabla V]^{(m+1)} - [\nabla V]^{(m)}, [\nabla W]^{(m+1)} - [\nabla W]^{(m)}) = G_2^m, \\ L^3([\nabla V]^{(m+1)} - [\nabla V]^{(m)}, [\nabla W]^{(m+1)} - [\nabla W]^{(m)}) = G_3^m, \end{cases}$$

where

$$\begin{aligned} G_0^m &:= -(\nabla\varpi)^T \cdot \left([\nabla\varrho]^{(m)} - [\nabla\varrho]^{(m-1)}\right) - \nabla\eta \sum_{i=1}^{3} \left([u_{x_i}]^{(m)} - [u_{x_i}]^{(m-1)}\right), \\ G_1^m &:= -\frac{1}{(\eta+1)^2} \left(\operatorname{div}\left([\nabla u]^{(m)} - [\nabla u]^{(m-1)}\right) + \nabla \sum_{i=1}^{3} \left([u_{x_i}]^{(m)} - [u_{x_i}]^{(m-1)}\right)\right) \\ &\qquad \cdot (\nabla\eta)^T, \\ G_2^m &:= \nabla A_{11} \operatorname{div}\left([\nabla V]^{(m)} - [\nabla V]^{(m-1)}\right) + \nabla A_{12} \operatorname{div}\left([\nabla W]^{(m)} - [\nabla W]^{(m-1)}\right), \\ G_3^m &:= \nabla A_{22} \operatorname{div}\left([\nabla W]^{(m)} - [\nabla W]^{(m-1)}\right) + \nabla A_{21} \operatorname{div}\left([\nabla V]^{(m)} - [\nabla V]^{(m-1)}\right). \end{aligned}$$

Then, by Lemma 3.6, we have

$$\begin{aligned} &\left\|(\mathcal{U}^{m+1} - \mathcal{U}^m)(t)\right\|_{H^2}^2 \\ &\quad \leqslant e^{C(1+E^2)t} \int_0^t CE^2 \left\|(\mathcal{U}^m - \mathcal{U}^{m-1})(\tau_1)\right\|_{H^2}^2 \, d\tau_1, \\ &\quad \leqslant e^{C(1+E^2)t} \int_0^t CE^2 e^{C(1+E^2)\tau_1} \int_0^{\tau_1} CE^2 \left\|(\mathcal{U}^{m-1} - \mathcal{U}^{m-2})(\tau_2)\right\|_{H^2}^2 \, d\tau_2 \, d\tau_1, \\ &\quad \leqslant \cdots \leqslant \left(\frac{E^2}{1+E^2}\right)^m e^{C(1+E^2)T} \left(4\left\|\nabla U_0\right\|_{H^2}^2 + C\int_0^T \left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 \, d\tau\right) \\ &\quad \to 0, \ m \to \infty. \end{aligned}$$

Hence, $\{\mathcal{U}^m(t)\}$ is a Cauchy sequence in $\mathcal{C}^0(0, T; H^2)$, and then there exists

$$\lim_{m\to\infty} \mathcal{U}^m(t) = \nabla U(t) \in \mathcal{C}^0(0, T; H^2),$$

which is the unique solution of (3.44), and so we obtain the solution $U(t) \in \mathcal{E}(0, T; H^3)$ to the Cauchy problem (3.32) with $f \equiv 0$. Then Lemma 3.6 gives (3.42), the energy estimate (3.43), and the uniqueness of the solution. This completes the proof of Proposition 3.10. ∎

3.3. Local solution for the nonlinear system. In this subsection, we will prove that the Cauchy problem (3.1) ((1.10)) admits a unique solution in $\mathcal{E}(0, T; H^3)$ for some $T > 0$ such that $\varrho > -1$.

From (3.3), we simply denote (3.2) as

$$\begin{cases} L^0_{\varrho_{n-1}, u_{n-1}}(\varrho_n, u_n) = 0, \\ L^1_{\varrho_{n-1}}(u_n) = g^1_{n-1}, \\ L^2_{V_{n-1}, W_{n-1}}(V_n, W_n) = g^2_{n-1}, \\ L^3_{V_{n-1}, W_{n-1}}(V_n, W_n) = g^3_{n-1}, \\ U_1 \equiv U_0, \ U_n |_{t=0} = U_0, \ n = 2, 3, 4, \ldots, \end{cases} \tag{3.46}$$

which satisfies

$$\inf_{x\in\mathbb{R}^3} \varrho_0(x) > -1. \tag{3.47}$$

Define

$$\begin{cases} E_0 := 2\|U_0\|_{H^3}, \\ \chi := (\inf_{x\in\mathbb{R}^3} \varrho_0(x) - 1)/2 > -1. \end{cases} \tag{3.48}$$

Next, we will estimate the approximate sequence $\{U_n(t)\}_{n=1}^{\infty}$.

LEMMA 3.11. *Assume that T is properly small. If $U_0 \in H^3$, then we have for all $n = 1, 2, 3, \ldots,$*

$$U_n(t) \in \mathcal{E}(0, T; H^3), \tag{3.49}$$

such that for any $t \in [0, T]$ and some $\nu > 0$,

$$\|U_n(t)\|_{H^3}, \ \left(\nu \int_0^t \|\nabla(u_n, V_n, W_n)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \leqslant E_0 \tag{3.50}$$

and

$$\varrho_n(t) \geqslant \chi > -1. \tag{3.51}$$

Proof. We use an induction to prove this lemma. First, it is trivial for $n = 1$ by (3.48). Assume (3.49)–(3.51) follow for $U_k(t)$, $k = 2, 3, \ldots, n-1$. Then by Proposition 3.10, we have

$$U_n(t) \in \mathcal{E}(0, T; H^3)$$

and

$$\begin{aligned} &\|U_n(t)\|_{H^3}, \ \left(\nu \int_0^t \|\nabla(u_n, V_n, W_n)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \\ &\leqslant e^{C(1+E_0)^2 t}\left(\|U_0\|_{H^3} + \left(\int_0^t \left\|(g_{n-1}^1, g_{n-1}^2, g_{n-1}^3)(\tau)\right\|_{H^2}^2 \, d\tau\right)^{1/2}\right) \\ &\leqslant e^{C(1+E_0)^2 t}\left(\|U_0\|_{H^3} + \left(\int_0^t C(E_0)\,\|U_{n-1}(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2}\right) \\ &\leqslant e^{C(1+E_0)^2 t}\left(\frac{E_0}{2} + \left(\int_0^t C(E_0) E_0^2 \, d\tau\right)^{1/2}\right) \leqslant E_0, \end{aligned} \tag{3.52}$$

provided that $t \in [0, T_1]$ for some small $T_1 > 0$. By $(3.46)_1$ and (3.52), we can obtain that the sequence $\{\partial_t \varrho_n(t)\}$ is uniformly bounded, which implies that the sequence $\{\varrho_n(t)\}$ is equicontinuous with respect to t. Then, there exists a small $T_2 > 0$ such that for all $n \geqslant 1$,

$$\varrho_n(t) \geqslant \chi > -1 \text{ for any } t \in [0, T_2]. \tag{3.53}$$

We choose $T = \min\{T_1, T_2\}$. Hence, the results (3.49)–(3.51) follow for $n = 1, 2, 3, \ldots$. The proof of Lemma 3.11 is completed. ∎

Now, we could give the local solution to the Cauchy problem (1.10).

Proposition 3.12 (local solution). *Assume $U_0 \in H^3$ and $\inf_{x\in\mathbb{R}^3} \varrho_0(x) > -1$. Then there exists a positive T (suitably small) such that the Cauchy problem* (1.10) *has a unique solution*

$$\begin{cases} U(t) \in \mathcal{E}(0,T;H^3), \\ (u,V,W)(t) \in \mathcal{L}_2(0,T;H^4), \end{cases} \tag{3.54}$$

which satisfies for any $t \in [0,T]$ and some $\nu > 0$,

$$\begin{cases} \varrho(t) \geqslant (\inf\limits_{x\in\mathbb{R}^3} \varrho_0(x) - 1)/2 > -1, \\ \|U(t)\|_{H^3}\,, \ \left(\nu \displaystyle\int_0^t \|\nabla(u,V,W)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \leqslant 2\,\|U_0\|_{H^3}\,. \end{cases} \tag{3.55}$$

Proof. First, we prove a convergence of the approximate sequence $\{U_n(t)\}$ constructed in Lemma 3.11. Subtracting the system (3.46) with $n = m \geqslant 2$ from that for $n = m+1$, we obtain

(3.56)
$$\begin{cases} L^0_{\varrho_m,u_m}(\varrho_{m+1}-\varrho_m, u_{m+1}-u_m) = L^0_{\varrho_{m-1},u_{m-1}}(\varrho_m,u_m) - L^0_{\varrho_m,u_m}(\varrho_m,u_m), \\ L^1_{\varrho_m}(u_{m+1}-u_m) = L^1_{\varrho_{m-1}}(u_m) - L^1_{\varrho_m}(u_m) + \left(g^1_m - g^1_{m-1}\right), \\ L^2_{V_m,W_m}(V_{m+1}-V_m, W_{m+1}-W_m) \\ \qquad = L^2_{V_{m-1},W_{m-1}}(V_m,W_m) - L^2_{V_m,W_m}(V_m,W_m) + \left(g^2_m - g^2_{m-1}\right), \\ L^3_{V_m,W_m}(V_{m+1}-V_m, W_{m+1}-W_m) \\ \qquad = L^3_{V_{m-1},W_{m-1}}(V_m,W_m) - L^3_{V_m,W_m}(V_m,W_m) + \left(g^3_m - g^3_{m-1}\right) \end{cases}$$

with the initial data

$$(U_{m+1} - U_m)\mid_{t=0} = 0, \quad m = 2,3,4,\ldots. \tag{3.57}$$

Next, we estimate the solution $(U_{m+1}-U_m)(t)$ to (3.56)–(3.57). For some fixed $m \geqslant 2$, by Lemma 3.11, (3.8) of Lemma 3.2, and Lemma 3.6, we obtain for some $T > 0$,

$$\|(U_{m+1}-U_m)(t)\|_{H^2}^2 \leqslant e^{C(1+E_0)^2T} C(E_0) \int_0^t \|(U_m - U_{m-1})(\tau_1)\|_{H^2}^2 \, d\tau_1, \tag{3.58}$$

which implies for any $t \in [0,T]$,

$$\begin{aligned} &\|(U_{m+1}-U_m)(t)\|_{H^2}^2 \\ &\quad \leqslant e^{C(1+E_0)^2T} C(E_0) \int_0^t \|(U_m - U_{m-1})(\tau_1)\|_{H^2}^2 \, d\tau_1 \\ &\quad \leqslant (e^{C(1+E_0)^2T} C(E_0))^{m-2} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-3}} \|(U_3 - U_2)(\tau_{m-2})\|_{H^2}^2 \, d\tau_{m-2} \\ &\quad \leqslant (e^{C(1+E_0)^2T} C(E_0))^{m-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-3}} \tau_{m-2} \, d\tau_{m-2} \\ &\quad \leqslant \frac{(e^{C(1+E_0)^2T} C(E_0)T)^{m-1}}{(m-1)!} \to 0, \quad m \to \infty. \end{aligned}$$

Hence, there exists $U(t)$ such that

$$U_n(t) \to U(t) \text{ strongly in } \mathcal{C}^0(0,T;H^2),\ n \to \infty. \tag{3.59}$$

By Lemma 3.11, there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$\nabla(u,V,W)_{n'}(t) \to \nabla(u,V,W)(t) \text{ weakly in } \mathcal{L}_2(0,T;H^3),\ n' \to \infty.$$

By Lemma 3.11 again, we have that for any fixed $t \in [0,T]$, there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$U_{n''}(t) \to U(t) \text{ weakly in } H^3,\ n'' \to \infty.$$

So far, we have proved that the Cauchy problem (3.1) has a solution $U(t) \in \mathfrak{L}(0,T;H^3)$ such that

$$\varrho(t) \geqslant \chi = \left(\inf_{x\in\mathbb{R}^3} \varrho_0(x) - 1\right)/2 > -1.$$

Next, we can show that such a solution $U(t)$ belongs to $\mathcal{E}(0,T;H^3)$. In fact, since $U(t) \in \mathcal{C}^0(0,T;H^2)$ by (3.59), we have $U_\epsilon(t) \in \mathcal{C}^0(0,T;H^\infty)$. Applying $J_\epsilon *$ to $(3.1)_1$–$(3.1)_4$, we obtain

$$\begin{cases} L^0_{\varrho,u}(\varrho_\epsilon,u_\epsilon) = R^0_\epsilon, \\ L^1_\varrho(u_\epsilon) = g^1_\epsilon + R^1_\epsilon, \\ L^2_{V,W}(V_\epsilon,W_\epsilon) = g^2_\epsilon + R^2_\epsilon, \\ L^3_{V,W}(V_\epsilon,W_\epsilon) = g^3_\epsilon + R^3_\epsilon, \\ U_\epsilon\mid_{t=0} = U_{0,\epsilon}, \end{cases}$$

where

$$R^0_\epsilon := L^0_{\varrho,u}(\varrho_\epsilon,u_\epsilon) - J_\epsilon * L^0_{\varrho,u}(\varrho,u),\ R^1_\epsilon := L^1_\varrho(u_\epsilon) - J_\epsilon * L^1_\varrho(u),$$
$$R^2_\epsilon := L^2_{V,W}(V_\epsilon,W_\epsilon) - J_\epsilon * L^2_{V,W}(V,W),\ R^3_\epsilon := L^3_{V,W}(V_\epsilon,W_\epsilon) - J_\epsilon * L^3_{V,W}(V,W).$$

By Lemmas 3.2 and 3.6, we can estimate the difference for any $\epsilon,\epsilon' > 0$,

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \|(U_\epsilon - U_{\epsilon'})(t)\|_{H^3} \\ &\leqslant e^{C(1+E_0)^2T}\Bigg(\|(U_{0,\epsilon} - U_{0,\epsilon'})\|_{H^3} + \int_0^T \left\|(R^0_\epsilon, R^0_{\epsilon'})(\tau)\right\|_{H^3}\, d\tau \\ &\qquad + \Bigg(C\int_0^T \left\|(g^1_\epsilon - g^1_{\epsilon'}, g^2_\epsilon - g^2_{\epsilon'}, g^3_\epsilon - g^3_{\epsilon'})(\tau)\right\|^2_{H^2} \\ &\qquad\quad + \sum_{i=1}^3 \left\|(R^i_\epsilon, R^i_{\epsilon'})(\tau)\right\|^2_{H^2}\, d\tau\Bigg)^{1/2}\Bigg). \end{aligned}$$

By (3.7) of Lemma 3.2 and the property of mollifier, we have as $\epsilon,\epsilon' \to 0$,

$$\sup_{0\leqslant t\leqslant T} \|(U_\epsilon - U_{\epsilon'})(t)\|_{H^3} \to 0.$$

Hence, the solution sequence $\{U_\epsilon(t)\}$ has a limit $U(t) \in \mathcal{E}(0,T;H^3)$ as $\epsilon \to 0$.

Finally, we prove that the solution $U(t)$ is unique. We assume that there is another solution $\widetilde{U}(t) := (\tilde{\varrho}, \tilde{u}, \widetilde{V}, \widetilde{W})(t)$ satisfying (3.54)–(3.55). Recalling (3.59), we have

$$U_n(t) \to U(t) \text{ strongly in } \mathcal{C}^0(0,T;H^2),\ n \to \infty. \tag{3.60}$$

For such a sequence $\{U_n(t)\}$ and $\widetilde{U}(t)$, we consider

$$\begin{cases} L^0_{\varrho_n,u_n}(\varrho_n - \tilde{\varrho}, u_n - \tilde{u}) = L^0_{\tilde{\varrho},\tilde{u}}(\tilde{\varrho},\tilde{u}) - L^0_{\varrho_n,u_n}(\tilde{\varrho},\tilde{u}),\\ L^1_{\varrho_n}(u_n - \tilde{u}) = L^1_{\tilde{\varrho}}(\tilde{u}) - L^1_{\varrho_n}(\tilde{u}) + \left(g^1_n - \tilde{g}^1\right),\\ L^2_{V_n,W_n}(V_n - \widetilde{V}, W_n - \widetilde{W}) = L^2_{\widetilde{V},\widetilde{W}}(\widetilde{V},\widetilde{W}) - L^2_{V_n,W_n}(\widetilde{V},\widetilde{W}) + \left(g^2_n - \tilde{g}^2\right),\\ L^3_{V_n,W_n}(V_n - \widetilde{V}, W_n - \widetilde{W}) = L^3_{\widetilde{V},\widetilde{W}}(\widetilde{V},\widetilde{W}) - L^3_{V_n,W_n}(\widetilde{V},\widetilde{W}) + \left(g^3_n - \tilde{g}^3\right) \end{cases}$$

with the initial data

$$(U_n - \widetilde{U}) \mid_{t=0} = 0.$$

By Lemma 3.6, (3.8) of Lemma 3.2, and (3.58), we have

$$\begin{aligned} &\left\|(U_n - \widetilde{U})(t)\right\|^2_{H^2}\\ &\quad\leqslant e^{C(1+E_0)^2 t} C(E_0) \int_0^t \left\|(U_n - \widetilde{U})(\tau)\right\|^2_{H^2} d\tau\\ &\quad\leqslant e^{C(1+E_0)^2 t} C(E_0) \int_0^t \left(\left\|(U_{n-1} - \widetilde{U})(\tau)\right\|^2_{H^2} + \|(U_n - U_{n-1})(\tau)\|^2_{H^2}\right) d\tau, \end{aligned}$$

which implies that

$$U_n(t) \to \widetilde{U}(t) \text{ strongly in } \mathcal{C}^0(0,T;H^2),\ n \to \infty. \tag{3.61}$$

In light of (3.60)–(3.61), we have $U(t) \equiv \widetilde{U}(t)$ for any $t \in [0,T]$. Hence, the proof of Proposition 3.12 is completed. ∎

4. A priori estimates. To obtain the a priori estimates effectively, we rewrite the system (1.10) as

$$\begin{cases} \varrho_t + \operatorname{div} u = -\operatorname{div}(\varrho u),\\ u_t + \nabla\varrho + \nabla V + \nabla W - \Delta u - \nabla \operatorname{div} u = G_1,\\ V_t + \operatorname{div} u - \frac{2}{3}\Delta V - \frac{1}{3}\Delta W + \frac{1}{3}\Delta\phi = G_2,\\ W_t + \operatorname{div} u - \frac{2}{3}\Delta W - \frac{1}{3}\Delta V - \frac{1}{3}\Delta\phi = G_3,\\ \Delta\phi = V - W,\\ (\varrho, u, V, W) \mid_{t=0} = (\varrho_0, u_0, V_0, W_0). \end{cases} \tag{4.1}$$

Here

$$\begin{aligned} G_1 &:= -u\cdot\nabla u - f_1\left(\Delta u + \nabla \operatorname{div} u\right) + f_1\left(\nabla V + \nabla W\right) - f_2 \nabla\varrho + \frac{1}{\varrho+1}\Delta\phi\nabla\phi,\\ G_2 &:= -\operatorname{div}(Vu) + 2\operatorname{div}(f_3\nabla V) + \operatorname{div}(f_3\nabla W) - \operatorname{div}(f_3\nabla\phi)\\ &\quad + \operatorname{div}\left(\frac{V\nabla V + V\nabla W}{V+W+3}\right) + \operatorname{div}\left(\frac{VW - V^2}{V+W+3}\nabla\phi\right) + \operatorname{div}\left(\frac{W-2V}{V+W+3}\nabla\phi\right), \end{aligned}$$

$$G_3 := -\operatorname{div}(Wu) + 2\operatorname{div}(f_3\nabla W) + \operatorname{div}(f_3\nabla V) + \operatorname{div}(f_3\nabla\phi) + \operatorname{div}\left(\frac{W\nabla V + W\nabla W}{V+W+3}\right) - \operatorname{div}\left(\frac{VW - W^2}{V+W+3}\nabla\phi\right) - \operatorname{div}\left(\frac{V-2W}{V+W+3}\nabla\phi\right),$$

where

$$f_1 := \frac{\varrho}{\varrho+1},\ f_2 := \frac{p'(\varrho+1)}{\varrho+1} - 1, \tag{4.2}$$

$$f_3 := \frac{1}{V+W+3} - \frac{1}{3}. \tag{4.3}$$

In what follows, we will derive the a priori estimates for the solutions to the PNP-NS system (4.1) by assuming that for some sufficiently small $\delta_1 > 0$ and any $t \in [0,T]$ with $T > 0$,

$$\|U(t)\|_{H^3} = \|(\varrho, u, V, W)(t)\|_{H^3} < \delta_1, \tag{4.4}$$

which implies

$$1/2 \leqslant \varrho+1, V+1, W+1 \leqslant 3/2 \tag{4.5}$$

and

$$\begin{cases} |f_1|, |f_2| \leqslant C|\varrho|, \\ |f_3| \leqslant C(|V| + |W|), \\ \sum_{i=1}^3 |f_i^{(k)}| \leqslant C \text{ for any } k \geqslant 1, \end{cases} \tag{4.6}$$

by Sobolev's inequality and Taylor's expansion. Noting that Hölder's, Sobolev's, and Cauchy's inequalities will be used almost everywhere, so we keep them in mind to avoid mentioning them repeatedly.

We first show the lower-order dissipation estimates for u, V, W, and ϕ.

LEMMA 4.1. *It holds that*

$$\frac{d}{dt}\|U\|_{L^2}^2 + C\|\nabla(u,V,W)\|_{L^2}^2 + C\|\Delta\phi\|_{L^2}^2 \lesssim \delta_1\|\nabla\varrho\|_{L^2}^2. \tag{4.7}$$

Proof. Multiplying the first four equations of (4.1) by ϱ, u, V, and W, respectively, summing them up, and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \frac{1}{3}\|\nabla(V,W)\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \frac{1}{3}\|\Delta\phi\|_{L^2}^2 \\ &\quad \leqslant -\int \operatorname{div}(\varrho u)\varrho + \int G_1\cdot u + G_2 V + G_3 W. \end{aligned} \tag{4.8}$$

We now estimate the RHS of (4.8). By the integration by parts, (4.4)–(4.6), and Lemma A.2, we can estimate some typical terms as

$$\begin{aligned} -\int f_1\Delta u\cdot u &= \int \nabla(f_1 u)\cdot\nabla u \\ &\lesssim (\|\nabla f_1\|_{L^2}\|u\|_{L^\infty} + \|f_1\|_{L^\infty}\|\nabla u\|_{L^2})\|\nabla u\|_{L^2} \\ &\lesssim \delta_1\|\nabla(\varrho,u)\|_{L^2}^2; \end{aligned} \tag{4.9}$$

$$\int \frac{1}{\varrho+1}\Delta\phi\nabla\phi\cdot u \lesssim \|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^6}\|u\|_{L^3} \lesssim \delta_1\|\Delta\phi\|_{L^2}^2; \tag{4.10}$$

$$\begin{aligned}\int \operatorname{div}(f_3\nabla W)V &= -\int f_3\nabla W\cdot\nabla V\\ &\lesssim \|f_3\|_{L^\infty}\|\nabla W\|_{L^2}\|\nabla V\|_{L^2} \lesssim \delta_1\|\nabla(V,W)\|_{L^2}^2;\end{aligned} \tag{4.11}$$

$$\begin{aligned}\int \operatorname{div}\left(\frac{W-2V}{V+W+3}\nabla\phi\right)V &= \int \frac{2V-W}{V+W+3}\nabla\phi\cdot\nabla V\\ &\lesssim \|(V,W)\|_{L^3}\|\nabla\phi\|_{L^6}\|\nabla V\|_{L^2}\\ &\lesssim \delta_1\left(\|\nabla V\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\right).\end{aligned} \tag{4.12}$$

Then, the other terms could be estimated as (4.9)–(4.12). So, we can obtain

$$\text{RHS of (4.8)} \lesssim \delta_1\left(\|\nabla(\varrho,u,V,W)\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\right). \tag{4.13}$$

Plugging (4.13) into (4.8), since δ_1 is small, we deduce (4.7). ∎

Next, we derive the higher-order dissipation estimates for u, V, W, and ϕ.

LEMMA 4.2. *Let $k \geqslant 3$. Then we have for $3 \leqslant \ell \leqslant k$,*

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla^\ell U\right\|_{L^2}^2 + C\left\|\nabla^{\ell+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2\\ &\quad\lesssim \delta_1\left(\left\|\nabla^\ell(\varrho,V,W)\right\|_{L^2}^2+\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2\right).\end{aligned} \tag{4.14}$$

Proof. Let $k \geqslant 3$. For $3 \leqslant \ell \leqslant k$, applying ∇^ℓ to the first four equations of (4.1) and then multiplying the resulting identities by $\nabla^\ell\varrho, \nabla^\ell u, \nabla^\ell V$, and $\nabla^\ell W$, respectively, summing them up, and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^\ell U\right\|_{L^2}^2 + \left\|\nabla^{\ell+1}u\right\|_{L^2}^2 + \frac{1}{3}\left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2 + \left\|\nabla^\ell \operatorname{div} u\right\|_{L^2}^2 + \frac{1}{3}\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2\\ &\quad= -\int \nabla^\ell\operatorname{div}(\varrho u)\cdot\nabla^\ell\varrho + \int \nabla^\ell G_1\cdot\nabla^\ell u + \nabla^\ell G_2\cdot\nabla^\ell V + \nabla^\ell G_3\cdot\nabla^\ell W.\end{aligned} \tag{4.15}$$

In the following, we only estimate some typical terms on the RHS of (4.15). Then, the remaining terms could be estimated in the same way. First, we spilt the first term as

$$-\int \nabla^\ell\operatorname{div}(\varrho u)\cdot\nabla^\ell\varrho = -\int \nabla^\ell\left(\varrho\operatorname{div} u\right)\cdot\nabla^\ell\varrho - \int \nabla^\ell\left(u\cdot\nabla\varrho\right)\cdot\nabla^\ell\varrho := J_1+J_2. \tag{4.16}$$

By the product estimates (A.11), we obtain

$$\begin{aligned}J_1 &\lesssim \left\|\nabla^\ell\left(\varrho\operatorname{div} u\right)\right\|_{L^2}\left\|\nabla^\ell\varrho\right\|_{L^2}\\ &\lesssim \left(\left\|\nabla^\ell\varrho\right\|_{L^2}\|\operatorname{div} u\|_{L^\infty} + \|\varrho\|_{L^\infty}\left\|\nabla^\ell\operatorname{div} u\right\|_{L^2}\right)\left\|\nabla^\ell\varrho\right\|_{L^2}\\ &\lesssim \delta_1\left(\left\|\nabla^\ell\varrho\right\|_{L^2}^2+\left\|\nabla^{\ell+1}u\right\|_{L^2}^2\right).\end{aligned} \tag{4.17}$$

By the commutator estimates (A.10) and the integration by parts, we obtain

$$
\begin{aligned}
J_2 &= -\int \left[\nabla^\ell, u\right] \cdot \nabla\varrho \cdot \nabla^\ell \varrho - \int u \cdot \nabla\nabla^\ell \varrho \cdot \nabla^\ell \varrho \\
&\lesssim \left(\|\nabla u\|_{L^\infty} \left\|\nabla^\ell \varrho\right\|_{L^2} + \left\|\nabla^\ell u\right\|_{L^6} \|\nabla\varrho\|_{L^3}\right) \left\|\nabla^\ell \varrho\right\|_{L^2} + \frac{1}{2}\int \operatorname{div} u \left|\nabla^\ell \varrho\right|^2 \\
&\lesssim \left(\|\nabla u\|_{L^\infty} \left\|\nabla^\ell \varrho\right\|_{L^2} + \left\|\nabla^{\ell+1} u\right\|_{L^2} \|\nabla\varrho\|_{L^3}\right) \left\|\nabla^\ell \varrho\right\|_{L^2} + \|\operatorname{div} u\|_{L^\infty} \left\|\nabla^\ell \varrho\right\|_{L^2}^2 \\
&\lesssim \delta_1 \left(\left\|\nabla^\ell \varrho\right\|_{L^2}^2 + \left\|\nabla^{\ell+1} u\right\|_{L^2}^2\right). \qquad (4.18)
\end{aligned}
$$

In light of (4.16)–(4.18), we obtain

$$
-\int \nabla^\ell \operatorname{div}(\varrho u) \cdot \nabla^\ell \varrho \lesssim \delta_1 \left(\left\|\nabla^\ell \varrho\right\|_{L^2}^2 + \left\|\nabla^{\ell+1} u\right\|_{L^2}^2\right). \qquad (4.19)
$$

By the integration by parts, (4.4)–(4.6), the product estimates (A.11), Lemma A.2, and Young's inequality, we obtain

$$
\begin{aligned}
&-\int \nabla^\ell \left(u \cdot \nabla u\right) \cdot \nabla^\ell u \\
&\quad= \int \nabla^{\ell-1} \left(u \cdot \nabla u\right) \cdot \nabla^{\ell+1} u \lesssim \left\|\nabla^{\ell-1} \left(u \cdot \nabla u\right)\right\|_{L^2} \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \left(\|u\|_{L^3} \left\|\nabla^\ell u\right\|_{L^6} + \left\|\nabla^{\ell-1} u\right\|_{L^6} \|\nabla u\|_{L^3}\right) \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \Bigg(\|u\|_{L^3} \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\qquad\quad + \|u\|_{L^2}^{\frac{1}{\ell+1}} \left\|\nabla^{\ell+1} u\right\|_{L^2}^{\frac{\ell}{\ell+1}} \left\|\nabla^{\frac{\ell+1}{2\ell}} u\right\|_{L^2}^{\frac{\ell}{\ell+1}} \left\|\nabla^{\ell+1} u\right\|_{L^2}^{\frac{1}{\ell+1}} \Bigg) \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \delta_1 \left\|\nabla^{\ell+1} u\right\|_{L^2}^2 ; \qquad (4.20)
\end{aligned}
$$

$$
\begin{aligned}
&\int \nabla^\ell \left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right) \cdot \nabla^\ell u \\
&\quad= -\int \nabla^{\ell-1} \left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right) \cdot \nabla^{\ell+1} u \\
&\quad\lesssim \left\|\nabla^{\ell-1} \left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right)\right\|_{L^2} \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \Bigg(\left\|\nabla^{\ell-1} \left(\frac{1}{\varrho+1}\right)\right\|_{L^6} \|\Delta\phi\nabla\phi\|_{L^3} \\
&\qquad\quad + \left\|\frac{1}{\varrho+1}\right\|_{L^\infty} \left\|\nabla^{\ell-1} \left(\Delta\phi\nabla\phi\right)\right\|_{L^2} \Bigg) \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \left\|\nabla^\ell \varrho\right\|_{L^2} \|\Delta\phi\|_{L^3} \|\nabla\phi\|_{L^\infty} \left\|\nabla^{\ell+1} u\right\|_{L^2} + \left\|\nabla^{\ell-1} \left(\Delta\phi\nabla\phi\right)\right\|_{L^2} \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\quad\lesssim \delta_1 \left\|\nabla^\ell \varrho\right\|_{L^2} \left\|\nabla^{\ell+1} u\right\|_{L^2} \\
&\qquad + \left(\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2} \|\nabla\phi\|_{L^\infty} + \|\Delta\phi\|_{L^3} \left\|\nabla^{\ell-1}\nabla\phi\right\|_{L^6}\right) \left\|\nabla^{\ell+1} u\right\|_{L^2}
\end{aligned}
$$

$$\lesssim \delta_1\left(\left\|\nabla^\ell\varrho\right\|_{L^2}^2+\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{\ell+1}u\right\|_{L^2}^2\right); \tag{4.21}$$

$$\begin{aligned}&\int\nabla^\ell\operatorname{div}\left(\frac{VW-V^2}{V+W+3}\nabla\phi\right)\nabla^\ell V\\&=-\int\nabla^\ell\left(\frac{VW-V^2}{V+W+3}\nabla\phi\right)\nabla^{\ell+1}V\\&\lesssim\left\|\nabla^\ell\left(\frac{VW-V^2}{V+W+3}\right)\right\|_{L^2}\|\nabla\phi\|_{L^\infty}\left\|\nabla^{\ell+1}V\right\|_{L^2}\\&\quad+\left\|\frac{VW-V^2}{V+W+3}\right\|_{L^3}\left\|\nabla^\ell\nabla\phi\right\|_{L^6}\left\|\nabla^{\ell+1}V\right\|_{L^2}\end{aligned}$$

$$\lesssim\delta_1\left(\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2\right). \tag{4.22}$$

Here we specially mention the following four terms:

$$\begin{aligned}&-\int\nabla^\ell\operatorname{div}(f_3\nabla\phi)\cdot\nabla^\ell V+\int\nabla^\ell\operatorname{div}(f_3\nabla\phi)\cdot\nabla^\ell W\\&\quad+\int\nabla^\ell\operatorname{div}\left(\frac{W-2V}{V+W+3}\nabla\phi\right)\cdot\nabla^\ell V-\int\nabla^\ell\operatorname{div}\left(\frac{V-2W}{V+W+3}\nabla\phi\right)\cdot\nabla^\ell W\\&=\int\nabla^\ell\operatorname{div}\left(\frac{4W-5V}{3(V+W+3)}\nabla\phi\right)\cdot\nabla^\ell V+\int\nabla^\ell\operatorname{div}\left(\frac{5W-4V}{3(V+W+3)}\nabla\phi\right)\cdot\nabla^\ell W\\&:=\tilde{S}.\end{aligned}$$

We hope to control $\tilde{S}$ in terms of $(\ell+1)$-order of V and W, that is,

$$\tilde{S}\lesssim\delta_1\left(\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2\right). \tag{4.23}$$

But it failed: we only could estimate

$$\tilde{S}\lesssim\delta_1\left(\left\|\nabla^\ell(V,W,\Delta\phi)\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2\right). \tag{4.24}$$

In fact, we could control $\tilde{S}$ like (4.23) (see Lemma 5.2) if we additionally assume that the initial electric filed $\|\nabla\phi_0\|_{L^2}$ is sufficiently small.

In light of (4.19)–(4.24), we can obtain

$$\text{RHS of (4.15)}\lesssim\delta_1\left(\left\|\nabla^\ell(\varrho,V,W,\Delta\phi)\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(u,V,W)\right\|_{L^2}^2+\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2\right). \tag{4.25}$$

Plugging (4.25) into (4.15), since δ_1 is small, we obtain (4.14). ∎

Next, we will derive the dissipation estimate for ϱ.

LEMMA 4.3. *Let $k\geqslant 3$ and $3\leqslant\ell\leqslant k$. Then we have for $0\leqslant l\leqslant\ell-1$,*

$$\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l\varrho+C\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2\lesssim\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2+\left\|\nabla^l\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{l+2}u\right\|_{L^2}^2. \tag{4.26}$$

Proof. For $0\leqslant l\leqslant\ell-1$, applying ∇^l to $(4.1)_2$ and then multiplying the resulting identity by $\nabla\nabla^l\varrho$ and integrating over $\mathbb{R}^3$, by the integration by parts, the Poisson equation $(4.1)_5$, and the product estimates (A.11), we obtain

$$\begin{aligned}\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \leqslant -\int \nabla^l u_t\cdot\nabla\nabla^l\varrho + C\left\|\nabla^{l+1}(V,W)\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2}\\ + C\left\|\nabla^{l+2}u\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2} + C\left\|\nabla^l G_1\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2}.\end{aligned} \tag{4.27}$$

By the integration by parts, $(4.1)_1$, and the product estimates (A.11), we obtain

$$\begin{aligned}-\int \nabla^l u_t\cdot\nabla\nabla^l\varrho &= -\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l\varrho - \int\nabla^l \operatorname{div} u\nabla^l\varrho_t\\ &= -\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l\varrho + \left\|\nabla^l\operatorname{div} u\right\|_{L^2}^2 + \int\nabla^l\operatorname{div} u\nabla^l\operatorname{div}(\varrho u)\\ &\leqslant -\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l\varrho + C\left\|\nabla^{l+1}u\right\|_{L^2}^2 + \delta_1\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2.\end{aligned} \tag{4.28}$$

By the product estimates (A.11), we easily obtain

$$\left\|\nabla^l G_1\right\|_{L^2} \lesssim \delta_1\left(\left\|\nabla^{l+1}(\varrho,u,V,W)\right\|_{L^2} + \left\|\nabla^l\Delta\phi\right\|_{L^2} + \left\|\nabla^{l+2}u\right\|_{L^2}\right). \tag{4.29}$$

Plugging (4.28)–(4.29) into (4.27), by Cauchy's inequality, since δ_1 is small, we obtain (4.26). ∎

Now, we could make good use of Lemmas 4.1–4.3 to obtain the a priori estimates.

PROPOSITION 4.4 (a priori estimates). *Let $T>0$ and $k\geqslant 3$. Assume $U_0\in H^k$ such that for some sufficiently small $\delta_1>0$,*

$$\sup_{t\in[0,T]}\|U(t)\|_{H^3}<\delta_1.$$

Then, we have for any $t\in[0,T]$ and $3\leqslant \ell\leqslant k$,

$$\begin{aligned}\|U(t)\|_{H^\ell} + \left(\int_0^t \|\nabla\varrho(\tau)\|_{H^{\ell-1}}^2 + \|\nabla(u,V,W)(\tau)\|_{H^\ell}^2 + \|\Delta\phi(\tau)\|_{H^\ell}^2\,d\tau\right)^{1/2}\\ \leqslant c\|U_0\|_{H^\ell},\end{aligned} \tag{4.30}$$

where c is some fixed positive constant.

Proof. Let $k\geqslant 3$ and $3\leqslant\ell\leqslant k$. Adding the estimates (4.7) of Lemma 4.1 to the estimates (4.14) of Lemma 4.2, by the interpolation estimates and Young's inequality, since δ_1 is small, we obtain

$$\frac{d}{dt}\|U\|_{H^\ell}^2 + C_1\left(\|\nabla(u,V,W)\|_{H^\ell}^2 + \|\Delta\phi\|_{H^\ell}^2\right) \leqslant C_2\delta_1\|\nabla\varrho\|_{H^{\ell-1}}^2, \tag{4.31}$$

where we have used the equivalent relation by the interpolation estimates

$$\|h\|_{H^\ell}^2 \sim \|h\|_{L^2}^2 + \left\|\nabla^\ell h\right\|_{L^2}^2.$$

Summing up the estimates (4.26) of Lemma 4.3 from $l = 0$ to $\ell - 1$, we obtain

$$\frac{d}{dt}\sum_{0\leqslant l\leqslant \ell-1}\int \nabla^l u\cdot\nabla\nabla^l\varrho + C_3\|\nabla\varrho\|_{H^{\ell-1}}^2 \leqslant C_4\left(\|\nabla u\|_{H^\ell}^2+\|\nabla(V,W)\|_{H^{\ell-1}}^2+\|\Delta\phi\|_{H^{\ell-1}}^2\right). \tag{4.32}$$

Multiplying (4.32) by $2C_2\delta_1/C_3$ and then adding it to (4.31), since δ_1 is small, we deduce that there exists a constant $\tilde{c} > 0$ such that

$$\frac{d}{dt}\left(\|U\|_{H^\ell}^2+\frac{2C_2\delta_1}{C_3}\sum_{0\leqslant l\leqslant \ell-1}\int \nabla^l u\cdot\nabla\nabla^l\varrho\right) + \tilde{c}\left(\|\nabla\varrho\|_{H^{\ell-1}}^2+\|\nabla(u,V,W)\|_{H^\ell}^2+\|\Delta\phi\|_{H^\ell}^2\right)\leqslant 0. \tag{4.33}$$

Next, we define

$$\mathcal{E}_0^\ell(t) := \frac{1}{\tilde{c}}\left(\|U\|_{H^\ell}^2+\frac{2C_2\delta_1}{C_3}\sum_{0\leqslant l\leqslant \ell-1}\int \nabla^l u\cdot\nabla\nabla^l\varrho\right)\sim\|U(t)\|_{H^\ell}^2$$

since δ_1 is small, that is, there exists a constant $c > 0$ such that

$$\frac{1}{c}\|U(t)\|_{H^\ell}^2\leqslant\mathcal{E}_0^\ell(t)\leqslant c\|U(t)\|_{H^\ell}^2. \tag{4.34}$$

Then we could rewrite (4.33) as

$$\frac{d}{dt}\mathcal{E}_0^\ell(t)+\|\nabla\varrho\|_{H^{\ell-1}}^2+\|\nabla(u,V,W)\|_{H^\ell}^2+\|\Delta\phi\|_{H^\ell}^2\leqslant 0. \tag{4.35}$$

Integrating (4.35) in time, by (4.34), we obtain (4.30). ∎

5. Proof of Theorem 1.1.

5.1. Global solution. We combine Proposition 3.12 (local solution) and Proposition 4.4 (a priori estimates) to obtain the global solution in Theorem 1.1.

Assume $U_0 \in H^k$ with an integer $k \geqslant 3$. And we let

$$\|U_0\|_{H^3} < \min\{\delta_1/2, \delta_1/2c\}, \tag{5.1}$$

where $c, \delta_1 > 0$ are the given constants in Proposition 4.4. If

$$\|U_0\|_{H^3} < \delta_1/2,$$

by Proposition 3.12, the Cauchy problem (1.10) has a unique local solution

$$U(t) \in \mathcal{E}(0, T_1; H^3)$$

with $T_1 > 0$ such that for any $t \in [0, T_1]$

$$\|U(t)\|_{H^3} \leqslant 2\|U_0\|_{H^3} < \delta_1. \tag{5.2}$$

By (5.2) and Proposition 4.4, we obtain for any $t \in [0, T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|U(t)\|_{H^\ell} \leqslant c\|U_0\|_{H^\ell}. \tag{5.3}$$

In particular, we have for any $t \in [0, T_1]$,

$$\|U(t)\|_{H^3} \leqslant c\,\|U_0\|_{H^3}\,. \tag{5.4}$$

By (5.1) and (5.4), we have

$$U(T_1) \in H^3 \text{ and } \|U(T_1)\|_{H^3} < \delta_1/2. \tag{5.5}$$

Then, by Proposition 3.12 again, the Cauchy problem (1.10) has a unique local solution

$$U(t) \in \mathcal{E}(T_1, 2T_1; H^3) \tag{5.6}$$

such that for any $t \in [T_1, 2T_1]$

$$\|U(t)\|_{H^3} \leqslant 2\,\|U(T_1)\|_{H^3} < \delta_1. \tag{5.7}$$

From the above, we have obtained that the Cauchy problem (1.10) has a unique local solution

$$U(t) \in \mathcal{E}(0, 2T_1; H^3) \tag{5.8}$$

such that for any $t \in [0, 2T_1]$

$$\|U(t)\|_{H^3} < \delta_1. \tag{5.9}$$

By Proposition 4.4, we also obtain for any $t \in [0, 2T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|U(t)\|_{H^\ell} \leqslant c\,\|U_0\|_{H^\ell}\,. \tag{5.10}$$

Next, by repeating the procedures (5.4)–(5.10), we could extend the local solution to the global one only if we assume that the initial $H^k (k \geqslant 3)$ data satisfy that $\|U_0\|_{H^3}$ is sufficiently small, as (5.1). The energy estimates (1.11) could be obtained by Proposition 4.4. Hence, the proof of the global solution in Theorem 1.1 is completed.

5.2. Time-decay rates. To obtain the time-decay rates of the global solution in Theorem 1.1, we need to prove some extra energy estimates shown in the following.

5.2.1. Energy estimates. Assume that $(U, \nabla\phi)(t)$ is the solution to the PNP-NS system (4.1) proved in the above. To prove some extra energy estimates, we need to set the initial data

$$\|U_0\|_{H^3} < \delta_0'$$

so that for any $t \geqslant 0$,

$$\|U(t)\|_{H^3}\,,\ \left(\int_0^t \|\nabla\varrho(\tau)\|_{H^2}^2 + \|\nabla(u, V, W)(\tau)\|_{H^3}^2 + \|\Delta\phi(\tau)\|_{H^3}^2\, d\tau\right)^{1/2} < \delta_0 \tag{5.11}$$

by (1.11) with $\ell = 3$. Here $\delta_0', \delta_0 > 0$ are two sufficiently small constants. And then the estimates (4.5)–(4.6) follow still.

First, we give the estimates for the electric field $\nabla\phi$.

Lemma 5.1. *Assume* $0 \leqslant l \leqslant k$. *If* $\nabla^{-1}(V_0 - W_0) \in L^2$, *then it holds that for any* $t \geqslant 0$ *and some constant* $\alpha > 0$,

$$\left\|\nabla^l \nabla\phi(t)\right\|_{L^2}^2 + \alpha \int_0^t \left\|\nabla^l \nabla\phi(\tau)\right\|_{L^2}^2\, d\tau \leqslant C_0 \tag{5.12}$$

and

$$\left\|\nabla^l\nabla\phi(t)\right\|_{L^2}^2 \leqslant C_0e^{-\alpha t} + C\int_0^t e^{-\alpha(t-\tau)}\left\|\nabla^{l+1}(u,V,W)(\tau)\right\|_{L^2}^2\, d\tau. \tag{5.13}$$

In particular, if $\|\nabla^{-1}(V_0-W_0)\|_{L^2}$ *is sufficiently small, then for any* $t\geqslant 0$,

$$\|\nabla\phi(t)\|_{L^2} < \tilde{\delta}, \tag{5.14}$$

where $\tilde{\delta}>0$ *is some sufficiently small constant.*

Proof. Subtracting $(4.1)_4$ from $(4.1)_3$ and using the Poisson equation $(4.1)_5$, we obtain

$$\Delta\phi_t + \frac{2}{3}\Delta\phi - \frac{1}{3}\Delta\Delta\phi = G_2 - G_3. \tag{5.15}$$

For $0\leqslant l\leqslant k$, applying ∇^l to (5.15) and then multiplying the resulting identity by $-\nabla^l\phi$ and integrating over $\mathbb{R}^3$, by the integration by parts and the Poisson equation $(4.1)_5$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \frac{2}{3}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \frac{1}{3}\left\|\nabla^l\Delta\phi\right\|_{L^2}^2\\
&\quad= -\int\nabla^l(\Delta\phi u)\cdot\nabla^l\nabla\phi - \int\nabla^l\left(\frac{2W\nabla W-2V\nabla V}{3(V+W+3)}\right)\cdot\nabla^l\nabla\phi\\
&\qquad-\int\nabla^l\left(\frac{4W\nabla V-4V\nabla W}{3(V+W+3)}\right)\cdot\nabla^l\nabla\phi - \int\nabla^l\left(\frac{V+W}{3(V+W+3)}\nabla\phi\right)\cdot\nabla^l\nabla\phi\\
&\qquad-\int\nabla^l\left(\frac{V^2-2VW+W^2}{V+W+3}\nabla\phi\right)\cdot\nabla^l\nabla\phi.
\end{aligned} \tag{5.16}$$

For $l=0$, by (5.11), we easily obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\phi\|_{L^2}^2 + \frac{2}{3}\|\nabla\phi\|_{L^2}^2 + \frac{1}{3}\|\Delta\phi\|_{L^2}^2\\
&\quad\lesssim \|\Delta\phi\|_{L^3}\|u\|_{L^6}\|\nabla\phi\|_{L^2} + \|(V,W)\|_{L^\infty}\|\nabla(V,W)\|_{L^2}\|\nabla\phi\|_{L^2}\\
&\qquad+\|(V,W)\|_{L^\infty}\|\nabla\phi\|_{L^2}^2 + \|(V,W)\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2\\
&\quad\lesssim \delta_0\|\nabla\phi\|_{L^2}^2 + \delta_0\|\nabla(u,V,W)\|_{L^2}^2.
\end{aligned} \tag{5.17}$$

Since δ_0 is small, we deduce from (5.17) that there exists some constant $\alpha>0$ such that

$$\frac{d}{dt}\|\nabla\phi\|_{L^2}^2 + \alpha\|\nabla\phi\|_{L^2}^2 \leqslant C\|\nabla(u,V,W)\|_{L^2}^2. \tag{5.18}$$

Integrating (5.18) in time and using (5.11) and $(4.1)_5$, we obtain

$$\|\nabla\phi(t)\|_{L^2}^2 + \alpha\int_0^t\|\nabla\phi(\tau)\|_{L^2}^2\, d\tau \leqslant \|\nabla\phi_0\|_{L^2}^2 + C\delta_0^2. \tag{5.19}$$

For $1\leqslant l\leqslant k$, by the product estimates (A.11) and (5.11), we could bound the RHS of (5.16) by

$$(\delta_0+\varepsilon)\left(\left\|\nabla^l\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right) + C_\varepsilon\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2$$

for any $\varepsilon > 0$, where we have used $\|\nabla\phi(t)\|_{L^2} \leqslant C_0$ from (5.19). Then, taking ε properly small, since δ_0 is small, we obtain for $1 \leqslant l \leqslant k$ and some constant $\alpha > 0$,

$$\frac{d}{dt}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \alpha\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 \leqslant C\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2. \tag{5.20}$$

By (1.11) with $\ell = k$, we have

$$\int_0^t \|\nabla(u,V,W)(\tau)\|_{H^k}^2\, d\tau \leqslant C_0. \tag{5.21}$$

Integrating (5.20) in time and using (5.21) and $(4.1)_5$, we obtain

$$\left\|\nabla^l\nabla\phi(t)\right\|_{L^2}^2 + \alpha\int_0^t \left\|\nabla^l\nabla\phi(\tau)\right\|_{L^2}^2\, d\tau \leqslant \left\|\nabla^l\nabla\phi_0\right\|_{L^2}^2 + C_0. \tag{5.22}$$

So, the estimate (5.12) follows from (5.19) and (5.22). And the estimate (5.13) follows from (5.18) and (5.20) by Gronwall's inequality. If $\|\nabla^{-1}(V_0 - W_0)\|_{L^2}$ is sufficiently small, then the estimate (5.19) implies (5.14). ∎

Next, we derive the energy estimates of ℓ-level.

LEMMA 5.2. *Let $k \geqslant 3$ and $0 \leqslant \ell \leqslant k-1$. If $\|\nabla^{-1}(V_0 - W_0)\|_{L^2}$ is sufficiently small, then there exists a functional $\mathcal{E}_\ell^k(t)$ which is equivalent to $\|\nabla^\ell(U,\nabla\phi)\|_{H^{k-\ell}}^2$ such that*

$$\frac{d}{dt}\mathcal{E}_\ell^k(t) + \left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2 + \left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2 + \left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2 \leqslant 0. \tag{5.23}$$

Proof. Let $k \geqslant 3$. For the case $\ell = 0$, the inequality (5.23) is trivial by (4.35), (5.18), and (5.20). Next, we will prove (5.23) for $1 \leqslant \ell \leqslant k-1$. We notice

$$\begin{aligned}
&\int -\nabla^\ell \operatorname{div}\left(\frac{V\nabla\phi}{V+W+3}\right)\cdot\nabla^\ell V \\
&\quad = \int \nabla^\ell\left(\frac{V\nabla\phi}{V+W+3}\right)\cdot\nabla^{\ell+1}V \\
&\quad \lesssim \left(\left\|\nabla^\ell\left(\frac{1}{V+W+3}\right)\right\|_{L^6}\|V\nabla\phi\|_{L^3} \right.\\
&\qquad \left. + \left\|\frac{1}{V+W+3}\right\|_{L^\infty}\left\|\nabla^\ell(V\nabla\phi)\right\|_{L^2}\right)\left\|\nabla^{\ell+1}V\right\|_{L^2} \\
&\quad \lesssim (\delta_0 + \tilde{\delta})\left(\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^{\ell+1}V\right\|_{L^2}^2\right),
\end{aligned} \tag{5.24}$$

where we have used $\|\nabla\phi(t)\|_{L^2} < \tilde{\delta}$. Then, the estimate of the form of (4.23) follows. Thus, the terms $\|\nabla^k(V,W)\|_{L^2}^2$ do not appear on the RHS of the estimates (4.14) with $\ell = k$ in Lemma 4.2, i.e.,

$$\begin{aligned}
&\frac{d}{dt}\left\|\nabla^k U\right\|_{L^2}^2 + C\left\|\nabla^{k+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^k\Delta\phi\right\|_{L^2}^2 \\
&\quad \lesssim (\delta_0 + \tilde{\delta})\left(\left\|\nabla^k\varrho\right\|_{L^2}^2 + \left\|\nabla^k\nabla\phi\right\|_{L^2}^2\right).
\end{aligned} \tag{5.25}$$

And in the proof of Lemma 4.2, using some different interpolation estimates, we easily obtain for $1 \leqslant l \leqslant k-1$,

$$\frac{d}{dt}\left\|\nabla^l U\right\|_{L^2}^2 + C\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^l \Delta\phi\right\|_{L^2}^2 \lesssim (\delta_0+\tilde{\delta})\left(\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 + \left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right). \tag{5.26}$$

By (5.20) and (5.25)–(5.26), since $\delta_0, \tilde{\delta}$ are small, we easily obtain for $1 \leqslant l \leqslant k-1$,

$$\frac{d}{dt}\left\|\nabla^l(U,\nabla\phi)\right\|_{L^2}^2 + C\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^l\nabla\phi\right\|_{H^1}^2 \lesssim (\delta_0+\tilde{\delta})\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \tag{5.27}$$

and

$$\frac{d}{dt}\left\|\nabla^k(U,\nabla\phi)\right\|_{L^2}^2 + C\left\|\nabla^{k+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^k\nabla\phi\right\|_{H^1}^2 \lesssim (\delta_0+\tilde{\delta})\left\|\nabla^k\varrho\right\|_{L^2}^2. \tag{5.28}$$

Summing up the estimates (5.27) from $l=\ell$ to $k-1$ and then adding it to (5.28), we obtain

$$\frac{d}{dt}\left\|\nabla^\ell(U,\nabla\phi)\right\|_{H^{k-\ell}}^2 + C_1\left(\left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2 + \left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2\right) \leqslant C_2(\delta_0+\tilde{\delta})\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2. \tag{5.29}$$

On the other hand, Lemma 4.3 gives for $1 \leqslant l \leqslant k-1$,

$$\frac{d}{dt}\int \nabla^l u\cdot\nabla\nabla^l\varrho + C\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \lesssim \left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + \left\|\nabla^l\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^{l+2}u\right\|_{L^2}^2. \tag{5.30}$$

Summing up the estimates (5.30) from $l=\ell$ to $k-1$, we obtain

$$\frac{d}{dt}\sum_{\ell\leqslant l\leqslant k-1}\int \nabla^l u\cdot\nabla\nabla^l\varrho + C_3\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2 \leqslant C_4\left(\left\|\nabla^{\ell+1}u\right\|_{H^{k-\ell}}^2 + \left\|\nabla^{\ell+1}(V,W)\right\|_{H^{k-\ell-1}}^2 + \left\|\nabla^\ell\Delta\phi\right\|_{H^{k-\ell-1}}^2\right). \tag{5.31}$$

Multiplying (5.31) by $2C_2(\delta_0+\tilde{\delta})/C_3$ and adding it to (5.29), since $\delta_0, \tilde{\delta}$ are small, we deduce that there exists a constant $c>0$ such that for $1 \leqslant \ell \leqslant k-1$,

$$\frac{d}{dt}\left(\left\|\nabla^\ell(U,\nabla\phi)\right\|_{H^{k-\ell}}^2 + \frac{2C_2(\delta_0+\tilde{\delta})}{C_3}\sum_{\ell\leqslant l\leqslant k-1}\int\nabla^l u\cdot\nabla\nabla^l\varrho\right) + c\left(\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2 + \left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2 + \left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2\right) \leqslant 0. \tag{5.32}$$

By defining $\mathcal{E}_\ell^k(t)$ to be c^{-1} times the expression under the time derivative in (5.32), we obtain (5.23). ◻

Finally, we show the evolution of the negative Besov norms of $U(t)$.

Lemma 5.3. *For $s\in(0,3/2]$, we have*

$$\|U(t)\|_{\dot{B}^{-s}_{2,\infty}} \leqslant C_0. \tag{5.33}$$

Proof. Similar to [73, Theorem 1.2, (1.8)], we omit the details. ◻

5.2.2. Proof of the time-decay rates. In this subsection, we will prove the time-decay rates by using the previous estimates.

By Lemma A.7 and (5.33) of Lemma 5.3, we have for $0 \leqslant \ell \leqslant k-1$,

$$\left\|\nabla^{\ell} U\right\|_{L^2} \leqslant \|U\|_{\dot{B}^{-s}_{2,\infty}}^{\frac{1}{\ell+1+s}} \left\|\nabla^{\ell+1} U\right\|_{L^2}^{\frac{\ell+s}{\ell+1+s}} \leqslant C_0 \left\|\nabla^{\ell+1} U\right\|_{L^2}^{\frac{\ell+s}{\ell+1+s}}. \tag{5.34}$$

This together with (1.11) implies that for $0 \leqslant \ell \leqslant k-1$,

$$\begin{aligned} &\left\|\nabla^{\ell+1}\varrho\right\|^2_{H^{k-\ell-1}} + \left\|\nabla^{\ell+1}(u,V,W)\right\|^2_{H^{k-\ell}} + \left\|\nabla^{\ell}\nabla\phi\right\|^2_{H^{k-\ell+1}} \\ &\quad \geqslant C_0 \left(\left\|\nabla^{\ell}(U,\nabla\phi)\right\|^2_{H^{k-\ell}}\right)^{1+\frac{1}{\ell+s}}. \end{aligned} \tag{5.35}$$

In view of (5.23) of Lemma 5.2 and (5.35), we obtain for $0 \leqslant \ell \leqslant k-1$,

$$\frac{d}{dt}\mathcal{E}^k_\ell(t) + C_0\left(\mathcal{E}^k_\ell(t)\right)^{1+\frac{1}{\ell+s}} \leqslant 0. \tag{5.36}$$

We solve the above inequality directly to obtain for $0 \leqslant \ell \leqslant k-1$,

$$\mathcal{E}^k_\ell(t) \leqslant C_0(1+t)^{-\ell-s}, \tag{5.37}$$

which implies (1.12). Then, (5.13) of Lemma 5.1 and (1.12) give the higher time-decay (1.13) for the electric field. So, the proof of time-decay rates is completed.

Appendix A. Some analytic remarks. In the appendix, we list some lemmas which are often used in the previous sections. We first recall the Gagliardo–Nirenberg inequality of Sobolev type.

LEMMA A.1. *Let $2 \leqslant p \leqslant \infty$ and $\alpha, \beta, \gamma \geqslant 0$. Then we have*

$$\|\nabla^{\alpha} f\|_{L^p} \lesssim \left\|\nabla^{\beta} f\right\|_{L^2}^{1-\theta} \|\nabla^{\gamma} f\|_{L^2}^{\theta}. \tag{A.1}$$

Here $0 \leqslant \theta \leqslant 1$ (if $p=\infty$, then we require that $0<\theta<1$) and α satisfy

$$\alpha + 3\left(\frac{1}{2}-\frac{1}{p}\right) = \beta(1-\theta) + \gamma\theta. \tag{A.2}$$

Proof. We omit the details by referring to [60, Theorem, p. 125] or [30, Lemma A.1]. ∎

LEMMA A.2. *Assume that $\|V\|_{L^\infty} \leqslant 1$ and $\|W\|_{L^\infty} \leqslant 1$. Let $g(V,W)$ be a smooth function of n, θ with bounded derivatives of any order, and then for any integer $k \geqslant 1$ and $2 \leqslant p \leqslant \infty$, we have*

$$\left\|\nabla^k(g(V,W))\right\|_{L^p} \lesssim \left\|\nabla^k V\right\|_{L^p} + \left\|\nabla^k W\right\|_{L^p}. \tag{A.3}$$

Proof. For $k \geqslant 1$, we have

$$\begin{aligned} \nabla^k(g(V,W)) = &\sum_{\alpha_1+\cdots+\alpha_j=k} \partial^{\alpha_1,\ldots,\alpha_j} g(V,W)\nabla^{\alpha_1}V\nabla^{\alpha_2}V\cdots\nabla^{\alpha_j}V \\ &+ \sum_{\beta_1+\cdots+\beta_m=k} \partial^{\beta_1,\ldots,\beta_m} g(V,W)\nabla^{\beta_1}W\nabla^{\beta_2}W\cdots\nabla^{\beta_m}W \\ &+ \sum_{\gamma_1+\cdots+\gamma_l=k} \partial^{\gamma_1,\ldots,\gamma_l} g(V,W)\nabla^{\gamma_1}V\cdots\nabla^{\gamma_s}V\nabla^{\gamma_{s+1}}W\cdots\nabla^{\gamma_l}W \\ := &\ Q_1+Q_2+Q_3. \end{aligned} \tag{A.4}$$

By Hölder's and Gagliardo–Nirenberg's inequalities, we obtain

$$\begin{aligned}\|Q_1\|_{L^p} &\lesssim \|\nabla^{\alpha_1} V \nabla^{\alpha_2} V \cdots \nabla^{\alpha_j} V\|_{L^p} \\ &\lesssim \|\nabla^{\alpha_1} V\|_{L^{\frac{kp}{\alpha_1}}} \|\nabla^{\alpha_2} V\|_{L^{\frac{kp}{\alpha_2}}} \cdots \|\nabla^{\alpha_j} V\|_{L^{\frac{kp}{\alpha_j}}} \\ &\lesssim \|V\|_{L^\infty}^{1-\frac{\alpha_1}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\alpha_1}{k}} \|V\|_{L^\infty}^{1-\frac{\alpha_2}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\alpha_2}{k}} \cdots \|V\|_{L^\infty}^{1-\frac{\alpha_j}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\alpha_j}{k}} \\ &\lesssim \|V\|_{L^\infty}^{j-1} \left\|\nabla^k V\right\|_{L^p}. \end{aligned} \tag{A.5}$$

Similarly, we have

$$\|Q_2\|_{L^p} \lesssim \|W\|_{L^\infty}^{m-1} \left\|\nabla^k W\right\|_{L^p}. \tag{A.6}$$

For the third term Q_3, by Hölder's, Gagliardo–Nirenberg's, and Young's inequalities, we obtain

$$\begin{aligned}\|Q_3\|_{L^p} &\lesssim \|\nabla^{\gamma_1} V \cdots \nabla^{\gamma_s} V \nabla^{\gamma_{s+1}} W \cdots \nabla^{\gamma_l} W\|_{L^p} \\ &\lesssim \|\nabla^{\gamma_1} V\|_{L^{\frac{kp}{\gamma_1}}} \cdots \|\nabla^{\gamma_s} V\|_{L^{\frac{kp}{\gamma_s}}} \|\nabla^{\gamma_{s+1}} W\|_{L^{\frac{kp}{\gamma_{s+1}}}} \cdots \|\nabla^{\gamma_l} W\|_{L^{\frac{kp}{\gamma_l}}} \\ &\lesssim \|V\|_{L^\infty}^{1-\frac{\gamma_1}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_1}{k}} \cdots \|V\|_{L^\infty}^{1-\frac{\gamma_s}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_s}{k}} \\ &\quad \cdot \|W\|_{L^\infty}^{1-\frac{\gamma_{s+1}}{k}} \left\|\nabla^k W\right\|_{L^p}^{\frac{\gamma_{s+1}}{k}} \cdots \|W\|_{L^\infty}^{1-\frac{\gamma_l}{k}} \left\|\nabla^k W\right\|_{L^p}^{\frac{\gamma_l}{k}} \\ &\lesssim \|V\|_{L^\infty}^{s-\frac{\gamma_1+\cdots+\gamma_s}{k}} \left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_1+\cdots+\gamma_s}{k}} \|W\|_{L^\infty}^{l-s-\frac{\gamma_{s+1}+\cdots+\gamma_l}{k}} \left\|\nabla^k W\right\|_{L^p}^{\frac{\gamma_{s+1}+\cdots+\gamma_l}{k}} \\ &\lesssim \|(V,W)\|_{L^\infty}^{l-1} \left(\left\|\nabla^k V\right\|_{L^p} + \left\|\nabla^k W\right\|_{L^p}\right). \end{aligned} \tag{A.7}$$

Substituting (A.5)–(A.7) into (A.4), we deduce (A.3) since $\|n\|_{L^\infty} \leqslant 1$ and $\|W\|_{L^\infty} \leqslant 1$. ∎

As a byproduct of Lemma A.2, we immediately have the following.

COROLLARY A.3. *Assume that $\|\varrho\|_{L^\infty} \leqslant 1$. Let $g(\varrho)$ be a smooth function of n with bounded derivatives of any order, and then for any integer $k \geqslant 1$ and $2 \leqslant p \leqslant \infty$, we have*

$$\left\|\nabla^k(g(\varrho))\right\|_{L^p} \lesssim \left\|\nabla^k \varrho\right\|_{L^p}. \tag{A.8}$$

We then recall the following commutator and product estimates.

LEMMA A.4. *Let $l \geqslant 1$ be an integer and define the commutator*

$$\left[\nabla^l, g\right] h = \nabla^l(gh) - g\nabla^l h. \tag{A.9}$$

Then we have

$$\left\|\left[\nabla^l, g\right] h\right\|_{L^{p_0}} \lesssim \|\nabla g\|_{L^{p_1}} \left\|\nabla^{l-1} h\right\|_{L^{p_2}} + \left\|\nabla^l g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}. \tag{A.10}$$

In addition, we have that for $l \geqslant 0$,

$$\left\|\nabla^l(gh)\right\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \left\|\nabla^l h\right\|_{L^{p_2}} + \left\|\nabla^l g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}. \tag{A.11}$$

In the above, $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ such that

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. Referring to [55, Lemma 3.4, p. 129], we give a complete and simple proof in the following. We first prove (A.11). Let $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ such that

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Assume $\ell = 0, 1, \ldots, l$. We choose q_1, q_2 by

$$\frac{1}{q_1} = \frac{1}{p_1}\left(1 - \frac{\ell}{l}\right) + \frac{1}{p_3}\frac{\ell}{l}, \quad \frac{1}{q_2} = \frac{1}{p_2}\left(1 - \frac{\ell}{l}\right) + \frac{1}{p_4}\frac{\ell}{l}.$$

Thus, we have

$$\frac{1}{p_0} = \frac{1}{q_1} + \frac{1}{q_2}.$$

By Hölder's, Gagliardo–Nirenberg's, and Young's inequalities, we have for $l \geqslant 0$,

$$\begin{aligned}
\left\|\nabla^l (gh)\right\|_{L^{p_0}} &= \left\|\sum_{\ell=0}^{l} \nabla^\ell g \nabla^{l-\ell} h\right\|_{L^{p_0}} \\
&\lesssim \sum_{\ell=0}^{l} \left\|\nabla^\ell g\right\|_{L^{q_1}} \left\|\nabla^{l-\ell} h\right\|_{L^{q_2}} \\
&\lesssim \|g\|_{L^{p_1}}^{1-\frac{\ell}{l}} \left\|\nabla^l g\right\|_{L^{p_3}}^{\frac{\ell}{l}} \|h\|_{L^{p_4}}^{\frac{\ell}{l}} \left\|\nabla^l h\right\|_{L^{p_2}}^{1-\frac{\ell}{l}} \\
&= \left(\|g\|_{L^{p_1}} \left\|\nabla^l h\right\|_{L^{p_2}}\right)^{1-\frac{\ell}{l}} \left(\left\|\nabla^l g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}\right)^{\frac{\ell}{l}} \\
&\lesssim \|g\|_{L^{p_1}} \left\|\nabla^l h\right\|_{L^{p_2}} + \left\|\nabla^l g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}. \qquad \text{(A.12)}
\end{aligned}$$

Note that for $l \geqslant 1$,

$$\left[\nabla^l, g\right] h = \sum_{\ell=1}^{l} \nabla^\ell g \nabla^{l-\ell} h.$$

We can prove (A.10) in the same way as (A.12). ∎

Now, we introduce the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3) (s \in \mathbb{R}, 1 \leqslant p, r \leqslant \infty)$; cf. [3].

DEFINITION A.5. *Let $\phi \in C_0^\infty(\mathbb{R}^3_\xi)$ be such that $\phi(\xi) = 1$ when $|\xi| \leqslant 1$ and $\phi(\xi) = 0$ when $|\xi| \geqslant 2$. Let $\varphi(\xi) = \phi(\xi) - \phi(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then by the construction, $\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1$ if $\xi \neq 0$.*

Then for $s \in \mathbb{R}$ and $1 \leqslant p, r \leqslant \infty$, we define the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}^s_{p,r}}$ defined by

$$\|f\|_{\dot{B}^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{rsj} \|\dot{\Delta}_j f\|_{L^p}^r\right)^{\frac{1}{r}},$$

where $\dot{\Delta}_j f := \mathscr{F}^{-1}(\varphi_j) * f$. Particularly, if $r = \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{sj} \left\|\dot{\Delta}_j f\right\|_{L^p}.$$

We have the following L^p embedding lemmas.

LEMMA A.6. *Let $1 \leqslant p < 2$, and $1/2 + s/3 = 1/p$, and then $0 < s \leqslant 3/2$ and*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}. \tag{A.13}$$

Proof. See [72, Lemma 4.1]. ∎

We will give the special interpolation estimate.

LEMMA A.7. *Let $k \geqslant 0$ and $s > 0$, and then we have*

$$\left\|\nabla^k f\right\|_{L^2} \lesssim \left\|\nabla^{k+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta}, \text{ where } \theta = \frac{1}{\ell + 1 + s}. \tag{A.14}$$

Proof. We refer to [72, Lemma 4.2] by noting that $\dot{B}^{-s}_{2,p} \subset \dot{B}^{-s}_{2,q}$ for $p \leqslant q$. ∎

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