On transport of ionic solutions: from kinetic laws to continuum descriptions

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Abstract

The Poisson–Nernst–Planck (PNP) system is a conventional macroscopic continuum model to describe the transport and distribution of ionic species in different media and solvents. In order to justify such a model for dilute solutions of multi-species charged particles, rather than employing the spatial coarse graining (averaging) we study a diffusion limit of Vlasov–Poisson–Fokker–Planck (VPFP) systems on a bounded domain with reflection boundary conditions of charge distributions. Here the VPFP system has a small parameter coming from the hypotheses of the scaled thermal velocity and mean free path of charged particles. Under the global neutrality assumption, we prove that as the small parameter tends to zero, solutions of VPFP systems converge to a global weak solution of the PNP system. The arguments use the renormalization techniques and the results support the PNP system as a model of multi-species charged particles.

Keywords: Ionic solutions, Diffusion limit, Vlasov–Poisson–Fokker–Planck system, Poisson–Nernst–Planck system, renormalized solutions.
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1 Introduction

The studies of solutions with charged-particles have attracted more and more attentions recently. They play crucial roles in many physical and biological problems, such as physical plasma [4], semiconductors [49, 50, 55, 56], electro-kinetic fluids [70, 71, 77] and ion channels in cell membranes [29,43,44,64,65]. The multiscale-multiphysics nature of these problems is closed tied to specific physical situations and applications.

Ionic fluids and the transport of ions through different biological environments, such as those in our cells, tissues and organs, are responsible to or involved in almost all biological activities in our life and also many different diseases [72]. These ionic solutions are mixtures consisting

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of 'bio-ions' (e.g., sodium Na^+, potassium K^+, calcium Ca^{2+} and chloride Cl^-), along with possibly many other charged components (e.g., bicarbonate HCO_3^- and proton H^+). These solutions are very different from the pure water and have dramatic effects on the cells and molecules of biological systems.

Classical theories of ideal models has been employed (consciously and in many times, unconsciously) by chemists and biologists to investigate the properties of ionic solutions, which, however, have evident differences in density, charge, and interactions. While the electrostatic interactions between particles in dilute (e.g., $< 0.3mM$) ionic solutions in the equilibrium state can be approximated by the Poisson–Boltzmann (PB) equation [42, 53, 69], the electrolyte solutions even those as dilute and nearly ideal as $1mM$ NaCl are dominated by the electrical interactions called shielding or screening [29–31]. Notice that the salt concentration of our body is around $500mM$ (like the sea ocean).

For the time evolution of ionic transport in solutions, one of the fundamental macroscopic models is the so-called Poisson–Nernst–Planck (PNP) system [23, 31, 32, 62, 67], which consists of coupled diffusion–convection equations. From the traditional view of nonlinear PDE theories, the PNP system is often referred to as a drift–diffusion–Poisson (DDP) system and it has been extensively used in the study of transport of carriers in semiconductors [36, 50, 55, 56]. The multispecies PNP system is a coupled system of parabolic-elliptic equations for the densities c_i $(i = 1, 2, ..., n)$ of charged particles in the solution and the self-consistent electric potential ϕ

$$\begin{cases} \partial_t c_i = \nabla \cdot J_i, \\ J_i = d_i \left(\nabla c_i + \dfrac{q_i}{k_B T} c_i \nabla \phi \right), \\ -\nabla \cdot (\epsilon \nabla \phi) = \sum_{i=1}^n q_i c_i + D(x), \end{cases} \tag{1.1}$$

where $D(x)$ is the permanent (fixed) charge density in the domain, q_i are the charges (positive or negative) of particles, J_i are the ionic flux densities, d_i are their diffusion coefficients, ϵ is the dielectric coefficient, k_B is the Boltzmann constant and T is the temperature. The PNP system associated with proper initial and boundary conditions has been extensively studied in the literature, we refer to [2, 6, 7, 34, 36, 37, 49, 56] and the reference cited therein.

It should be pointed out that the PNP system can be viewed as a special case of general diffusion which involves the nonlocal particle interactions (the Coulomb interactions). It can be reformulated and derived from the general energetic variational approaches (cf. e.g., [76]). In particular, the approaches allow one to include other physical ingredients, such as ion size (steric) effects, in the models for non-ideal, non-diluted solutions (cf. [54]).

The PNP system (1.1) provides a continuum description of the evolution of charged particles. It can be regarded as the one via macroscopic (averaged) quantities of an ensemble of charged particles in a spacial domain, e.g., the particle density, the particle current density etc. It is basically the coupling between the mass conservation laws with force balance equations. One of the advantages for continuum models is the cheaper cost for numerics. On the other hand, it would be a challenging task to predict the correct macroscopic description of the microscopic attributes, especially when nontrivial interactions between particles are considered. For this reason, continuum models often involve phenomenological assumptions (that is, written directly

at the continuum level rather than derived from their discrete counterparts) based on ideal situations or by making assumptions that cannot be related to individual behavior of particles [9].

The kinetic theories provide an alternative way to describe the motion of (charged) particles. In these approaches, the state of charged particles is given by a distribution function $f(t, x, v)$, i.e. a probability density in the (x, v)-phase space at time t. The distribution function contains immense amount of information of the particles so that we can use it to calculate macroscopic properties. For the collisionless (dilute) plasma, if we assume that the motion is governed by an external electric field, the classical Vlasov equation can be derived from the Liouville equation [17]. Suppose that the motion is governed by the (self-consistent) Coulomb field generated by the plasma itself, then we arrive at the Vlasov–Poisson (VP) system. Besides, if the magnetic effect is also considered, we have the Vlasov–Maxwell system [56]. The Vlasov equation does not account for short-range particle interactions, like collisions of the particles with others or with the crystal lattice. If collisions between particles are taken into account, proper collision operator should be introduced into the system and we arrive at the Boltzmann equation (and furthermore, the Vlasov–Poisson–Boltzmann (VPB) system [59,61]). If the collisions between the charged particles are approximated by their Brownian motions modeled by the Fokker–Planck term [22], then the evolution of particles can be described by the Vlasov–Poisson–Fokker–Planck (VPFP) system, on which we shall have the main interest in this paper. There are many works dealing with the existence and uniqueness of solutions to the initial value problem or the initial boundary value problem of the VPFP system. We refer to [11, 68, 75] for the classical solutions and to [12, 14, 15, 74] for weak solutions and their regularity. Concerning the long-time behavior of the VPFP system, we refer to [10, 13, 16]. Nevertheless, mathematical analysis and numerical simulations of the kinetic equations are usually very difficult due to the complex structure of the collision operator and the high number of independent variables (e.g., three position plus three wave vector plus one time variable). The relations to the continuum (macroscopic) models and the coupling between them are important and desirable in practics. Moreover, these will also be the keys to develop multiscale models and numerical algorithms for more realistic situations.

The continuum models can be (formally) derived from the kinetic models by several coarse graining methods, for instance, the moment method (which assume independence between individuals at some stage), or the Hilbert expansion method etc [49, 56, 58]. On the other hand, rigorous derivation of the continuum models as suitable hydrodynamical limits from certain kinetic equations has been investigated by many researchers [33, 38, 39, 57, 63, 66]. Diffusion limit of the VPFP system has been extensively studied in the literature, see [33, 38, 39, 63, 66] and the references therein. In these works, VPFP system for a single species of particles was considered such that only the evolution of the negative particles (electrons) is described in terms of a distribution function in phase space, while the positive charged particles (ions) are supposed to be static, namely, their charge and current are given functions, due to their (relatively) very heavy mass. In [33,38,66], the authors proved the convergence of suitable solutions to the VPFP system towards a solution to the (one species) DDP model, by taking the so-called parabolic limit (or low-field limit). More precisely, in [66], under a suitable regularity assumption on the initial data, the convergence result was obtained in two dimension and locally in time for the three dimensional case. Later in [38], the author proved a global convergence result in two di-

mensional case, without any restriction on the time interval and the assumptions on the initial data were weakened with only bounds on entropy and energy. Recently, the previous results were generalized by [33], where the authors established a global convergence result, without any restriction on the time interval or on the spatial dimension. They achieved their goal by working with the renormalized solutions (or free energy solutions) in the terminology of [26,28]. As pointed in [33], the notion of renormalized solutions is natural for the problem, because it seems that the free energy of the VPFP system is the only quantity that is uniformly bounded with respect to the parameter ε (related to the mean free path, which will tend to zero in the diffusion limit). Even one works with more regular initial data such that the solutions can be defined in the weak sense without the need of renormalizing, one still has to use renormalization techniques to pass to the limit. Besides, the use of renormalization techniques together with an averaging lemma helps to remove the restriction on spatial dimensions and treat the nonlinear term $\nabla_x\phi \cdot \nabla_v f$, where the main difficulty comes from (cf. [33] for more details).

The work in this paper had been motivated by our current study of ions passing through ion channels (transmembrane proteins) [31,45–47,54,76]. While the geometric sizes of the channels are extremely small (comparable to the ion sizes), the measured results in the experiments are all in 10^{-3} seconds, a very long time for single particle transports. Although the PNP systems had been extensively applied successfully, it lacks the justification from the microscopic description of ionic particles. In this paper, we prove rigorously that for the multi-species cases, the PNP system (1.1) is a diffusion limit of the VPFP system (2.4)–(2.6) as the small renomalized parameter ε tends to zero. We generalized the techniques introduced previously by other work, such as those in [33], to cases involving multiple species of charged particles in a bounded region with reflection boundary conditions [14,19,61] that also recover the common used no-flux boundary conditions of the PNP system. Different from the single species case in the literature, the previous arguments have to be modified in order to deal with the nonlocal interactions between different species of particles through the Poisson equation for electric potential ϕ. Furthermore, in the biological systems, the positively charged particles (e.g., Na^+, K^+, Ca^{2+}) and negative charged particles (e.g., Cl^-) have comparable but different masses, valencies. The effects of these differences will become obvious in our mathematical analysis.

The remaining part of the paper is organized as follows. In Section 2, we describe the multi-species VFFP system with proper boundary conditions and perform a suitable non-dimensionless analysis. In Section 3, we present the definition of renormalized solutions and derive the energy dissipation of the VFFP system (Proposition 3.1), which yields the necessary uniform estimates needed in Section 4 (Lemmas 3.1, 4.1, 4.2). In the final Section 4, we prove the main result of the paper on the diffusion limit (Theorem 4.1).

2 Description of the problem

Without loss of generality, in the remaining part of this paper, we shall consider two species of charged particles (with opposite sign of electric charges). The general multi-species case can be treated in a similar way.

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a sufficiently smooth bounded domain. For instance, the outward

unit normal vector $\mathbf{n}(x)$ at $x \in \partial\Omega$ satisfies $\mathbf{n} \in W^{2,\infty}(\Omega, \mathbb{R}^d)$. The functions $f(t,x,v) \geq 0$, $g(t,x,v) \geq 0$ denote the scalar distribution of negative/positive particles (e.g., $f dx dv$ is the number of negatively charged particles at time t located at a volume element dx about the position x and having velocities in a volume dv about the value v). We consider a dilute ionic solution such that the evolution of distribution functions of negative/positive charged particles are subject to the electrostatic force coming from their (self-consistent) Coulomb interaction. The electrostatic force is responsible for the self-consistent force term where ϕ solves the Poisson equation. The low density of the particles implies that their collisions with one another may be neglected. Besides, we make the assumption that their momentum changes little when colliding with the particles of the environment. Then the collision term in the kinetic equation may be approximated by using the Brownian force modeled by the Fokker–Planck term. As a consequence, we consider the following VPFP system on $(0,T) \times \Omega \times \mathbb{R}^d$, $T > 0$:

$$\partial_t f + v \cdot \nabla_x f - \frac{z_f q}{m_f} \nabla_x \phi \cdot \nabla_v f = \frac{1}{\tau_f} \mathcal{L}^f_{FP}(f), \tag{2.1}$$

$$\partial_t g + v \cdot \nabla_x g - \frac{z_g q}{m_g} \nabla_x \phi \cdot \nabla_v g = \frac{1}{\tau_g} \mathcal{L}^g_{FP}(g), \tag{2.2}$$

$$-\epsilon_0 \Delta_x \phi = q \left(z_f \int_{\mathbb{R}^d} f dv + z_g \int_{\mathbb{R}^d} g dv + D(x) \right), \tag{2.3}$$

where $\mathcal{L}^i_{FP}$ ($i \in \{f, g\}$) are the Fokker–Planck operators such that

$$\mathcal{L}^f_{FP}(f) = \nabla_v \cdot (vf + \theta_f \nabla_v f), \quad \mathcal{L}^g_{FP}(g) = \nabla_v \cdot (vg + \theta_g \nabla_v g).$$

Here, $\epsilon_0 > 0$ is the vacuum permittivity, $q > 0$ is the positive elementary charge, z_f, z_g are the valencies (which are positive integers for cations and negative integers for anions), m_f, m_g are the masses for the two species of charged particles, τ_f, τ_g are relaxation time due to collisions of the particles with the thermal bath, $\sqrt{\theta_f}, \sqrt{\theta_g}$ are the thermal velocities given by $\sqrt{\theta_i} = \sqrt{2 k_B T_b m_i^{-1}}$, $i \in \{f, g\}$ and T_b is the temperature of the thermal bath. The function $D(x)$ (doping profile) is the density of background charge that is assumed to be fixed in time for the sake of simplicity.

2.1 Dimensionless analysis

We now present the suitable scalings of the VPFP system (2.1)–(2.3). Let L be the characteristic length. Then we introduce a characteristic value for the concentration of the particles N and a characteristic variation of the electric potential Φ_0 over L. We denote the reference magnitude for the drift velocities given by $U = -\tau \frac{q}{m} E$ with $E = -\nabla \phi$. Since we may treat more than one species of charged particles in the ionic solution (e.g., Na^+, K^+, Ca^{2+}, Cl^-) that have different masses and charges, it is convenient to introduce, as the unit, a "reference particle" (for instance, Na^+) with mass m_{ref}, electric charge $z_{ref} q$ (with $z_{ref} = 1$), relaxation time τ_{ref} and thermal velocity θ_{ref}. The microscopic variation as well as the drift velocity for the reference particle are given by

$$V_{ref} = \sqrt{\theta_{ref}}, \quad U_{ref} = \tau_{ref} \frac{q}{m_{ref}} \frac{\Phi_0}{L},$$

respectively.

Concerning the two different species of particles in the ionic solution, we define the ratios

$$\kappa_i = \frac{m_{ref}}{m_i}, \quad \zeta_i = \frac{\tau_{ref}}{\tau_i}, \quad i \in \{f, g\}.$$

Recalling the definition of thermal velocity, we see that

$$\frac{\theta_i}{\theta_{ref}} = \kappa_i, \quad i \in \{f, g\}.$$

The microscopic variations of v for the two species and their drift velocities are given by

$$V_f = \sqrt{\theta_f}, \quad V_g = \sqrt{\theta_g}, \quad U_f = \tau_f \frac{q}{m_f}\frac{\Phi_0}{L}, \quad U_g = \tau_g \frac{q}{m_g}\frac{\Phi_0}{L},$$

respectively. Next, we choose the following scaling (with respect to the reference particle) such that

$$t \to T_0 t', \quad x \to L x', \quad v \to V_{ref} v', \quad \text{with } T_0 = \frac{L}{U_{ref}}.$$

Then we apply the change of unknowns

$$f(t,x,v) = \frac{N}{V_{ref}^d} f'(t',x',v'), \qquad g(t,x,v) = \frac{N}{V_{ref}^d} g'(t',x',v')$$
$$\phi(t,x,v) = \Phi_0 \phi'(t',x',v'), \qquad D(x) = N D'(x').$$

First, we obtain the rescaled Poisson equation (drop the prime)

$$-\varpi \Delta_x \phi = z_f \int_{\mathbb{R}^d} f dv + z_g \int_{\mathbb{R}^d} g dv + D,$$

where ϖ is the dimensionless parameter

$$\varpi = \frac{\epsilon_0 \Phi_0}{qNL^2}.$$

Next, we can write down the rescaled equations for f, g (drop the prime again):

$$f_t + \nu v \cdot \nabla_x f - \frac{\kappa_f z_f}{\varepsilon} \nabla_x \phi \cdot \nabla_v f = \frac{\zeta_f \nu}{\varepsilon} \nabla_v \cdot (vf + \kappa_f \nabla_v f),$$
$$g_t + \nu v \cdot \nabla_x g - \frac{\kappa_g z_g}{\varepsilon} \nabla_x \phi \cdot \nabla_v g = \frac{\zeta_g \nu}{\varepsilon} \nabla_v \cdot (vg + \kappa_g \nabla_v g).$$

where ν (the 'scaled' thermal velocity) and ε (the 'scaled' thermal mean free path) given by

$$\nu = \frac{V_{ref}}{U_{ref}}, \qquad \varepsilon = \frac{\tau_{ref} V_{ref}}{L}$$

are dimensionless parameters. We recall that the (thermal) mean free path of a particle is the average distance the particle travels between collisions with other moving particles, which is usually given by $\tau\sqrt{\theta}$ (cf. e.g., [39]).

In this paper, we will consider the so-called low field scaling, which means that the drift velocity U is small comparing with the thermal velocity V, while the thermal velocity V is small

comparing to the relaxation velocity $L\tau^{-1}$, and the two ratios have the same order of magnitude (cf. [1, 33, 38, 66]):

$$\nu \simeq \varepsilon^{-1} \quad \text{and} \quad \varepsilon << 1.$$

For $\varepsilon > 0$, let $f^\varepsilon(t,x,v) \geq 0$, $g^\varepsilon(t,x,v) \geq 0$ denote the scalar distribution of charged particles. Taking $\nu = \varepsilon^{-1}$ (just for the sake of simplicity, otherwise a finite parameter $\tilde{\nu} = \nu\varepsilon$ will enter the equation, which does not affect the subsequent analysis but leads to different coefficient in the resulting PDE system), we arrive at the rescaled VPFP system under low field scaling, which will be investigated in the remaining part of this paper:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon - \frac{\kappa_f z_f}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v f^\varepsilon = \frac{\zeta_f}{\varepsilon^2} L^f_{FP}(f^\varepsilon), \tag{2.4}$$

$$\partial_t g^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g^\varepsilon - \frac{\kappa_g z_g}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v g^\varepsilon = \frac{\zeta_g}{\varepsilon^2} L^g_{FP}(g^\varepsilon), \tag{2.5}$$

$$-\varpi \Delta_x \phi^\varepsilon = z_f \int_{\mathbb{R}^d} f^\varepsilon(t,x,v) dv + z_g \int_{\mathbb{R}^d} g^\varepsilon(t,x,v) dv + D(x), \tag{2.6}$$

where the rescaled Fokker–Planck operators are given by

$$L^f_{FP}(f) = \nabla_v \cdot (vf + \kappa_f \nabla_v f), \quad L^g_{FP}(g) = \nabla_v \cdot (vg + \kappa_g \nabla_v g). \tag{2.7}$$

Remark 2.1. *We remark that different types of scalings can be chosen for the VPFP system. For instance, if we assume that the drift and thermal velocities are comparable, but both are small comparing with the relaxation velocity $L\tau^{-1}$, e.g.,*

$$\nu = \mathcal{O}(1) \quad \textit{and} \quad \varepsilon << 1,$$

then we arrive at a different rescaled VPFP system

$$\begin{aligned}
&f^\varepsilon_t + v \cdot \nabla_x f^\varepsilon - \frac{\kappa_f z_f}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v f^\varepsilon = \frac{\zeta_f}{\varepsilon} L^f_{FP}(f^\varepsilon), \\
&g^\varepsilon_t + v \cdot \nabla_x g^\varepsilon - \frac{\kappa_g z_g}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v g^\varepsilon = \frac{\zeta_g}{\varepsilon} L^g_{FP}(g^\varepsilon), \\
&-\varpi \Delta_x \phi^\varepsilon = z_f \int_{\mathbb{R}^d} f^\varepsilon dv + z_g \int_{\mathbb{R}^d} g^\varepsilon dv + D.
\end{aligned}$$

This is usually called drift-collision balance scaling or high field scaling in the literature. Taking the hydrodynamic limit as $\varepsilon \to 0$ (the high field limit or the hyperbolic limit), the VPFP system will lead to a (first-order) hyperbolic system for the density of particles coupled with the Poisson equation, cf. e.g., [1, 15, 39, 63].

2.2 Boundary conditions

Next, we describe the boundary conditions for the distribution functions and electric potential on $\partial\Omega$ (in x-space). Recall that $\partial\Omega$ is assumed to be regular enough such that the outward unit normal vector on $\partial\Omega$ satisfies $\mathbf{n}(x) \in W^{2,\infty}(\Omega; \mathbb{R}^d)$.

For the electric potential ϕ^ε, we simply assume the zero-outward electric field condition such that

$$\nabla_x \phi^\varepsilon \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.8}$$

In order to uniquely determine the value of ϕ^ε from the Poisson equation, we require that $\int_\Omega \phi^\varepsilon dx = 0$.

The boundary conditions for the distribution functions that we will consider in this paper allow us to preserve mass conservation and obtain proper energy and entropy balance laws of the VPFP system (2.4)–(2.6). As in [18–20] (see also [10, 61]), we define

$$\Sigma_\pm^x := \{v \in \mathbb{R}^d : \ \pm v \cdot \mathbf{n}(x) > 0\}$$

the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at point $x \in \partial\Omega$. Besides, we denote

$$\Sigma_\pm = \{(x, v) : \ x \in \Omega, \ v \in \Sigma_\pm^x\}.$$

The Lebesgue surface measure on $\partial\Omega$ will be denoted by dS. Let γh be the trace of function h (when this makes sense) and $\gamma_\pm h = \mathbf{1}_{(0,+\infty)\times\Sigma_\pm}\gamma h$. Boundary conditions for the kinetic equations take into account how the particles are reflected by the wall (the boundary $\partial\Omega$) and take the form of integral (balance) relations between the densities of the particles on the outgoing and incoming velocity subsets of the boundary $\partial\Omega$ at a given time [18–20]:

$$\gamma_- f(t, x, v) = \mathcal{R}_x(\gamma_+ f(t, x, v)) \quad \text{on } (0, \infty) \times \Sigma_-.$$

Here, the reflection operator $\mathcal{R}$ is independent of time, local in position but can be either local or nonlocal in velocity. The phenomenological expression of $\mathcal{R}$ was first introduced in [58]. Given $x \in \partial\Omega$ and $t > 0$, we can write the boundary condition into the following integral form (cf. [10]):

$$\gamma_- f(t, x, v) = \int_{\Sigma_+^x} R(t, x; v, v^*)\gamma_+ f(t, x, v^*)dv^*, \quad v \in \Sigma_-^x, \tag{2.9}$$

where R represents the probability that a particle with velocity v^* at time t striking the boundary on x reemerges at the same instant and location with velocity v. We refer to [10] for possible minimal assumptions on R such that (2.9) is well-defined, i.e., R is nonnegative and verifies the normalization condition and the reciprocity principle. Detailed discussions on the boundary conditions can be found in Cercignani's works [18–20].

If we consider $v' = -v$ for any $v \in \Sigma_-^x$ and take $R(t, x; v, v^*) = \delta_{v'}$ being the Dirac measure centered at $v^* = v'$, then we have $\gamma_- f(t, x, v) = \gamma_- f(t, x, -v)$ on Σ_-, which is the classical (local) inverse reflection boundary condition. Similarly, if we take $v' = v - 2(v \cdot \mathbf{n}(x))\mathbf{n}(x)$, then we arrive at the classical (local) specular reflection boundary condition, see [10, 14].

Here, we are more interested in the so-called diffuse reflection according to a Maxwellian profile M with temperature of the thermal bath, which is nonlocal. For our current case, we denote by $M_f(v)$, $M_g(v)$ the Maxwellians for the two species of particles

$$M_i(v) = \frac{1}{(2\pi)^{\frac{d-1}{2}}\kappa_i^{\frac{d+1}{2}}} e^{-\frac{1}{2\kappa_i}|v|^2}, \quad i \in \{f, g\}. \tag{2.10}$$

We note that M_i are chosen as the zeros of the rescaled Fokker–Planck operators L_{FP}^i (2.7), i.e., $L_{FP}^i(M_i) = 0$, $(i \in \{f, g\})$. Then we can choose the special form of R in (2.9) and propose

the following boundary conditions for the distribution functions (cf. [19,61]) such that for given $x \in \partial\Omega$ and $t > 0$,

$$\gamma_- f^\varepsilon = \frac{M_f(v)}{\int_{v\cdot\mathbf{n}(x)<0} |v\cdot\mathbf{n}(x)| M_f(v) dv} \int_{v^*\cdot\mathbf{n}(x)>0} (\gamma_+ f^\varepsilon) v^*\cdot\mathbf{n}(x) dv^*, \quad \text{on } \Sigma_-^x, \tag{2.11}$$

$$\gamma_- g^\varepsilon = \frac{M_g(v)}{\int_{v\cdot\mathbf{n}(x)<0} |v\cdot\mathbf{n}(x)| M_g(v) dv} \int_{v^*\cdot\mathbf{n}(x)>0} (\gamma_+ g^\varepsilon) v^*\cdot\mathbf{n}(x) dv^*, \quad \text{on } \Sigma_-^x. \tag{2.12}$$

Boundary conditions (2.11) and (2.12) are special cases of the so-called Maxwell boundary condition, which combines both local and nonlocal reflections phenomenologically (cf. [61]).

Let us now define the current densities associated to the distribution functions

$$J_f^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} v f^\varepsilon dv, \quad J_g^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} v g^\varepsilon dv. \tag{2.13}$$

Multiplying the boundary conditions (2.11) and (2.12) by $v\cdot\mathbf{n}(x)$ and integrating over Σ_-^x, we easily deduce the macroscopic boundary conditions for the fluxes such that

$$J_f^\varepsilon \cdot \mathbf{n} = J_g^\varepsilon \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.14}$$

which imply that all the particles that reach the boundary are reflected (no particle goes out nor enters in the domain Ω).

3 Renormalized solutions, energy dissipation and uniform estimates

Set functions

$$n^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon(t,x,v) dv, \quad p^\varepsilon(t,x) = \int_{\mathbb{R}^d} g^\varepsilon(t,x,v) dv$$

as the densities of negative and positive charge particles, respectively. In the rest of this paper, we assume the global neutrality:

$$z_f \int_\Omega n^\varepsilon dx + z_g \int_\Omega p^\varepsilon dx + \int_\Omega D(x) dx = 0. \tag{3.1}$$

We impose the initial data for the VPFP system (2.4)–(2.6) (depending on the parameter ε) as follows:

$$f^\varepsilon(t,x,v)|_{t=0} = f_0^\varepsilon(x,v), \quad g^\varepsilon(t,x,v)|_{t=0} = g_0^\varepsilon(x,v)\,. \tag{3.2}$$

According to (3.1), we also suppose the global neutrality assumption for the initial data

$$\int_\Omega \left(z_f \int_{\mathbb{R}^d} f_0^\varepsilon dv + z_g \int_{\mathbb{R}^d} g_0^\varepsilon dv + D(x) \right) dx = 0, \quad \forall\, \varepsilon > 0. \tag{3.3}$$

Besides, we make the following assumptions:

$$f_0^\varepsilon \geq 0, \quad g_0^\varepsilon \geq 0, \tag{3.4}$$

$$\int_\Omega \int_{\mathbb{R}^d} f_0^\varepsilon (1 + |v|^2 + |\log f_0^\varepsilon|) dv dx \leq C_0, \tag{3.5}$$

$$\int_\Omega \int_{\mathbb{R}^d} g_0^\varepsilon (1+|v|^2+|\log g_0^\varepsilon|) dv dx \le C_0, \tag{3.6}$$

$$\|\phi_0^\varepsilon\|_{H^1(\Omega)} \le C_0, \tag{3.7}$$

for some constant $C_0 > 0$ independent of the parameter ε. For the sake of simplicity, we always assume in the remaining part of the paper that the background charge is independent of time and satisfies

$$D(x) \in L^\infty(\Omega).$$

In the spirit of [33, 57], we now introduce the definition of renormalized solutions:

Definition 3.1. *The triple $(f^\varepsilon, g^\varepsilon, \phi^\varepsilon) \in L^\infty(0,T; L^1(\Omega\times\mathbb{R}^d)\times L^1(\Omega\times\mathbb{R}^d)\times H^1(\Omega))$ is a renormalized solution to the VPFP system (2.4)–(2.6) if*

(1) For all functions $\beta_i \in C^2(\mathbb{R})$, $i \in \{f, g\}$ satisfying

$$|\beta_i(s)| \le C(s^{\frac{1}{2}}+1), \quad |\beta_i'(s)| \le C(1+s)^{-\frac{1}{2}}, \quad |\beta_i''(s)| \le C(1+s)^{-1}, \quad s \ge 0,$$

the triple $(\beta_f(f^\varepsilon), \beta_g(g^\varepsilon), \phi^\varepsilon)$ is a weak solution to

$$\varepsilon\partial_t \beta_f(f^\varepsilon) + v\cdot\nabla_x\beta_f(f^\varepsilon) - \kappa_f z_f \nabla_x\phi^\varepsilon\cdot\nabla_v\beta_f(f^\varepsilon) = \frac{\zeta_f}{\varepsilon} L_{FP}^f(f^\varepsilon)\beta_f'(f^\varepsilon), \tag{3.8}$$

$$\varepsilon\partial_t \beta_g(g^\varepsilon) + v\cdot\nabla_x\beta_g(g^\varepsilon) - \kappa_g z_g \nabla_x\phi^\varepsilon\cdot\nabla_v\beta_g(g^\varepsilon) = \frac{\zeta_g}{\varepsilon} L_{FP}^g(g^\varepsilon)\beta_g'(g^\varepsilon), \tag{3.9}$$

$$-\varpi\Delta\phi^\varepsilon = z_f n^\varepsilon + z_g p^\varepsilon + D(x), \tag{3.10}$$

with initial data

$$\beta_f(f^\varepsilon)|_{t=0} = \beta_f(f_0^\varepsilon), \quad \beta_g(g^\varepsilon)|_{t=0} = \beta_g(g_0^\varepsilon) \tag{3.11}$$

and boundary conditions

$$\gamma_-\beta_f(f^\varepsilon) = \frac{M_f(v)}{\int_{v\cdot\mathbf{n}(x)<0}|v\cdot\mathbf{n}(x)|M_f(v)dv}\int_{v\cdot\mathbf{n}(x)>0}\gamma_+\beta_f(f^\varepsilon)v\cdot\mathbf{n}(x)dv, \tag{3.12}$$

$$\gamma_-\beta_g(g^\varepsilon) = \frac{M_g(v)}{\int_{v\cdot\mathbf{n}(x)<0}|v\cdot\mathbf{n}(x)|M_g(v)dv}\int_{v\cdot\mathbf{n}(x)>0}\gamma_+\beta_g(g^\varepsilon)v\cdot\mathbf{n}(x)dv, \tag{3.13}$$

$$\nabla_x\phi^\varepsilon\cdot\mathbf{n} = 0. \tag{3.14}$$

(2) For any $\lambda > 0$, $\theta_{\varepsilon,\lambda} = (f^\varepsilon + \lambda\widetilde{M}_f)^{\frac{1}{2}}$, $\eta_{\varepsilon,\lambda} = (g^\varepsilon + \lambda\widetilde{M}_g)^{\frac{1}{2}}$ satisfy

$$\begin{aligned} &\varepsilon\partial_t\theta_{\varepsilon,\lambda} + v\cdot\nabla_x\theta_{\varepsilon,\lambda} - \kappa_f z_f\nabla_v\cdot(\nabla_x\phi^\varepsilon\theta_{\varepsilon,\lambda}) \\ = \;& \frac{\zeta_f}{2\varepsilon\theta_{\varepsilon,\lambda}}L_{FP}^f(f^\varepsilon) + \frac{z_f\lambda\widetilde{M}_f}{2\theta_{\varepsilon,\lambda}}v\cdot\nabla_x\phi^\varepsilon, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\varepsilon\partial_t\eta_{\varepsilon,\lambda} + v\cdot\nabla_x\eta_{\varepsilon,\lambda} - \kappa_g z_g\nabla_v\cdot(\nabla_x\phi^\varepsilon\eta_{\varepsilon,\lambda}) \\ = \;& \frac{\zeta_g}{2\varepsilon\eta_{\varepsilon,\lambda}}L_{FP}^g(g^\varepsilon) + \frac{z_g\lambda\widetilde{M}_g}{2\eta_{\varepsilon,\lambda}}v\cdot\nabla_x\phi^\varepsilon, \end{aligned} \tag{3.16}$$

where $\widetilde{M}_f$, $\widetilde{M}_g$ are the normalized Maxwellians (comparing with (2.10))

$$\widetilde{M}_i(v) = \Big(\frac{\kappa_i}{2\pi}\Big)^{\frac{1}{2}} M_i(v), \quad i \in \{f, g\} \tag{3.17}$$

such that

$$\int_{\mathbb{R}^d} \widetilde{M}_f(v)dv = \int_{\mathbb{R}^d} \widetilde{M}_g(v)dv = 1.$$

Remark 3.1. *Due to the regularity of renormalized functions β_i, the corresponding boundary conditions* (3.12) *and* (3.13) *for the renormalized distribution functions make sense. We refer to [3, 21, 73] (see also [10, 61]) for more detailed discussions about the traces of distribution functions on the boundary.*

For the sake of simplicity, we introduce the function

$$\mathcal{H}(s) = s\log s, \quad \forall\, s \geq 0.$$

Then the free energy of the VPFP system (2.4)–(2.6) is given by

$$\begin{aligned}\mathcal{E}(t) &= \int_\Omega \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2\kappa_f} f^\varepsilon + \mathcal{H}(f^\varepsilon)\right) dvdx + \int_\Omega \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2\kappa_g} g^\varepsilon + \mathcal{H}(g^\varepsilon)\right) dvdx \\ &\quad + \frac{\varpi}{2}\int_\Omega |\nabla_x \phi^\varepsilon|^2 dx. \end{aligned} \tag{3.18}$$

The entropy productions of the VPFP system are given by

$$\begin{aligned}\mathcal{D}^i(w) &= \int_\Omega \int_{\mathbb{R}^d} (v\sqrt{w} + 2\kappa_i \nabla_v \sqrt{w})^2 dvdx \\ &= 4\int_\Omega \int_{\mathbb{R}^d} \left|\nabla_v \sqrt{w e^{\frac{1}{2\kappa_i}|v|^2}}\right|^2 e^{-\frac{1}{2\kappa_i}|v|^2} dvdx, \quad i \in \{f, g\}. \end{aligned} \tag{3.19}$$

Besides, we introduce the Darrozès–Guiraud information $\mathcal{I}$ (cf. e.g., [25]) for the two species on the boundary such that

$$\mathcal{I}^i(w) = \int_{\Sigma_+^x} \mathcal{H}(w)\, d\mu_x^i - \mathcal{H}\left(\int_{\Sigma_+^x} w d\mu_x^i\right), \quad i \in \{f, g\},$$

where

$$d\mu_x^i(v) = M_i(v)|v \cdot \mathbf{n}(x)|dv, \quad i \in \{f, g\}$$

are probability measures on $\Sigma_\pm^x$ by the particular choice of the normalized Maxwellians M_f, M_g (cf. (2.10)).

First, we state the energy dissipation property of the VPFP system (2.4)–(2.6).

Proposition 3.1. *The renormalized solution of the VPFP system* (2.4)–(2.6) *with described initial data and boundary conditions satisfies*

$$\partial_t n^\varepsilon + \nabla_x \cdot J_f^\varepsilon = 0, \tag{3.20}$$

$$\partial_t p^\varepsilon + \nabla_x \cdot J_g^\varepsilon = 0, \tag{3.21}$$

as well as the free energy inequality (energy dissipation property)

$$\begin{aligned}&\mathcal{E}(t) + \frac{\zeta_f}{\kappa_f \varepsilon^2}\int_0^t \mathcal{D}^f(f^\varepsilon)ds + \frac{\zeta_g}{\kappa_g \varepsilon^2}\int_0^t \mathcal{D}^g(g^\varepsilon)ds \\ &+ \frac{1}{\varepsilon}\int_0^t \int_{\partial\Omega} \mathcal{I}^f\left(\frac{\gamma f_+^\varepsilon}{M_f(v)}\right) dSds + \frac{1}{\varepsilon}\int_0^t \int_{\partial\Omega} \mathcal{I}^g\left(\frac{\gamma g_+^\varepsilon}{M_g(v)}\right) dSds \\ \leq\ &\mathcal{E}(0), \quad \forall\, t \geq 0. \end{aligned} \tag{3.22}$$

Proof. We just present a formal calculation which leads to (3.22). Multiplying the first equation (2.4) of the VPFP system by $\frac{1}{2}|v|^2$ and integrating the result with respect to x and v, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2}|v|^2 f^\varepsilon dvdx + \int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}|v|^2 v\cdot\nabla_x f^\varepsilon dvdx \\
&\quad - \int_\Omega\int_{\mathbb{R}^d}\frac{\kappa_f z_f}{2\varepsilon}|v|^2\nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon dvdx \\
=\;& \int_\Omega\int_{\mathbb{R}^d}\frac{\zeta_f}{2\varepsilon^2}|v|^2 L^f_{FP}(f^\varepsilon)dvdx, \qquad (3.23)
\end{aligned}$$

integrating by parts, we see that

$$\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}|v|^2 v\cdot\nabla_x f^\varepsilon dvdx = \frac{1}{2\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})|v|^2\gamma f^\varepsilon dvdS, \qquad (3.24)$$

$$-\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}|v|^2\nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon dvdx = \frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^d}(v\cdot\nabla_x\phi^\varepsilon)f^\varepsilon dvdx \qquad (3.25)$$

$$\begin{aligned}
=\;& -\int_\Omega\phi^\varepsilon\nabla_x\cdot J^\varepsilon_f dx + \int_{\partial\Omega}\gamma\phi^\varepsilon J^\varepsilon_f\cdot\mathbf{n}dS \\
=\;& \int_\Omega\phi^\varepsilon\partial_t n^\varepsilon dx, \qquad (3.26)
\end{aligned}$$

and

$$\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\varepsilon^2}|v|^2 L^f_{FP}(f^\varepsilon)dvdx = -\frac{1}{\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vf^\varepsilon+\kappa_f\nabla_v f^\varepsilon)\cdot v dvdx. \qquad (3.27)$$

As a result,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\kappa_f}|v|^2 f^\varepsilon dvdx + z_f\int_\Omega\phi^\varepsilon\partial_t n^\varepsilon dx \\
=\;& -\frac{1}{2\kappa_f\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})|v|^2\gamma f^\varepsilon dvdS - \frac{\zeta_f}{\kappa_f\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vf^\varepsilon+\kappa_f\nabla_v f^\varepsilon)\cdot v dvdx. \qquad (3.28)
\end{aligned}$$

In a similar way, we have the equality for g^ε such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\frac{1}{2\kappa_g}|v|^2 g^\varepsilon dvdx + z_g\int_\Omega\phi^\varepsilon\partial_t p^\varepsilon dx \\
=\;& -\frac{1}{2\kappa_g\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})|v|^2\gamma g^\varepsilon dvdS - \frac{\zeta_g}{\kappa_g\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vg^\varepsilon+\kappa_g\nabla_v g^\varepsilon)\cdot v dvdx. \qquad (3.29)
\end{aligned}$$

Next, multiplying the first equation (2.4) of the VPFP system by $\log f^\varepsilon$ and integrating the result with respect to x and v, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\mathcal{H}(f^\varepsilon)dvdx + \int_\Omega\int_{\mathbb{R}^d}\frac{1}{\varepsilon}(v\cdot\nabla_x f^\varepsilon)\log f^\varepsilon dvdx \\
&\quad - \int_\Omega\int_{\mathbb{R}^d}\frac{\kappa_f z_f}{\varepsilon}(\nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon)\log f^\varepsilon dvdx \\
=\;& \int_\Omega\int_{\mathbb{R}^d}\frac{\zeta_f}{\varepsilon^2}L^f_{FP}(f^\varepsilon)\log f^\varepsilon dvdx,
\end{aligned}$$

after integrating by parts, we see that

$$\begin{aligned}
&\int_\Omega\int_{\mathbb{R}^d}\frac{1}{\varepsilon}(v\cdot\nabla_x f^\varepsilon)\log f^\varepsilon dvdx\\
=\;& -\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^d}(v\cdot\nabla_x f^\varepsilon)dvdx+\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\gamma f^\varepsilon\log\gamma f^\varepsilon dvdS\\
=\;& -\int_{\partial\Omega}J_f^\varepsilon\cdot\mathbf{n}dS+\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\gamma f^\varepsilon\log\gamma f^\varepsilon dvdS\\
=\;& \frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\gamma f^\varepsilon\log\gamma f^\varepsilon dvdS,
\end{aligned}$$

$$-\int_\Omega\int_{\mathbb{R}^d}\frac{1}{\varepsilon}(\nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon)\log f^\varepsilon dvdx=0,$$

and

$$\int_\Omega\int_{\mathbb{R}^d}\frac{\zeta_f}{\varepsilon^2}L^f_{FP}(f^\varepsilon)\log f^\varepsilon dvdx=-\frac{\zeta_f}{\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vf^\varepsilon+\kappa_f\nabla_v f^\varepsilon)\cdot\frac{\nabla_v f^\varepsilon}{f^\varepsilon}dvdx.$$

As a consequence,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\mathcal{H}(f^\varepsilon)dvdx &= -\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\gamma f^\varepsilon\log\gamma f^\varepsilon dvdS\\
&\quad -\frac{\zeta_f}{\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vf^\varepsilon+\kappa_f\nabla_v f^\varepsilon)\cdot\frac{\nabla_v f^\varepsilon}{f^\varepsilon}dvdx. \qquad (3.30)
\end{aligned}$$

Similarly for g^ε, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\mathcal{H}(g^\varepsilon)dvdx &= -\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\gamma g^\varepsilon\log\gamma g^\varepsilon dvdS\\
&\quad -\frac{\zeta_g}{\varepsilon^2}\int_\Omega\int_{\mathbb{R}^d}(vg^\varepsilon+\kappa_g\nabla_v g^\varepsilon)\cdot\frac{\nabla_v g^\varepsilon}{g^\varepsilon}dvdx. \qquad (3.31)
\end{aligned}$$

Moreover, we see that for f^ε and g^ε

$$\int_\Omega\int_{\mathbb{R}^d}(vf^\varepsilon+\kappa_f\nabla_v f^\varepsilon)\cdot\left(v+\frac{\kappa_f\nabla_v f^\varepsilon}{f^\varepsilon}\right)dvdx=\mathcal{D}^f(f^\varepsilon), \qquad (3.32)$$

$$\int_\Omega\int_{\mathbb{R}^d}(vg^\varepsilon+\kappa_g\nabla_v g^\varepsilon)\cdot\left(v+\frac{\kappa_g\nabla_v g^\varepsilon}{g^\varepsilon}\right)dvdx=\mathcal{D}^g(g^\varepsilon). \qquad (3.33)$$

Due to the Poisson equation (2.6), we have

$$\int_\Omega\phi^\varepsilon\partial_t(z_f n^\varepsilon+z_g p^\varepsilon)dx=-\varpi\int_\Omega\phi^\varepsilon\partial_t\Delta_x\phi^\varepsilon dx=\frac{\varpi}{2}\frac{d}{dt}\int_\Omega|\nabla_x\phi^\varepsilon|^2dx. \qquad (3.34)$$

Then we conclude from (3.28)–(3.34) that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t)+\frac{\zeta_f}{\kappa_f\varepsilon^2}\mathcal{D}^f(f^\varepsilon)+\frac{\zeta_g}{\kappa_g\varepsilon^2}\mathcal{D}^g(g^\varepsilon)\\
=\;& -\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\left(\frac{1}{2\kappa_f}|v|^2+\log\gamma f^\varepsilon\right)\gamma f^\varepsilon dvdS\\
&-\frac{1}{\varepsilon}\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\left(\frac{1}{2\kappa_g}|v|^2+\log\gamma g^\varepsilon\right)\gamma g^\varepsilon dvdS.
\end{aligned}$$

Recall that $d\mu_x^i(v) = M_i(v)|v\cdot\mathbf{n}(x)|dv$, $i\in\{f,g\}$ are probability measures on $\Sigma_\pm^x$ (see the definition of $M_i(v)$ (2.10) and (3.17)). Then for the boundary terms, we can apply the Darrozès–Guiraud inequality [25], namely, we deduce from (2.11) that

$$\begin{aligned}
&\int_{\partial\Omega}\int_{\mathbb{R}^d}(v\cdot\mathbf{n})\left(\frac{1}{2\kappa_f}|v|^2+\log\gamma f^\varepsilon\right)\gamma f^\varepsilon dvdS\\
=\;&\int_{\partial\Omega}\int_{\Sigma_+^x}\mathcal{H}\left(\frac{\gamma f_+^\varepsilon}{M_f(v)}\right)d\mu_x^f dS-\int_{\partial\Omega}\int_{\Sigma_-^x}\mathcal{H}\left(\frac{\gamma f_-^\varepsilon}{M_f(v)}\right)d\mu_x^f dS\\
=\;&\int_{\partial\Omega}\mathcal{I}^f\left(\frac{\gamma f_+^\varepsilon}{M_f(v)}\right)dS\\
\geq\;&0,
\end{aligned}$$

thanks to the convectity of $\mathcal{H}(s)=s\log s$ and the Jensen inequality (see also [61]). Similar result holds for the boundary term associated with g^ε. As a consequence,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t)+\frac{\zeta_f}{\kappa_f\varepsilon^2}\mathcal{D}^f(f^\varepsilon)+\frac{\zeta_g}{\kappa_g\varepsilon^2}\mathcal{D}^g(g^\varepsilon)\\
&\quad+\frac{1}{\varepsilon}\int_{\partial\Omega}\mathcal{I}^f\left(\frac{\gamma f_+^\varepsilon}{M_f(v)}\right)dS+\frac{1}{\varepsilon}\int_{\partial\Omega}\mathcal{I}^g\left(\frac{\gamma g_+^\varepsilon}{M_g(v)}\right)dS\leq 0.
\end{aligned}\tag{3.35}$$

Integrating (3.35) with respect to time, we arrive at our conclusion (3.22). □

Due to the energy dissipation property Proposition 3.1, the existence of renormalized solutions to the VPFP system (2.4)–(2.6) can be obtained by using the argument as in [13,57,59,61].

Proposition 3.2. *Suppose that assumptions* (3.4)–(3.3) *are satisfied. For arbitrary but fixed* $\varepsilon>0$*, the initial boundary value problem of the VPFP system* (2.4)–(2.6) *admits at least one (renormalized) solution* $(f^\varepsilon,g^\varepsilon,\phi^\varepsilon)$ *in the sense of Definition 3.1, which satisfies Proposition 3.1.*

Remark 3.2. *We note that the initial boundary value problem of a full Vlasov–Poisson–Fokker–Planck–Boltzmann system (subject to more general reflection boundary conditions for the distribution function but only for one species of charged particles) has been studied in the recent paper [61]. The author proved the existence of DiPerna–Lions renormalized solutions by using the approximation procedure in [59] with crucial trace theorems previously introduced by the same author for the Vlasov equations [60] and some new results concerning weak-weak convergence (the renormalized convergence and the biting* L^1*-weak convergence). In the current case with multiple species of charged particles, the coupling between different species is weak, i.e., only via the Poisson equation. As a result, based on the uniform estimates derived from the energy dissipation Proposition 3.1, we can prove the existence result Proposition 3.2 following the proofs in [61] with minor modifications. The details are thus omitted.*

The energy dissipation (3.22) yields the following estimates that are uniform in the parameter ε, which enable us to take the diffusion limit as $\varepsilon\to 0$ in the next section:

Lemma 3.1. *For any* $T>0$*, there exists a constant* C *depending on* C_0*,* ζ_f*,* κ_g*,* ζ_g*,* κ_g*,* ϖ*, but independent of* ε *and* $t\in[0,T]$*:*

$$\int_\Omega\int_{\mathbb{R}^d}(1+|v|^2+|\log(f^\varepsilon)|)f^\varepsilon dvdx\leq C,$$

$$\int_\Omega \int_{\mathbb{R}^d} (1+|v|^2+|\log(g^\varepsilon)|)g^\varepsilon dvdx \le C,$$
$$\int_\Omega |\nabla_x \phi^\varepsilon|^2 dx \le C,$$
$$\frac{1}{\varepsilon^2}\int_0^t \mathcal{D}^f(f^\varepsilon)ds \le C, \quad \frac{1}{\varepsilon^2}\int_0^t \mathcal{D}^g(g^\varepsilon)ds \le C,$$
$$\frac{1}{\varepsilon}\int_0^t \int_{\partial\Omega} \mathcal{I}^f\left(\frac{\gamma f_+^\varepsilon}{M_f(v)}\right) dSds \le C, \quad \frac{1}{\varepsilon}\int_0^t \int_{\partial\Omega} \mathcal{I}^g\left(\frac{\gamma g_+^\varepsilon}{M_g(v)}\right) dSds \le C.$$

The functions $f^\varepsilon, g^\varepsilon$ are weakly relatively compact in $L^1((0,T)\times\Omega\times\mathbb{R}^d)$. Concerning the fluxes, we have

$$\|J_f^\varepsilon(t,\cdot)\|_{L^1(\Omega)} \le \frac{1}{2\varepsilon^2}\mathcal{D}^f(f^\varepsilon) + \frac{1}{2}\|f_0^\varepsilon\|_{L^1(\Omega\times\mathbb{R}^d)},$$
$$\|J_g^\varepsilon(t,\cdot)\|_{L^1(\Omega)} \le \frac{1}{2\varepsilon^2}\mathcal{D}^g(g^\varepsilon) + \frac{1}{2}\|g_0^\varepsilon\|_{L^1(\Omega\times\mathbb{R}^d)}.$$

Moreover,

$$\|\nabla_v\sqrt{f^\varepsilon}\|_{L^2((0,T)\times\Omega\times\mathbb{R}^d} + \|\nabla_v\sqrt{g^\varepsilon}\|_{L^2((0,T)\times\Omega\times\mathbb{R}^d} \le C.$$

Proof. The proof is similar to [33, Propositions 5.1, 5.2, 5.3], based on the energy inequality (3.22). Since we are now dealing with bounded domain, we do not need to estimate $\int_\Omega\int_{\mathbb{R}^d}|x|f^\varepsilon dvdx$ as well as $\int_\Omega\int_{\mathbb{R}^d}|x|g^\varepsilon dvdx$ like there. The L^1 weak compactness of f^ε and g^ε follows from the well-known Dunford–Pettis theorem. □

4 Diffusion limit as $\varepsilon \to 0$

In this section, we shall show that as ε go to zero, the limiting system of the VPFP system (2.4)–(2.6) recovers the following PNP system:

$$\partial_t n + \nabla_x \cdot J_f = 0, \tag{4.1}$$
$$\partial_t p + \nabla_x \cdot J_g = 0, \tag{4.2}$$
$$-\varpi\Delta_x\phi = z_f n + z_g p + D(x), \tag{4.3}$$

with density currents

$$J_f = -\frac{1}{\zeta_f}\nabla_x n - \frac{z_f}{\zeta_f} n\nabla_x\phi, \quad J_g = -\frac{1}{\zeta_g}\nabla_x p - \frac{z_g}{\zeta_g} p\nabla_x\phi. \tag{4.4}$$

The PNP system (4.1)–(4.3) is subject to the following boundary conditions as well as initial conditions:

$$J_f\cdot\mathbf{n} = J_g\cdot\mathbf{n} = \nabla_x\phi\cdot\mathbf{n} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{4.5}$$
$$n|_{t=0} = n_0, \quad p|_{t=0} = p_0, \quad \text{in } \Omega. \tag{4.6}$$

Moreover, we assume that

$$\int_\Omega \phi dx = 0 \quad \text{and} \quad \int_\Omega (z_f n + z_g p + D(x))dx = 0.$$

First, we introduce the weak formulation of the PNP system (4.1)–(4.6).

Definition 4.1. *We say that the triple (n, p, ϕ) is a weak solution to the initial boundary value problem of the PNP system* (4.1)–(4.6)*, if*

$$\begin{aligned} n, p &\in L^\infty(0,T; L\log L(\Omega)), \\ \sqrt{n}, \sqrt{p} &\in L^2(0,T; H^1(\Omega)), \\ \partial_t n, \partial_t p &\in L^1(0,T; W^{-1,1}(\Omega)), \\ \phi &\in L^2(0,T; H^1(\Omega)), \end{aligned}$$

where the function space $L\log L(\Omega)$ is given by

$$L\log L(\Omega) := \left\{ n : n \geq 0, \int_\Omega n(1 + |\log n|)dx < +\infty \right\}$$

and the PNP system (4.1)–(4.3) *is satisfied in the weak sense: for any $u \in C_0^\infty([0,T); C^\infty(\overline{\Omega}))$ and $\psi \in L^2(0,T; (H^1(\Omega))')$,*

$$\begin{aligned} &\int_0^T \int_\Omega n\partial_t u dxdt + \frac{1}{\zeta_f}\int_0^T \int_\Omega (\nabla_x n + z_f n \nabla_x \phi)\cdot \nabla_x u dxdt = \int_\Omega n_0 u(0,\cdot)dx, \\ &\int_0^T \int_\Omega p\partial_t u dxdt + \frac{1}{\zeta_g}\int_0^T \int_\Omega (\nabla_x p + z_g p \nabla_x \phi)\cdot \nabla_x u dxdt = \int_\Omega p_0 u(0,\cdot)dx, \\ &\varpi \int_0^T \int_\Omega \nabla_x \phi \cdot \nabla_x \psi dxdt = \int_0^T \int_\Omega (z_f n + z_g p + D(x))\psi dxdt. \end{aligned}$$

Now we state the main result of this section.

Theorem 4.1. *Suppose that the assumptions* (3.4)–(3.3) *are satisfied. Let $(f^\varepsilon, g^\varepsilon, \phi^\varepsilon)$ be a free energy (renormalized) solution of the VPFP system* (2.4)–(2.6) *with corresponding initial and boundary conditions. Then, as ε tends to zero, up to a subsequence if necessary, we have the strong convergence results*

$$f^\varepsilon \to n\widetilde{M}_f(v) \in L^1(0,T; L^1(\Omega \times \mathbb{R}^d)), \tag{4.7}$$

$$g^\varepsilon \to p\widetilde{M}_g(v) \in L^1(0,T; L^1(\Omega \times \mathbb{R}^d)), \tag{4.8}$$

$$\phi^\varepsilon \to \phi \in L^2(0,T; W^{1,q}(\Omega)), \quad 1 < q < 2. \tag{4.9}$$

In particular, $n^\varepsilon, p^\varepsilon$ strongly converge in $L^1(0,T; L^1(\Omega))$ towards (n,p) and (n,p,ϕ) is a weak solution to the PNP system (4.1)–(4.6) *(cf. Definition 4.1) with initial data*

$$n|_{t=0} = n_0 = \int_{\mathbb{R}^d} f_0 dv, \quad p|_{t=0} = p_0 = \int_{\mathbb{R}^d} g_0 dv$$

such that f_0, g_0 are the weak limits of f_0^ε and g_0^ε. Moreover, the weak solution (n,p,ϕ) satisfy the following energy inequality

$$\begin{aligned} &e(t) + \int_0^t \int_\Omega \left(\frac{1}{\zeta_f} n \Big| \nabla\Big(\ln n + z_f \phi \Big) \Big|^2 + \frac{1}{\zeta_g} p \Big| \nabla\Big(\ln p + z_g \phi \Big) \Big|^2 \right) dxdt \leq e(0), \\ &\text{with} \;\; e(t) := \int_\Omega \Big(n\ln n + p\ln p + \frac{\varpi}{2}|\nabla\phi|^2 \Big) dx. \end{aligned}$$

Proof. The proof of Theorem 4.1 mainly follows the arguments in [33] for the VPFP system that concerns only one single species of particles in the whole space. However, for the present problem involving multiple species of charged particles, we need to modify the previous argument to deal with nonlocal interactions between particles as well as the boundary conditions. In what follows, we state the essential steps and point out the possible differences.

Step 1. Strong convergence of electric potential. Based on the uniform estimates in Lemma 3.1, it is straightforward to argue as [57, Propisition 3.3] to conclude that

Lemma 4.1. *The renormalized solution $(f^\varepsilon, g^\varepsilon, \phi^\varepsilon)$ satisfies the following properties:*

(1) $n^\varepsilon, p^\varepsilon$ are weakly relatively compact in $L^1((0,T)\times\Omega)$,

(2) ϕ^ε is relatively compact in $L^2(0,T;W^{1,p}(\Omega))$ with $1\le p<2$.

In particular, the strong convergence of ϕ^ε (4.9) (up to a subsequence) is a direct consequnece of Lemma 4.1.

Step 2. Strong convergence of densities. Lemma 4.1 also implies the weak compactness of densities n^ε, p^ε. Indeed, we can show the convergence of density functions in strong sense. By using the definition of renormalized solutions (cf. Definition 3.1) and a velocity averaging lemma (cf. [57, Lemma 4.2], also [27]), we are able to obtain the compactness of the densities (cf. [33, Proposition 6.1]) such that the densities n^ε, p^ε are relatively compact in $L^1((0,T)\times\Omega)$, namely, there exist $n, p\in L^1((0,T)\times\Omega)$ and up to a subsequence if necessary,

$$n^\varepsilon\to n,\quad p^\varepsilon\to p,\quad \text{in } L^1((0,T)\times\Omega) \text{ and a.e. as } \varepsilon\to 0. \tag{4.10}$$

The above result and the simple inequality $(\sqrt{a}-\sqrt{b})^2\le|a-b|$ imply that

$$\sqrt{n^\varepsilon}\to\sqrt{n},\quad \sqrt{p^\varepsilon}\to\sqrt{p},\quad \text{in } L^2((0,T)\times\Omega) \text{ and a.e. as } \varepsilon\to 0. \tag{4.11}$$

Step 3. Strong convergence of distribution functions. Recalling the logarithmic Sobolev inequality (cf. e.g., [41, Corollary 4.2]) such that

$$\int_{\mathbb{R}^d}|h(v')|^2\log|h(v')|d\mu(v')\le\int_{\mathbb{R}^d}|\nabla_{v'}h(v')|^2d\mu(v')+\|h(v')\|^2_{L^2(\mathbb{R}^d,d\mu)}\log\|h(v')\|_{L^2(\mathbb{R}^d,d\mu)},$$

where $d\mu(v')$ is the Gauss measure $d\mu(v')=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v'|^2}{2}}dv$. Making the simple change of variable $v'\to\frac{v}{\sqrt{\kappa}}$ and denoting $h_\kappa(v)=h(v')$, we have

$$d\mu(v')=\Big(\frac{\kappa}{2\pi}\Big)^{\frac{d}{2}}e^{-\frac{1}{2\kappa}|v|^2}dv:=d\mu_\kappa(v),\quad \|h(v')\|_{L^2(\mathbb{R}^d,d\mu)}=\|h_\kappa(v)\|_{L^2(\mathbb{R}^d,d\mu_\kappa)}.$$

As a result,

$$\begin{aligned}&\int_{\mathbb{R}^d}|h_\kappa(v)|^2\log|h_\kappa(v)|d\mu_\kappa(v)\\ \le\ &\kappa\int_{\mathbb{R}^d}|\nabla_v h_\kappa(v)|^2d\mu_\kappa(v)+\|h_\kappa(v)\|^2_{L^2(\mathbb{R}^d,d\mu_\kappa(v))}\log\|h_\kappa(v)\|_{L^2(\mathbb{R}^d,d\mu_\kappa(v))}.\end{aligned}$$

In the above inequality, we set

$$\kappa = \kappa_f, \quad h_\kappa(v) = \sqrt{\frac{f^\varepsilon}{\widetilde{M}_f(v)}}, \quad d\mu_\kappa(v) = \widetilde{M}_f(v)dv.$$

Then we infer from the definition $n^\varepsilon = \int_{\mathbb{R}^n} f^\varepsilon(t,x,v)dv$ that $\|h_\kappa\|^2_{L^2(\mathbb{R}^d, d\mu_\kappa(v))} = n^\varepsilon$, which yields

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} f^\varepsilon \log\left(\frac{f^\varepsilon}{n^\varepsilon \widetilde{M}_f(v)}\right) dvdx \\
= &\int_\Omega \left(2\int_{\mathbb{R}^d} |h_\kappa(v)|^2 \log|h_\kappa(v)| d\mu_\kappa(v) - 2\|h_\kappa(v)\|^2_{L^2(\mathbb{R}^d, d\mu_\kappa(v))} \log \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}\right) dx \\
\leq &\ 2\kappa_f \int_\Omega \int_{\mathbb{R}^d} \left|\nabla_v \sqrt{\frac{f^\varepsilon}{\widetilde{M}_f(v)}}\right|^2 \widetilde{M}_f(v) dvdx \\
= &\ \frac{\kappa_f}{2}\mathcal{D}^f(f^\varepsilon). \qquad (4.12)
\end{aligned}$$

On the other hand, we recall the classical Csiszar–Kullback inequality (cf. [24, Theorem 3.1, and Section 4, pp. 314], see also [51]) that for all non-negative $u \in L^1(\mathbb{R}^d, d\mu)$ (where $d\mu$ is a probability measure) with $\int_{\mathbb{R}^d} u d\mu = 1$, it holds

$$\|u-1\|_{L^1(\mathbb{R}^d, d\mu)} \leq 2\left(\int_{\mathbb{R}^d} (u\log u - u + 1) d\mu\right)^{\frac{1}{2}}.$$

Now we choose in the above inequality

$$u = \frac{f^\varepsilon}{n^\varepsilon \widetilde{M}_f(v)}, \quad d\mu = \widetilde{M}_f(v)dv,$$

which easily implies

$$\begin{aligned}
&\left(\int_\Omega \int_{\mathbb{R}^d} |f^\varepsilon - n^\varepsilon \widetilde{M}_f(v)| dvdx\right)^2 \\
\leq &\ 4\left(\int_\Omega n^\varepsilon dx\right) \int_\Omega \int_{\mathbb{R}^d} f^\varepsilon \log\left(\frac{f^\varepsilon}{n^\varepsilon \widetilde{M}_f(v)}\right) dvdx. \qquad (4.13)
\end{aligned}$$

We keep in mind that similar versions of estimates hold for g^ε. As a consequence, we infer from the entropy dissipation in (3.22), the uniform estimates in Lemma 3.1 and the estimates (4.12) and (4.13) that when $\varepsilon \to 0$, (keeping in mind that similar versions hold for g^ε)

$$f^\varepsilon - n^\varepsilon \widetilde{M}_f \to 0, \quad g^\varepsilon - p^\varepsilon \widetilde{M}_g \to 0, \quad \text{in } L^1((0,T)\times\Omega\times\mathbb{R}^d) \text{ and a.e.}$$

Combing the above results with (4.10), we conclude that as $\varepsilon \to 0$

$$f^\varepsilon \to n\widetilde{M}_f, \quad g^\varepsilon \to p\widetilde{M}_g, \quad \text{in } L^1((0,T)\times\Omega\times\mathbb{R}^d) \text{ and a.e.} \qquad (4.14)$$

Step 4. Weak convergence of fluxes. We introduce the auxiliary functions

$$r_f^\varepsilon = \frac{\sqrt{f^\varepsilon} - \sqrt{n^\varepsilon \widetilde{M}_f(v)}}{\varepsilon\sqrt{\widetilde{M}_f(v)}}, \quad r_g^\varepsilon = \frac{\sqrt{g^\varepsilon} - \sqrt{p^\varepsilon \widetilde{M}_g(v)}}{\varepsilon\sqrt{\widetilde{M}_g(v)}}. \qquad (4.15)$$

In analogy to [57, Proposition 3.4] and [33, Proposition 5.5], we have

Lemma 4.2. *For arbitrary $T>0$, the following uniform estimates hold*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^d}\left(|r^\varepsilon|^2\widetilde{M}_f+\varepsilon|r_f^\varepsilon|^2|v|^2\widetilde{M}_f+\sqrt{\varepsilon}|r_f^\varepsilon|^2|v|\widetilde{M}_f\right)dvdxdt\le C,$$
$$\int_0^T\int_\Omega\int_{\mathbb{R}^d}\left(|r_g^\varepsilon|^2\widetilde{M}_g+\varepsilon|r_g^\varepsilon|^2|v|^2\widetilde{M}_g+\sqrt{\varepsilon}|r_g^\varepsilon|^2|v|\widetilde{M}_g\right)dvdxdt\le C,$$

where C is a constant that may depend on C_0, ζ_f, κ_f, ζ_g, κ_g, ϖ, but independent of ε and $t\in[0,T]$.

Using the expressions of r_f^ε and r_g^ε (cf. (4.15)), we have

$$f^\varepsilon=n^\varepsilon\widetilde{M}_f+2\varepsilon\widetilde{M}_f\sqrt{n^\varepsilon}r_f^\varepsilon+\varepsilon^2(r_f^\varepsilon)^2\widetilde{M}_f, \tag{4.16}$$
$$g^\varepsilon=p^\varepsilon\widetilde{M}_g+2\varepsilon\widetilde{M}_g\sqrt{p^\varepsilon}r_g^\varepsilon+\varepsilon^2(r_g^\varepsilon)^2\widetilde{M}_g. \tag{4.17}$$

Due to the simple facts

$$\int_{\mathbb{R}^d}v\widetilde{M}_i(v)dv=0,\quad i\in\{f,g\},$$

it follows from (2.13), (4.11) and Lemma 4.2 that as $\varepsilon\to0$

$$\begin{aligned}J_f^\varepsilon&=2\sqrt{n^\varepsilon}\int_{\mathbb{R}^d}r_f^\varepsilon v\widetilde{M}_f dv+\sqrt{\varepsilon}\int_{\mathbb{R}^d}\sqrt{\varepsilon}|r_f^\varepsilon|^2v\widetilde{M}_f dv\\&\to 2\sqrt{n}\int_{\mathbb{R}^d}r_f v\widetilde{M}_f dv,\quad\text{weakly in } L^1((0,T)\times\Omega),\end{aligned}$$

and

$$\begin{aligned}J_g^\varepsilon&=2\sqrt{p^\varepsilon}\int_{\mathbb{R}^d}r_g^\varepsilon v\widetilde{M}_g dv+\sqrt{\varepsilon}\int_{\mathbb{R}^d}\sqrt{\varepsilon}|r_g^\varepsilon|^2v\widetilde{M}_g dv\\&\to 2\sqrt{p}\int_{\mathbb{R}^d}r_g v\widetilde{M}_g dv,\quad\text{weakly in } L^1((0,T)\times\Omega),\end{aligned}$$

where r_f, r_g are the weak limits of r_f^ε and r_g^ε, respectively.

It remains to identify the limit functions of J_f^ε and J_g^ε, which can be done by using the same argument as in [33, Proposition 7.2]. The strong convergence of f^ε and g^ε (see (4.14)) implies that

$$\theta_{\varepsilon,\lambda}\to\sqrt{(n+\lambda)\widetilde{M}_f},\quad \eta_{\varepsilon,\lambda}\to\sqrt{(p+\lambda)\widetilde{M}_g},\quad\text{for } \lambda>0 \text{ and } \varepsilon\to0.$$

On the other hand, it follows from (4.16) and (4.17) that for any $\lambda>0$ as $\varepsilon\to0$,

$$\frac{\zeta_f}{2\varepsilon}L_{FP}^f(f^\varepsilon)=\zeta_f L_{FP}^f(\widetilde{M}_f\sqrt{n^\varepsilon}r_f^\varepsilon+\frac{\varepsilon}{2}(r_f^\varepsilon)^2\widetilde{M}_f)\to\zeta_f\sqrt{n}L_{FP}^f(r_f\widetilde{M}_f),$$
$$\frac{\zeta_g}{2\varepsilon}L_{FP}^g(g^\varepsilon)=\zeta_g L_{FP}^g(\widetilde{M}_g\sqrt{p^\varepsilon}r_g^\varepsilon+\frac{\varepsilon}{2}(r_g^\varepsilon)^2\widetilde{M}_g)\to\zeta_g\sqrt{p}L_{FP}^g(r_g\widetilde{M}_g).$$

As a consequence, first passing to the limit for any $\lambda>0$ as $\varepsilon\to0$ in the renormalized formula (3.15) and (3.16) and then letting $\lambda\to0$, we obtain that

$$(\nabla_x\sqrt{n}+\frac{z_f}{2}\nabla_x\phi\sqrt{n})\cdot v\widetilde{M}_f=\zeta_f L_{FP}^f(r_f\widetilde{M}_f), \tag{4.18}$$
$$(\nabla_x\sqrt{p}+\frac{z_g}{2}\nabla_x\phi\sqrt{p})\cdot v\widetilde{M}_g=\zeta_g L_{FP}^g(r_g\widetilde{M}_g), \tag{4.19}$$

where ϕ is the limit of ϕ^ε (recall (4.9)). The convergence results obtained above are always understood to be up to a subsequence.

On the other hand, it follows from [33, Proposition 3.1] that $\chi_i = -v_i\widetilde{M}_f$ $(i = 1, ..., d)$ is the unique solution to the equation $L^f_{FP}\chi = v_i\widetilde{M}_f$ in $R(L^f_{FP}) \cap D(L^f_{FP})$, where

$$\begin{aligned}
L^2_{\widetilde{M}_f}(\mathbb{R}^d) &= L^2(\mathbb{R}^d; \widetilde{M}_f^{-1}dv), \\
R(L^f_{FP}) &= \left\{ f \in L^2_{\widetilde{M}_f}(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(v)dv = 0 \right\}, \\
D(L^f_{FP}) &= \left\{ f \in L^2_{\widetilde{M}_f}(\mathbb{R}^d) : \nabla_v \cdot \left(e^{-\frac{1}{2\kappa_f}|v|^2} \nabla_v (e^{\frac{1}{2\kappa_f}|v|^2} f) \right) \in L^2_{\widetilde{M}_f}(\mathbb{R}^d) \right\}.
\end{aligned}$$

Since $-L^f_{FP}$ is a self-adjoint operator on $L^2_{\widetilde{M}_f}(\mathbb{R}^d)$, using (4.18), we have

$$\begin{aligned}
J_f &= 2\sqrt{n}\int_{\mathbb{R}^d} r_f v\widetilde{M}_f dv \\
&= 2\sqrt{n}\int_{\mathbb{R}^d} (r_f\widetilde{M}_f) L^f_{FP}(-v\widetilde{M}_f)\widetilde{M}_f^{-1}dv \\
&= 2\sqrt{n}\int_{\mathbb{R}^d} L^f_{FP}(r_f\widetilde{M}_f)(-v\widetilde{M}_f)\widetilde{M}_f^{-1}dv \\
&= \frac{2}{\zeta_f}\sqrt{n}\int_{\mathbb{R}^d} \left[(\nabla_x\sqrt{n} + \frac{z_f}{2}\nabla_x\phi\sqrt{n}) \cdot v\widetilde{M}_f \right] (-v\widetilde{M}_f)\widetilde{M}_f^{-1}dv \\
&= -\frac{2}{\zeta_f}\sqrt{n}\left(\int_{\mathbb{R}^d} v \otimes v\widetilde{M}_f dv \right) \left(\nabla_x\sqrt{n} + \frac{z_f}{2}\nabla_x\phi\sqrt{n} \right) \\
&= -\frac{2}{\zeta_f}\sqrt{n}\left(\nabla_x\sqrt{n} + \frac{z_f}{2}\nabla_x\phi\sqrt{n} \right).
\end{aligned}$$

where we use the fact that $\int_{\mathbb{R}^d} v \otimes v\widetilde{M}_f dv = \mathbb{I}$. In a similar way, we can deduce that

$$J_g = -\frac{2}{\zeta_g}\sqrt{p}\left(\nabla_x\sqrt{p} + \frac{z_g}{2}\nabla_x\phi\sqrt{p} \right).$$

Therefore, we can see that as $\varepsilon \to 0$

$$J^\varepsilon_f \to J_f := -\frac{2}{\zeta_f}\sqrt{n}\left(\nabla_x\sqrt{n} + \frac{z_f}{2}\nabla_x\phi\sqrt{n} \right), \tag{4.20}$$

$$J^\varepsilon_g \to J_g := -\frac{2}{\zeta_g}\sqrt{p}\left(\nabla_x\sqrt{p} + \frac{z_g}{2}\nabla_x\phi\sqrt{p} \right), \tag{4.21}$$

in the distribution sense.

Step 5. Passage to the limit in the PDE system. In order to recover the PNP system, we state a regularity result for the density functions n, p in the spirit of [57, Lemma 7.1]

Lemma 4.3. *Let Ω be a smooth bounded and open set in $\mathbb{R}^d$. Assume n, p are positive functions belonging to $L^\infty(0, T; L^1(\Omega))$ and $\phi \in L^2(0, T; H^1(\Omega))$ that satisfy*

$$\nabla_x\sqrt{n} + \frac{z_f}{2}\nabla_x\phi\sqrt{n} = G_f \in L^2(0, T; L^2(\Omega)), \tag{4.22}$$

$$\nabla_x\sqrt{p} + \frac{z_g}{2}\nabla_x\phi\sqrt{p} = G_g \in L^2(0, T; L^2(\Omega)), \tag{4.23}$$

$$-\varpi\Delta_x\phi = z_f n + z_g p + D(x).$$

Then we have

$$\sqrt{n}, \sqrt{p} \in L^2(0,T;H^1(\Omega)), \quad z_f n + z_g p \in L^2(0,T;L^2(\Omega)),$$
$$\nabla_x\phi\sqrt{n}, \quad \nabla_x\phi\sqrt{p} \in L^2(0,T;L^2(\Omega)).$$

Proof. As in [57, Corollary 3.2], we take $\beta_\delta(s) = \delta^{-1}\beta(\delta s)$ where $\beta \in C^\infty(\mathbb{R})$ satisfying $\beta(s) = s$ for $-1 \le s \le 1$, $0 \le \beta'(s) \le 1$ for $s \in \mathbb{R}$ and $\beta(s) = 2$ for $|s| \ge 3$. Then we renormalize the equations (4.22), (4.23) for $\sqrt{n}, \sqrt{p}$ such that

$$\nabla_x\beta_\delta(\sqrt{n}) + \frac{z_f}{2}\nabla_x\phi\beta'_\delta(\sqrt{n})\sqrt{n} = G_f\beta'_\delta(\sqrt{n}) \in L^2(0,T;L^2(\Omega)), \tag{4.24}$$
$$\nabla_x\beta_\delta(\sqrt{p}) + \frac{z_g}{2}\nabla_x\phi\beta'_\delta(\sqrt{p})\sqrt{p} = G_g\beta'_\delta(\sqrt{p}) \in L^2(0,T;L^2(\Omega)). \tag{4.25}$$

For any $\delta > 0$, due to our choice of β and the given regularity for $\nabla_x\phi$, we have

$$\|\nabla_x\phi\beta'_\delta(\sqrt{n})\sqrt{n}\|_{L^2(0,T;L^2(\Omega))} \le \frac{3}{\delta}\|\nabla_x\phi\|_{L^2(0,T;L^2(\Omega))},$$
$$\|\nabla_x\phi\beta'_\delta(\sqrt{p})\sqrt{p}\|_{L^2(0,T;L^2(\Omega))} \le \frac{3}{\delta}\|\nabla_x\phi\|_{L^2(0,T;L^2(\Omega))},$$

which implies that

$$\nabla_x\beta_\delta(\sqrt{n}), \quad \nabla_x\beta_\delta(\sqrt{n}) \in L^2(0,T;L^2(\Omega)).$$

Then we can take L^2 norm on both sides of the two equations (4.24), (4.25). Adding the resultants together, we have

$$\begin{aligned}
&\|\nabla_x\beta_\delta(\sqrt{n})\|^2_{L^2(0,T;L^2(\Omega))} + \frac{z_f^2}{4}\|\nabla_x\phi\beta'_\delta(\sqrt{n})\sqrt{n}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|\nabla_x\beta_\delta(\sqrt{p})\|^2_{L^2(0,T;L^2(\Omega))} + \frac{z_g^2}{4}\|\nabla_x\phi\beta'_\delta(\sqrt{p})\sqrt{p}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \int_0^T\int_\Omega [z_f\nabla_x\beta_\delta(\sqrt{n})\beta'_\delta(\sqrt{n})\sqrt{n} + z_g\nabla_x\beta_\delta(\sqrt{p})\beta'_\delta(\sqrt{p})\sqrt{p}]\cdot\nabla_x\phi dxdt \\
\le\ & \|G_f\|^2_{L^2(0,T;L^2(\Omega))} + \|G_g\|^2_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

where the right-hand side is independent of δ. For the crossing term, using integration by parts, we have

$$\begin{aligned}
&\int_0^T\int_\Omega [z_f\nabla_x\beta_\delta(\sqrt{n})\beta'_\delta(\sqrt{n})\sqrt{n} + z_g\nabla_x\beta_\delta(\sqrt{p})\beta'_\delta(\sqrt{p})\sqrt{p}]\cdot\nabla_x\phi dxdt \\
=\ & \int_0^T\int_\Omega \nabla_x[z_f\tilde{\beta}_\delta(\sqrt{n}) + z_g\tilde{\beta}_\delta(\sqrt{p})]\cdot\nabla_x\phi dxdt \\
=\ & \frac{1}{\varpi}\int_0^T\int_\Omega (z_f n + z_g p + D(x))[z_f\tilde{\beta}_\delta(\sqrt{n}) + z_g\tilde{\beta}_\delta(\sqrt{p})]dxdt
\end{aligned}$$

where $\tilde{\beta}(s) = \int_0^s \tau\beta'(\tau)^2 d\tau$, $\tilde{\beta}_\delta(s) = \delta^{-2}\tilde{\beta}(\delta s)$ and $\tilde{\beta}_\delta(s) \to \frac{s^2}{2}$ as $\delta \to 0$. Let $\delta \to 0$ go to zero, we have

$$\|\nabla_x\sqrt{n}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{z_f^2}{4}\|\nabla_x\phi\sqrt{n}\|^2_{L^2(0,T;L^2(\Omega))}$$

$$
\begin{aligned}
&+\|\nabla_x\sqrt{p}\|^2_{L^2(0,T;L^2(\Omega))}+\frac{z_g^2}{4}\|\nabla_x\phi\sqrt{p}\|^2_{L^2(0,T;L^2(\Omega))}\\
&+\frac{1}{2\varpi}\int_0^T\int_\Omega (z_f n+z_g p)^2 dxdt\\
\leq\ & \|G_f\|^2_{L^2(0,T;L^2(\Omega))}+\|G_g\|^2_{L^2(0,T;L^2(\Omega))}\\
&+\frac{1}{2\varpi}\left|\int_0^T\int_\Omega D(x)(z_f n+z_g p)dxdt\right|\\
\leq\ & \|G_f\|^2_{L^2(0,T;L^2(\Omega))}+\|G_g\|^2_{L^2(0,T;L^2(\Omega))}\\
&+\frac{1}{4\varpi}\int_0^T\int_\Omega |D(x)|^2+(z_f n+z_g p)^2 dxdt,
\end{aligned}
$$

which yields the required regularity. The lemma is proved. □

Finally, using the above regularity lemma and the facts (4.20), (4.21), we are able to write the currents J_f and J_g as in (4.4). Then we can pass to the limit as $\varepsilon \to 0$ in the weak form of equations (3.20), (3.21) as well as in the Poisson equation (3.10). The proof of Theorem 4.1 is complete. □

5 Conclusion and future work

In the classical PNP treatment for the ionic solutions, the charged particles are assumed to be points such that the effect of size exclusion is not incorporated. However, such an assumption would be oversimplified for solutions with crowded ions because ions size effect are significant to their properties. For instance, the most obvious difference between the biologically crucial ions Na^+ and K^+ is their diameter and they are otherwise somewhat indistinguishable [46]. The size effects are crucial in the study of the selectivity of ion channels in cell membranes, due to the narrow size of ion channels [23, 43, 52, 64]. Modelling size exclusion (at high concentrations) is a difficult problem even if ions are treated as simple hard spheres. There is a large literature involving different attempts and treatments on the finite size effects (cf. [8, 9, 31, 35, 43, 46, 47] and the references therein). Recently, a modified PNP system for ionic dynamics including the Brownian motion of ions, electrostatic interactions among charged ions, and finite size effects was introduced in [31,46], and employed in various situations [45,47,54]. The derivation is based on an energetic variational approach [48], which combines the maximum dissipation principle (for long time dynamics) and least action principle (for intrinsic and short time dynamics) into a force balance law that expands the law of conservation of momentum to include dissipation, using the generalized forces in the variational formulation of mechanics [31]. The total energy for the modified PNP system consists of the entropic energy induced by the Brownian motion of ions, the electrostatic potential energy representing the coulomb interaction between the charged ions, and in particular, the repulsive potential energy caused by the excluded volume effect (e.g., the Lennard–Jones potential). We note that the classical PNP system (1.1) can be easily recovered by using the same variational principle, if we include only the entropy and electrostatic potential for the total energy.

As we had achieved in this paper, for the case of crowded ions, it would be interesting to study the diffusion-limit of certain suitable VPFP type systems to get the modified PNP system [31, 46] in the future. At last, we would like to mention that the PNP system can also be derived from diffusion limits of other kinetic equations, e.g., the Boltzmann–Poisson system. We refer to [57] for the one species case and we believe that their argument can also be applied to the multi-species case.

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