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CHAPTER 6

Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

In the preceding chapters we dealt mostly with steady-state problems in electricity and in magnetism. Similar mathematical techniques were employed, but electric and magnetic phenomena were treated as independent. The only link between them was that the currents that produce magnetic fields are basically electrical in character, being charges in motion. The almost independent nature of electric and magnetic phenomena disappears when we consider time-dependent problems. Faraday's discovery of induction (Section 5.15) destroyed the independence. Time-varying magnetic fields give rise to electric fields and vice versa. We then must speak of *electromagnetic fields*, rather than electric or magnetic fields. The full import of the interconnection between electric and magnetic fields and their essential sameness becomes clear only within the framework of special relativity (Chapter 11). For the present we content ourselves with examining the basic phenomena and deducing the set of equations known as the *Maxwell equations*, which describe the behavior of electromagnetic fields. Vector and scalar potentials, gauge transformations, and Green functions for the wave equation are next discussed, including retarded solutions for the fields, as well as the potentials. There follows a derivation of the macroscopic equations of electromagnetism. Conservation laws for energy and momentum and transformation properties of electromagnetic quantities are treated, as well as the interesting topic of magnetic monopoles.

6.1 *Maxwell's Displacement Current; Maxwell Equations*

The basic laws of electricity and magnetism we have discussed so far can be summarized in differential form by these four (not yet Maxwell) equations:

$$\begin{array}{ll} \text{COULOMB'S LAW} & \nabla \cdot \mathbf{D} = \rho \\ \text{AMPÈRE'S LAW } (\nabla \cdot \mathbf{J} = 0) & \nabla \times \mathbf{H} = \mathbf{J} \\ \text{FARADAY'S LAW} & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \text{ABSENCE OF FREE MAGNETIC POLES} & \nabla \cdot \mathbf{B} = 0 \end{array} \quad (6.1)$$

Let us recall that all but Faraday's law were derived from steady-state observations. Consequently, from a logical point of view there is no a priori reason to

expect that the static equations will hold unchanged for time-dependent fields. In fact, the equations in set (6.1) are inconsistent as they stand.

It required the genius of J. C. Maxwell, spurred on by Faraday's observations, to see the inconsistency in equations (6.1) and to modify them into a consistent set that implied new physical phenomena, at the time unknown but subsequently verified in all details by experiment. For this brilliant stroke in 1865, the modified set of equations is justly known as the *Maxwell equations*.

The faulty equation is Ampère's law. It was derived for steady-state current phenomena with $\nabla \cdot \mathbf{J} = 0$. This requirement on the divergence of \mathbf{J} is contained right in Ampère's law, as can be seen by taking the divergence of both sides:

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{H}) \equiv 0 \quad (6.2)$$

While $\nabla \cdot \mathbf{J} = 0$ is valid for steady-state problems, the general relation is given by the continuity equation for charge and current:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (6.3)$$

What Maxwell saw was that the continuity equation could be converted into a vanishing divergence by using Coulomb's law (6.1). Thus

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad (6.4)$$

Then Maxwell replaced \mathbf{J} in Ampère's law by its generalization

$$\mathbf{J} \rightarrow \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

for time-dependent fields. Thus Ampère's law became

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (6.5)$$

still the same, experimentally verified, law for steady-state phenomena, but now mathematically consistent with the continuity equation (6.3) for time-dependent fields. Maxwell called the added term in (6.5) the *displacement current*. Its presence means that a changing *electric* field causes a magnetic field, even without a current—the converse of Faraday's law. This necessary addition to Ampère's law is of crucial importance for rapidly fluctuating fields. Without it there would be no electromagnetic radiation, and the greatest part of the remainder of this book would have to be omitted. It was Maxwell's prediction that light was an electromagnetic wave phenomenon, and that electromagnetic waves of all frequencies could be produced, that drew the attention of all physicists and stimulated so much theoretical and experimental research into electromagnetism during the last part of the nineteenth century.

The set of four equations,

$$\begin{aligned} \nabla \cdot \mathbf{D} = \rho & & \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 & & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \end{aligned} \quad (6.6)$$

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 $D = \epsilon E$
 $E = E_0 (1 + \chi_e)$

known as the *Maxwell equations*, forms the basis of all classical electromagnetic phenomena. When combined with the Lorentz force equation and Newton's second law of motion, these equations provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields (see Section 6.7 and Chapters 12 and 16). The range of validity of the Maxwell equations is discussed in the Introduction, as are questions of boundary conditions for the normal and tangential components of fields at interfaces between different media. Constitutive relations connecting \mathbf{E} and \mathbf{B} with \mathbf{D} and \mathbf{H} were touched on in the Introduction and treated for static phenomena in Chapters 4 and 5. More is said later in this chapter and in Chapter 7.

The units employed in writing the Maxwell equations (6.6) are those of the preceding chapters, namely, SI. For the reader more at home in other units, such as Gaussian, Table 2 of the Appendix summarizes essential equations in the commoner systems. Table 3 of the Appendix allows the conversion of any equation from Gaussian to SI units or vice versa, while Table 4 gives the corresponding conversions for given amounts of any variable.

6.2 Vector and Scalar Potentials

The Maxwell equations consist of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. They can be solved as they stand in simple situations. But it is often convenient to introduce potentials, obtaining a smaller number of second-order equations, while satisfying some of the Maxwell equations identically. We are already familiar with this concept in electrostatics and magnetostatics, where we used the scalar potential Φ and the vector potential \mathbf{A} .

Since $\nabla \cdot \mathbf{B} = 0$ still holds, we can define \mathbf{B} in terms of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (6.7)$$

Then the other homogeneous equation in (6.6), Faraday's law, can be written

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (6.8)$$

This means that the quantity with vanishing curl in (6.8) can be written as the gradient of some scalar function, namely, a scalar potential Φ :

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \quad (6.9)$$

or

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

The definition of \mathbf{B} and \mathbf{E} in terms of the potentials \mathbf{A} and Φ according to (6.7) and (6.9) satisfies identically the two homogeneous Maxwell equations. The dynamic behavior of \mathbf{A} and Φ will be determined by the two inhomogeneous equations in (6.6).

At this stage it is convenient to restrict our considerations to the vacuum

shapes of the nuclei themselves. The energy levels or states of a nucleus are described by the quantum numbers of total angular momentum J and its projection M along the z axis, as well as others, which we will denote by a general index α . A given nuclear state has associated with it a quantum-mechanical charge density* $\rho_{JM\alpha}(\mathbf{x})$, which depends on the quantum numbers (J, M, α) but is cylindrically symmetric about the z axis. Thus the only nonvanishing quadrupole moment is q_{20} in (4.6), or Q_{33} in (4.9).† The quadrupole moment of a nuclear state is defined as the value of $(1/e) Q_{33}$ with the charge density $\rho_{JM\alpha}(\mathbf{x})$, where e is the protonic charge:

$$Q_{JM\alpha} = \frac{1}{e} \int (3z^2 - r^2) \rho_{JM\alpha}(\mathbf{x}) d^3x \quad (4.25)$$

The dimensions of $Q_{JM\alpha}$ are consequently (length)². Unless the circumstances are exceptional (e.g., nuclei in atoms with completely closed electronic shells), nuclei are subjected to electric fields that possess field gradients in the neighborhood of the nuclei. Consequently, according to (4.24), the energy of the nuclei will have a contribution from the quadrupole interaction. The states of different M value for the same J will have different quadrupole moments $Q_{JM\alpha}$, and so a degeneracy in M value that may have existed will be removed by the quadrupole coupling to the “external” (crystal lattice, or molecular) electric field. Detection of these small energy differences by radiofrequency techniques allows the determination of the quadrupole moment of the nucleus.‡

The interaction energy between two dipoles \mathbf{p}_1 and \mathbf{p}_2 can be obtained directly from (4.24) by using the dipole field (4.20). Thus, the mutual potential energy is

$$W_{12} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{n} \cdot \mathbf{p}_1)(\mathbf{n} \cdot \mathbf{p}_2)}{4\pi\epsilon_0 |\mathbf{x}_1 - \mathbf{x}_2|^3} \quad (4.26)$$

where \mathbf{n} is a unit vector in the direction $(\mathbf{x}_1 - \mathbf{x}_2)$ and it is assumed that $\mathbf{x}_1 \neq \mathbf{x}_2$. The dipole-dipole interaction is attractive or repulsive, depending on the orientation of the dipoles. For fixed orientation and separation of the dipoles, the value of the interaction, averaged over the relative positions of the dipoles, is zero. If the moments are generally parallel, attraction (repulsion) occurs when the moments are oriented more or less parallel (perpendicular) to the line joining their centers. For antiparallel moments the reverse is true. The extreme values of the potential energy are equal in magnitude.

4.3 Elementary Treatment of Electrostatics with Ponderable Media

In Chapters 1, 2, and 3 we considered electrostatic potentials and fields in the presence of charges and conductors, but no other ponderable media. We there-

*See *Blatt and Weisskopf* (pp. 23 ff.) for an elementary discussion of the quantum aspects of the problem.

†Actually Q_{11} and Q_{22} are different from zero, but are not independent of Q_{33} , being given by $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$.

‡“The quadrupole moment of a nucleus,” denoted by Q , is defined as the value of $Q_{JM\alpha}$ in the state $M = J$. See *Blatt and Weisskopf, loc. cit.*

fore made no distinction between microscopic fields and macroscopic fields, although our treatment of conductors in an idealized fashion with *surface* charge densities implied a macroscopic description. Air is sufficiently tenuous that the neglect of its dielectric properties causes no great error; our results so far are applicable there. But much of electrostatics concerns itself with charges and fields in ponderable media whose respective electric responses must be taken into account. In the Introduction we indicated the need for averaging over macroscopically small, but microscopically large, regions to obtain the Maxwell equations appropriate for macroscopic phenomena. This is done in a careful fashion in Chapter 6, after the Maxwell equations with time variation have been discussed. For the present we merely remind the reader of the outlines of the elementary discussion of polarization in a fashion that glosses over difficult and sometimes subtle aspects of the averaging procedure and the introduction of the macroscopic quantities.

The first observation is that when an averaging is made of the homogeneous equation, $\nabla \times \mathbf{E}_{\text{micro}} = 0$, the same equation, namely,

$$\nabla \times \mathbf{E} = 0 \quad (4.27)$$

holds for the averaged, that is, the macroscopic, electric field \mathbf{E} . This means that the electric field is still derivable from a potential $\Phi(\mathbf{x})$ in electrostatics.

If an electric field is applied to a medium made up of a large number of atoms or molecules, the charges bound in each molecule will respond to the applied field and will execute perturbed motions. The molecular charge density will be distorted. The multipole moments of each molecule will be different from what they were in the absence of the field. In simple substances, when there is no applied field the multipole moments are all zero, at least when averaged over many molecules. The dominant molecular multipole with the applied fields is the dipole. There is thus produced in the medium an *electric polarization* \mathbf{P} (dipole moment per unit volume) given by

$$\mathbf{P}(\mathbf{x}) = \sum_i N_i \langle \mathbf{p}_i \rangle \quad (4.28)$$

where \mathbf{p}_i is the dipole moment of the *i*th type of molecule in the medium, the average is taken over a small volume centered at \mathbf{x} and N_i is the average number per unit volume of the *i*th type of molecule at the point \mathbf{x} . If the molecules have a net charge e_i and, in addition, there is macroscopic excess or free charge, the charge density at the macroscopic level will be

$$\rho(\mathbf{x}) = \sum_i N_i \langle e_i \rangle + \rho_{\text{excess}} \quad (4.29)$$

Usually the average molecular charge is zero. Then the charge density is the excess or free charge (suitably averaged).

If we now look at the medium from a macroscopic point of view, we can build up the potential or field by linear superposition of the contributions from each macroscopically small volume element ΔV at the variable point \mathbf{x}' . Thus the charge of ΔV is $\rho(\mathbf{x}') \Delta V$ and the dipole moment of ΔV is $\mathbf{P}(\mathbf{x}') \Delta V$. If there are no higher macroscopic multipole moment densities, the potential $\Delta\Phi(\mathbf{x}, \mathbf{x}')$

caused by the configuration of moments in ΔV can be seen from (4.10) to be given without approximation by

$$\Delta\Phi(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \left[\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \Delta V + \frac{\mathbf{P}(\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \Delta V \right] \quad (4.30)$$

provided \mathbf{x} is outside ΔV . We now treat ΔV as (macroscopically) infinitesimal, put it equal to d^3x' , and integrate over all space to obtain the potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{P}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \quad (4.31)$$

The second term is analogous to the dipole layer potential (1.25), but is for a volume distribution of dipoles. An integration by parts transforms the potential into

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} [\rho(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')] \quad (4.32)$$

This is just the customary expression for the potential caused by a charge distribution $(\rho - \nabla \cdot \mathbf{P})$. With $\mathbf{E} = -\nabla\Phi$, the first Maxwell equation therefore reads

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} [\rho - \nabla \cdot \mathbf{P}] \quad (4.33)$$

The presence of the divergence of \mathbf{P} in the effective charge density can be understood qualitatively. If the polarization is nonuniform there can be a net increase or decrease of charge within any small volume, as indicated schematically in Fig. 4.2.

With the definition of the *electric displacement* \mathbf{D} ,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (4.34)$$

(4.33) becomes the familiar

$$\nabla \cdot \mathbf{D} = \rho \quad (4.35)$$

Equations (4.27) and (4.35) are the macroscopic counterparts of (1.13) and (1.14) of Chapter 1.

As discussed in the Introduction, a *constitutive relation* connecting \mathbf{D} and \mathbf{E} is necessary before a solution for the electrostatic potential or fields can be obtained. In the subsequent sections of this chapter we assume that the response of the system to an applied field is linear. This excludes ferroelectricity from discussion, but otherwise is no real restriction provided the field strengths do not

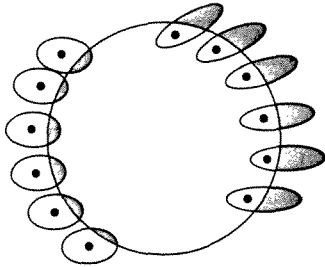


Figure 4.2 Origin of polarization-charge density. Because of spatial variation of polarization, more molecular charge may leave a given small volume than enters it. Only molecules near the boundary are shown.

become extremely large. As a further simplification we suppose that the *medium is isotropic*. Then the induced polarization \mathbf{P} is parallel to \mathbf{E} with a coefficient of proportionality that is independent of direction:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad (4.36)$$

The constant χ_e is called the *electric susceptibility* of the medium. The displacement \mathbf{D} is therefore proportional to \mathbf{E} ,

$$\mathbf{D} = \epsilon \mathbf{E} \quad (4.37)$$

where

$$\epsilon = \epsilon_0(1 + \chi_e) \quad (4.38)$$

is the electric permittivity; $\epsilon/\epsilon_0 = 1 + \chi_e$ is called the *dielectric constant* or relative electric permittivity.

If the dielectric is not only isotropic, but also uniform, then ϵ is independent of position. The divergence equations (4.35) can then be written

$$\nabla \cdot \mathbf{E} = \rho/\epsilon \quad (4.39)$$

All problems *in that medium* are reduced to those of preceding chapters, except that the electric fields produced by given charges are reduced by a factor ϵ_0/ϵ . The reduction can be understood in terms of a polarization of the atoms that produce fields in opposition to that of the given charge. One immediate consequence is that the capacitance of a capacitor is increased by a factor of ϵ/ϵ_0 if the empty space between the electrodes is filled with a dielectric with dielectric constant ϵ/ϵ_0 (true only to the extent that fringing fields can be neglected).

If the uniform medium does not fill all of the space where there are electric fields or, more generally, if there are different media juxtaposed, not necessarily linear in their responses, we must consider the question of boundary conditions on \mathbf{D} and \mathbf{E} at the interfaces between media. These boundary conditions are derived from the full set of Maxwell equations in Section I.5. The results are that the normal components of \mathbf{D} and the tangential components of \mathbf{E} on either side of an interface satisfy the *boundary conditions*, valid for time-varying as well as static fields,

$$\left. \begin{aligned} (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} &= \sigma \\ (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} &= 0 \end{aligned} \right\} \quad (4.40)$$

where \mathbf{n}_{21} is a unit normal to the surface, directed from region 1 to region 2, and σ is the macroscopic surface-charge density on the boundary surface (*not* including the polarization charge).

4.4 Boundary-Value Problems with Dielectrics

The methods of earlier chapters for the solution of electrostatic boundary-value problems can readily be extended to handle the presence of dielectrics. In this section we treat a few examples of the various techniques applied to dielectric media.

To illustrate the method of images for dielectrics we consider a point charge q embedded in a semi-infinite dielectric ϵ_1 a distance d away from a plane inter-