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POTENTIAL INDUCED BY A POINT SOURCE OF CURRENT INSIDE AN INFINITE CYLINDRICAL CELL

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1. INTRODUCTION

In this report we determine the potential inside a right circular cylinder produced by a point source at an arbitrary location inside the cylinder. The potential is given by a Green's function which is the solution of Laplace's equation inside a cylindrical boundary on which the potential is specified by homogeneous mixed boundary conditions (the potential on the boundary is proportional to its normal derivative). We develop a representation for the case in which the boundary acts as an almost perfect insulator; the representation has a simple physical interpretation and is easy to compute. This Green's function has not previously been determined in an easily interpretable or computable form and indeed few problems involving the mixed boundary condition have been studied.[1, 2, 3] We have, however, solved the spherical problem.[4] In cylindrical geometry computable Green's functions have not been obtained even for Dirichlet and Neumann boundary conditions, in which just the potential or just the normal derivative respectively, of the potential is specified at boundary.

The mixed boundary condition for Laplace's equation is of wide interest since it arises naturally in many physical situations. In problems of steady heat flow this boundary condition is common[5, 6] and is called the "radiation" boundary condition; in diffusion problems the boundary condition is called the "general boundary condition" and applies to a variety of specific situations including film limited diffusion, surface evaporation, and so on.[7] In electric potential problems the mixed boundary condition is used to describe the flow of current across a contact resistance between two materials of widely different conductivities. In physical problems, the contact resistance usually arises in a thin layer of resistive material; in biological problems, the resistance usually arises in a membrane.

One of the reasons so few problems involving the mixed boundary condition have been solved completely is that solutions have been sought for the general case in which any linear combination of potential and the normal derivative might occur at the boundary. From the point of view of applied mathematics, however, it is not necessary to consider such a general case: in most of the physical situations described above the boundary is known to act as a slightly imperfect insulator and there is only a small flow or gradient of potential normal to the boundary. In such cases the normal derivative, expressed with respect to a dimensionless spatial variable, is much less than the potential at the boundary, and so the problem has many features in common with the Neumann problem. We exploit the relative size of the potential and normal derivative by using the method of matched asymptotic expansions, a particular technique in singular perturbation theory.[8] The technique permits development of asymptotic expansions for the potential, one expansion (the near field or inner expansion) being valid in a region including the source; the other expansion (the far field or outer expansion) being valid in the entire region away from the source. The two expansions match in the intermediate overlap region in which both are valid and a composite expansion can be written which is valid everywhere. A quite simple formula is obtained for the far-field expansion, consisting of a known result, and correction terms. The near field expansion is more complicated. However, its leading term is merely a large constant potential. The higher order terms are obtained as eigenfunction expansions, double infinite series of Bessel functions or single infinite sums of infinite integrals of Bessel functions. We also derive these expansions directly from the exact solution to the problem and so establish the validity of the method of matched asymptotic expansions for this problem. This result is of some general interest since theorems are not available to establish the validity of the method of matched asymptotic expansions and since there are few boundary value problems of this complexity which permit the comparision of the exact solution with that derived by singular perturbation theory.[9]

In order to be specific, we now derive the equation and boundary condition, and will calculate the electric potential induced by a point source of current inside a long cylindrical cell. The model for the cell is a circular cylinder of infinite length. The cell interior has a conductivity σ_i, and is surrounded by a thin membrane with conductivity σ_m. The outside surface of the membrane is held at zero potential. The current density $\vec{j}'$ is related to the potential V' by

$$\vec{j}' = \begin{cases} -\sigma_i \ \nabla' V', & \text{inside cell,} \\ -\sigma_m \ \nabla' V', & \text{in membrane,} \end{cases} \tag{1.1}$$

where primes denote quantities in physical units; unprimed quantities will be defined later in nondimensional variables.

Continuity of current across the boundary between the cell interior and membrane requires that

$$\frac{\sigma_m V'}{\delta} = -\sigma_i \frac{\partial V'}{\partial n'} \quad \text{on } S, \tag{1.2}$$

where S is the inner surface of the membrane, $\partial/\partial n'$ is of the outward normal derivative, and δ is the thickness of the membrane, so that the potential gradient within the membrane is $-V'/\delta$.

Assuming a unit point source of current within the cell at $\vec{r}' = \vec{R}'$, the divergence of the current density is given by

$$\nabla' \cdot \vec{j}' = \delta(\vec{r}' - \vec{R}') \tag{1.3}$$

where $\delta(\vec{r}' - \vec{R}')$ is a Dirac "delta function."

Using (1.1) in (1.3),

$$- \sigma_i \nabla'^2 V' = \delta (\vec{r}' - \vec{R}') \tag{1.4}$$

Making the change of variables

$$\vec{r} = \frac{\vec{r}'}{a}, \tag{1.5}$$

where a is the radius of the cylinder, and

$$V = a \sigma_i V', \tag{1.6}$$

(1.4) becomes

$$\nabla^2 V = - \delta (\vec{r} - \vec{R}), \tag{1.7}$$

and the boundary condition (1.2) becomes

$$\frac{\partial V}{\partial n} + \epsilon V = 0, \text{ on } S, \tag{1.8}$$

where

$$\epsilon = \frac{\sigma_m a}{\sigma_i \delta} \tag{1.9}$$

is a dimensionless parameter which is small ($\lesssim 10^{-3}$) in living cells.

Defining cylindrical coordinates (x, r, θ), shown in Figure 1, the source may be placed at $\vec{R} = (0, R, 0)$ with no loss in generality. The boundary condition at large distances down the cylinder, $|x| \to \infty$, is that the inside potential approach the zero potential of the outer membrane surface. The problem for determining the potential may thus be written, in cylindrical coordinates,

$$\left\{ \begin{array}{ll} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial x^2} = - \frac{1}{r} \delta(x)\delta(r-R)\delta(\theta), & (1.10) \\ \frac{\partial V}{\partial r}(x, 1, \theta) + \epsilon V(x, 1, \theta) = 0, & (1.11) \\ V(\infty, r, \theta) = 0. & (1.12) \end{array} \right.$$

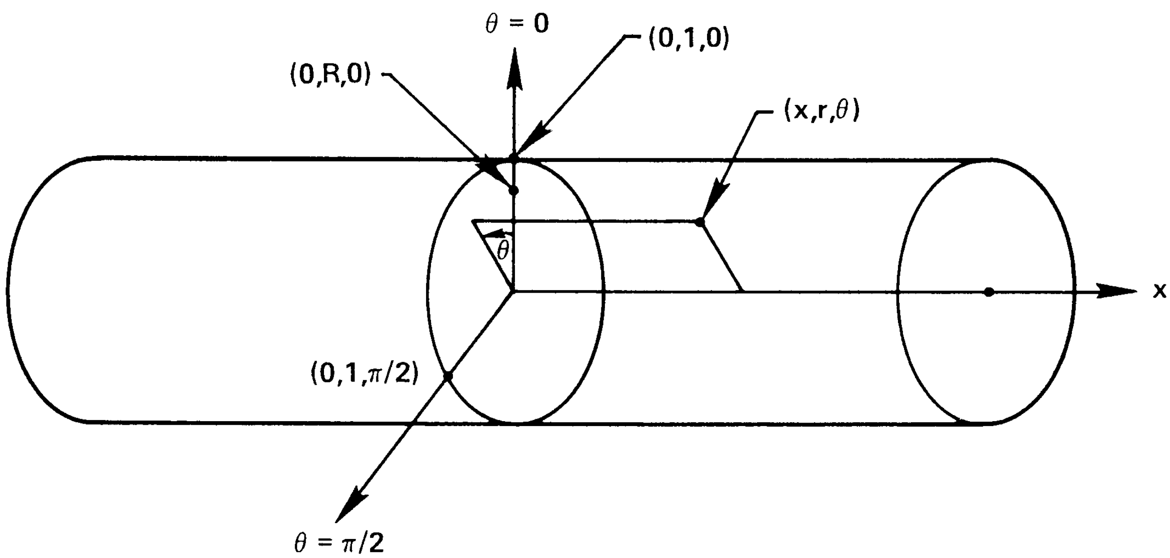

Figure 1. Coordinate System for Cylindrical Cell.

2. METHOD OF SOLUTION FOR SMALL ϵ

The problem posed by (1.10) - (1.12) can be solved exactly, for any ϵ, in the form of a double infinite sum or a single infinite sum of infinite integrals of Bessel functions.[10] The behavior of these exact solutions for small ϵ can then be obtained by taking appropriate limits of the solutions, letting $\epsilon \to 0$ in a region including the source and in a region some distance from the source. This procedure is carried out in Section 6, starting from the double infinite sum representation of the solution, given in Equations (6.1) and (6.2).

Since we are only interested in the small-ϵ behavior of the solution, we can alternatively bypass the exact solution entirely and apply the techniques of singular perturbation theory to (1.10) - (1.12). The procedure is to generate a sequence of problems from (1.10) - (1.12). Each problem in the sequence is simpler than the original problem; its solution corresponds to one term in the expansion of the exact solution in powers of ϵ. We find one sequence of problems, and its corresponding expansion of the potential, which is valid near the source; another which is valid far from the source, and use the technique of matching [8] to join the two together in the intermediate region where they are both valid. This mathematical approach is justified by the relative simplicity of each problem and the physical insight gained by obtaining each individual term in the expansions directly from the solution of a relatively simple problem.

Let us consider the basis for the singular nature of the problem. The physical problem under study is the potential distribution caused by a point current source inside an infinitely long cylinder. When ϵ is small, the boundary condition (1.11) implies that the current flow will be predominantly in the axial direct, i.e., only a small fraction of the local current, $O(\epsilon)$, crosses the membrane in an axial distance of O(1). We are tempted to try to find an expansion, in the small parameter ϵ, in which the leading term is the potential for $\epsilon = 0$. Denoting this potential by $V_1(x, r, \theta)$, from (1.10) and (1.11) V_1 satisfies the equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial \theta^2} + \frac{\partial^2 V_1}{\partial x^2} = -\frac{1}{r}\delta(x)\,\delta(r-R)\,\delta(\theta) \tag{2.1}$$

and the boundary condition at r = 1,

$$\frac{\partial V_1}{\partial r}(x, 1, \theta) = 0. \tag{2.2}$$

Physically, the boundary condition (2.2) means that no current crosses the membrane; all the current is confined to flow in the interior of the cell. Consequently, V_1 must contain a part which is linearly decreasing with increasing $|x|$. This would lead to a potential of $V_1 \to -\infty$ as $|x| \to \infty$, which would make it impossible to satisfy the boundary condition $V = 0$ at $|x| \to \infty$, required by (1.12). To avoid this divergence, any expansion which contains V_1 can be valid only over a limited range of x, designated the near field, which contains the source point. At large distances from the source, we must look for another, far-field, expansion.

We expect that as $\epsilon \to 0$, the region of validity of any near-field expansion of which V_1 is a part will become large. If there is a linearly decaying potential over a large distance, and in addition the potential is required to approach zero as $|x| \to \infty$, then the potential at $x = 0$ must be very large, i.e., $V(0, r, \theta) \to \infty$ as $\epsilon \to 0$. As usual, we will separate terms in the expansion according to their order in ϵ. Clearly, V_1 must be O(1), and therefore cannot be the leading term in the expansion.

This is about as far as we can go with these somewhat intuitive arguments. They are presented to give physical perspective to the problem. We turn now to a solution of the far-field problem and will see that the process of matching it to the near field will dictate in a precise way each term that is necessary in the near field, without any need for intuitive arguments.

3. FAR-FIELD POTENTIAL

In the near field, in order to conform to the boundary condition at $r = 1$ and the singularity at $(0, R, 0)$ the current-density field is complicated and its variation along the three coordinate axes are of comparable magnitudes. In the far field, a long distance from the source, the situation is different. The current is predominantly in the axial direction. Since only a small fraction of the current within the cell, at any value of x, leaks out of the cylinder in an axial distance of O(1), the variation in the x direction will be slow. We therefore, for convenience in ordering the far-field expansion, write the far-field potential in terms of a new slow variable x^*. Denoting the far field by W, we write the following expansion,

$$W(x^*, r, \theta; \epsilon) = \zeta_0(\epsilon)W_0(x^*, r, \theta) + \zeta_1(\epsilon)W_1(x^*, r, \theta) + \ldots, \tag{3.1}$$

where the slow variable is defined by

$$x^* = x\,\eta(\epsilon), \tag{3.2}$$

W_n is of O(1) and $\zeta_n/\zeta_{n-1} \to 0$ as $\epsilon \to 0$, and in accordance with the physical argument given above, it is expected that $\eta \to 0$ as $\epsilon \to 0$. We might expect initially that this expansion is valid only for $x^* > x_0^*$, say.

Substituting (3.1) and (3.2) in (1.10) and (1.11), and noting that the right-hand side of (1.10) is zero since $x = 0$ is outside the present domain, we have,

$$\begin{aligned}\nabla^2 W = 0 = \zeta_0\eta^2 \frac{\partial^2 W_0}{\partial x^{*2}} + \zeta_1\eta^2 \frac{\partial^2 W_1}{\partial x^{*2}} + \cdots \\ + \zeta_0 \nabla_t^2 W_0 + \zeta_1 \nabla_t^2 W_1 + \cdots,\end{aligned} \tag{3.3}$$

where $\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ is the transverse Laplacian, and on the boundary, $r = 1$, we have,

$$\frac{\partial W}{\partial r} + \epsilon W = 0 = \zeta_0\frac{\partial W_0}{\partial r} + \zeta_1\frac{\partial W_1}{\partial r} + \cdots + \epsilon\zeta_0 W_0 + \epsilon\zeta_1 W_1. \tag{3.4}$$

Requiring that (3.3) and (3.4) be satisfied to each order of ϵ, the lowest-order ϵ problem is

$$\begin{cases} \nabla_t^2 W_0 = 0 \\ \dfrac{\partial W_0}{\partial r}(x^*, 1, \theta) = 0 \\ W_0(\pm\infty, r, \theta) = 0 \ . \end{cases} \tag{3.5}$$

The second problem is

$$\begin{cases} \nabla_t^2 W_1 = -\dfrac{\partial^2 W_0}{\partial x^{*2}} \\ \dfrac{\partial W_1}{\partial r}(x^*, 1, \theta) = -W_0(x^*, 1, \theta) \\ W_1(\pm\infty, r, \theta) = 0, \end{cases} \tag{3.6}$$

where we have used

$$\epsilon \zeta_0 = \zeta_1 \tag{3.7}$$

to obtain the second of Equations (3.6).

Writing an expansion for $\eta(\epsilon)$ in the form

$$\eta(\epsilon) = \eta_0(\epsilon) + \eta_1(\epsilon) + \eta_2(\epsilon) + \cdots, \tag{3.8}$$

where the $\eta_i(\epsilon)$ are an ordered sequence, we further set

$$\zeta_0 \eta_0^2 = \zeta_1$$

to obtain the first of Equations (3.6). Thus,

$$\eta_0 = \sqrt{\epsilon}\ . \tag{3.9}$$

We could take $\eta = \eta_0$, with $\eta_1 = \eta_2 = \cdots = 0$ and still obtain a sequence of problems of increasing order in ϵ. It will be seen below, however, that we would not be able to maintain uniform validity of the asymptotic expansion for W at large x^*. Assuming $\eta(\epsilon)$ to have the more general form (3.8) makes it possible to obtain a uniform expansion.

The third problem is found by collecting terms of the next higher order in ϵ in Equation (3. 3) and (3. 4), and is

$$\begin{cases} \nabla_t^2 W_2 = - \dfrac{\partial^2}{\partial x^{*2}} (W_1 + 2\alpha_1 W_0) \\ \dfrac{\partial W_2}{\partial r} (x^*, 1, \theta) = - W_1 (x^*, 1, \theta) \\ W_2 (\pm \infty, r, \theta) = 0, \end{cases} \tag{3.10}$$

where we set

$$\epsilon \zeta_1 = \zeta_2 \tag{3.11}$$

to obtain the second of Equations (3. 10), and, in addition

$$\eta_1 = \alpha_1 \epsilon \eta_0 \tag{3.12}$$

to obtain the first of Equations (3. 10). Continuing this process of collecting terms of equal order in ϵ in Equations (3. 3) and (3. 4), we obtain for the fourth problem,

$$\begin{cases} \nabla_t^2 W_3 = - \dfrac{\partial^2}{\partial x^{*2}} \left(W_2 + 2\alpha_1 W_1 + (\alpha_1^2 + 2\alpha_2) W_0 \right) \\ \dfrac{\partial W_3}{\partial r} (x^*, 1, \theta) = - W_2 (x^*, 1, \theta) \\ W_3 (\pm \infty, r, \theta) = 0 \end{cases} \tag{3.13}$$

Combining (3. 1), (3. 7) - (3. 9), (3. 11), (3. 12) and the additional requirements on ζ_3 and η_2 needed to obtain (3. 13), we have, for the far-field expansion of the potential,

$$\begin{aligned} W(x^*, r, \theta; \epsilon) = \zeta_0 (\epsilon) \, [W_0 (x^*, r, \theta) + \epsilon W_1 (x^*, r, \theta) \\ + \epsilon^2 W_2 (x^*, r, \theta) + \cdots], \end{aligned} \tag{3.14}$$

where the axial coordinate variable is

$$x^* = \sqrt{\epsilon}\, x (1 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \cdots). \tag{3.15}$$

So far $\zeta_0(\epsilon)$, the order of the leading term in the W expansion, is unknown. It will be determined by matching to the near field. The constants $\alpha_1, \alpha_2, \ldots$ in the expansion of x^*, which appear as coupling constants between different orders of W in the sequence of problems, will be determined by requiring uniform validity of the W expansion (3.14) for all values of x^*.

We now return to (3.5) and begin to solve the sequence of problems. The solution to the first problem (3.5) is independent of r and θ. Thus we have

$$W_0(x^*, r, \theta) = F(x^*), \tag{3.16}$$

where $F(x^*)$ is an as yet arbitrary function of x^*. We must go to the second problem (3.6) to determine its functional form.

From (3.6) and (3.16), we obtain

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial W_1}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 W_1}{\partial \theta^2} = -F''(x^*) \\ \dfrac{\partial W_1}{\partial r}(x^*, 1, \theta) = -F(x^*) \\ W_1(\pm\infty, r, \theta) = 0, \end{cases} \tag{3.17}$$

where prime denotes differentiation with respect to x^*.

Since the inhomogenous term in the equation, and the boundary condition at r=1, are both independent of θ, clearly, W is independent of θ. Examining (3.10) and (3.13), the same reasoning then implies that W_2, $W_3\ldots$ are all independent of θ.

Integrating the equation in (3.17) twice (noting that $\partial^2 W_1/\partial\theta^2$ is zero) we obtain for the solution which is bounded at r = 0,

$$W_1(x^*, r) = -\frac{r^2}{4}F''(x^*) + G(x^*), \tag{3.18}$$

where $G(x^*)$ is an, as yet, arbitrary function of x^* which cannot be determined until we go to the next problem, (3.10), for W_2.

Substituting the result (3.18) in the r = 1 boundary condition of (3.17) yields,

$$F'' - 2F = 0 , \tag{3.19}$$

and hence,

$$W_0(x^*) = F(x^*) = A\, e^{-\sqrt{2}\, |x^*|} , \tag{3.20}$$

where A is a constant to be determined by matching to the near field.

Substituting W_0 and W_1, from (3.16) and (3.18) in (3.10), and using (3.19) to eliminate derivatives of F, and deleting the θ dependence in accordance with our findings, we obtain,

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\, r\, \dfrac{\partial W_2}{\partial r} = (r^2 - 4\alpha_1)\, F - G'' \\ \dfrac{\partial W_2}{\partial r}\,(x^*, 1) = \dfrac{1}{2}\, F - G \\ W_2\,(\pm\infty,\, r) = 0 , \end{cases} \tag{3.21}$$

for the next problem.

Integrating the equation in (3.21) twice and requiring the result to be finite at r = 0, we obtain

$$W_2(x^*,\ r) = \frac{r^4}{16}\, F - r^2\, (\alpha_1 F + \frac{1}{4}\, G'') + H(x^*), \tag{3.22}$$

where $H(x^*)$ is an arbitrary function of x^*.

Substituting the expression (3.22) for W_2 and (3.20) for F in the r = 1 boundary condition in (3.21) yields,

$$G'' - 2G = -\, 4A\, (\alpha_1 + \frac{1}{8})\, e^{-\sqrt{2}\, |x^*|} . \tag{3.23}$$

The right-hand side of (3.23) is an homogeneous solution of the equation. Therefore the particular solution contains a term proportional to x^* times

$\exp(-\sqrt{2}\,|x^*|)$. If such a term appears in G, and consequently, by (3.18), in W_1, then for sufficiently large $|x^*|$ (i.e., $|x^*| \gtrsim \epsilon^{-1}$) the equality used in the definition of an asymptotic development,

$$\lim_{\epsilon \to 0} \frac{\epsilon W_1}{W_0} = 0,$$

will not be satisfied and hence the expansion (3.14) will not be valid uniformly in x^*. To avoid this we require the right-hand side of (3.23) to vanish, which occurs if

$$\alpha_1 = -\frac{1}{8}. \tag{3.24}$$

It is now clear why we could not assume the simple relation $x^* = \sqrt{\epsilon}\, x$ but required the more general form (3.15). The freedom to choose $\alpha_1, \alpha_2, \cdots$ allows us to force all of the x^* dependence of W into $\exp(-\sqrt{2}\, x^*)$ eliminating nonuniformities in the expansion.

With the choice (3.24) of α_1, the solution to (3.23) is

$$G(x^*) = B\, e^{-\sqrt{2}\, x^*}, \tag{3.25}$$

where

$$x^* = \sqrt{\epsilon}\,(1 - \frac{\epsilon}{8} + \cdots), \tag{3.26}$$

and B will be determined by matching.

Substituting (3.20) and (3.25) for F and G in (3.18) we obtain for the second term in the far-field expansion,

$$W_1(x^*, r) = (-\frac{1}{2} A\, r^2 + B)\, e^{-\sqrt{2}\, x^*}. \tag{3.27}$$

We will now continue the same procedure in the next problem (3.13), to obtain an expression for W_2, and to find x^* to one more order in ϵ.

Substituting (3.16), (3.18) and (3.22) in (3.13) and using (3.19), (3.23) and (3.24), we obtain for the next problem,

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r} r \dfrac{\partial W_3}{\partial r} = -\left(\dfrac{r^4}{8} + \dfrac{r^2}{2} + \dfrac{1}{32} + 4\alpha_2\right)F + (r^2 + \dfrac{1}{2})G - H'' \\ \dfrac{\partial W_3}{\partial r}(x^*, 1) = -\dfrac{3}{16}F + \dfrac{1}{2}G - H \\ W_3(\pm\infty, r) = 0 \end{cases} \tag{3.28}$$

Integrating the equation in (3.28) once, setting r = 1 in the result, and substituting this in the r = 1 boundary condition yields the equation

$$H'' - 2H = (\frac{5}{96} - 4\alpha_2)F. \tag{3.29}$$

Using our earlier arguments, we require the right-hand side of (3.29) to be zero, in order to maintain uniformity of the expansion in x^*. This yields

$$\alpha_2 = \frac{5}{384}, \tag{3.30}$$

and

$$H(x^*) = C\, e^{-\sqrt{2}\, x^*}, \tag{3.31}$$

where C is to be determined by matching, and we now have x^* to one more order,

$$x^* = x\sqrt{\epsilon}\,(1 - \frac{\epsilon}{8} + \frac{5}{384}\epsilon^2 - \dots). \tag{3.32}$$

Substituting (3.20), (3.25) and (3.31) for F, G and H and (3.24) for α_1, in (3.22), we obtain

$$W_2(x^*, r) = \left[\frac{Ar^2}{8}(1 + \frac{r^2}{2}) - \frac{Br^2}{2} + C\right]e^{-\sqrt{2}\,|x^*|}. \tag{3.33}$$

Substituting (3.20), (3.27) and (3.33) in (3.14), we obtain for the far-field expansion,

$$W(x^*, r; \epsilon) = \zeta_0(\epsilon)\, e^{-\sqrt{2}\,|x^*|}\left[A + \epsilon\left(-\frac{Ar^2}{2} + B\right) + \epsilon^2\left(\frac{Ar^2}{8}(1 + \frac{r^2}{2}) - \frac{Br^2}{2} + C\right) + O(\epsilon^3)\right], \tag{3.34}$$

where x^* is given to $O(\epsilon^2)$ by (3.32). The three constants A, B, and C, and the function $\zeta_0(\epsilon)$ are to be determined by requiring that the $x^* \rightarrow 0$ limit of $W(x^*, r; \epsilon)$ be identical, in all orders of ϵ, to the $|x| \rightarrow \infty$ limit of the near-field expansion, $V(x, r, \theta; \epsilon)$. This procedure is motivated in the following way.

It is assumed that as $\epsilon \rightarrow 0$, the far-field expansion is asymptotic to the true potential if the far-field coordinate is greater than some value i.e., $x^* > x_0^*(\epsilon)$. Similarly, the near-field expansion is asymptotic to the true potential if $x < x_1(\epsilon)$. Furthermore, in order to be able to match the two expansions, there must be an overlap domain in which both expansions are valid, that is, we must have

$$x_1(\epsilon) > \frac{x_0^*(\epsilon)}{\sqrt{\epsilon}\left(1 - \frac{\epsilon}{8} + \frac{5\epsilon^2}{384} - \cdots\right)} \equiv x_0(\epsilon).$$

We expect that as $\epsilon \rightarrow 0$, $x_1(\epsilon) \rightarrow \infty$ and $x_0^*(\epsilon) \rightarrow 0$.

If we write both expansions in terms of an intermediate variable $x_\xi = \xi(\epsilon)x$, where $\xi(\epsilon)$ has the asymptotic behavior

$$\lim_{\epsilon \rightarrow 0} \xi(\epsilon) = 0,$$

$$\lim_{\epsilon \rightarrow 0} \frac{\xi(\epsilon)}{\sqrt{\epsilon}} = \infty,$$

then in the $\epsilon \rightarrow 0$ limit the two expansions must be identical in each order of ϵ. As $\epsilon \rightarrow 0$, for some fixed x_ξ in the range

$$\xi(\epsilon)x_1(\epsilon) > x_\xi > \xi(\epsilon)x_0(\epsilon),$$

we then have

$$x = \frac{x_\xi}{\xi(\epsilon)} \rightarrow \infty$$

and

$$x^* = \frac{x_\xi \sqrt{\epsilon}\left(1 - \frac{\epsilon}{8} + \cdots\right)}{\xi(\epsilon)} \rightarrow 0.$$

Thus the $x^* \to 0$ limit of the far-field expansion must coincide with the $x \to \infty$ limit of the near-field expansion. For compactness, rather than write each expansion in terms of some arbitrary intermediate variable and take the $\epsilon \to 0$ limit, we accomplish the same result by writing both in terms of x and taking the $x \to \infty$ limit of the near-field expansion and the $x^* \to 0$ limit of the far-field expansion. The procedure outlined here is discussed in greater detail, with many examples, elsewhere.[8]

Substituting the expression (3.32) for x^* in (3.34), expanding the exponential to $O(\epsilon^{5/2})$, multiplying by the expression in square brackets, and arranging terms in ascending powers of ϵ, we obtain

$$
\begin{aligned}
&W\left(x\sqrt{\epsilon}\left(1-\frac{\epsilon}{8}+\frac{5\epsilon}{384}-\cdots\right),\ r;\ \epsilon\right)\\
&\quad=\zeta_0(\epsilon)\Bigg[A-\epsilon^{1/2}A\sqrt{2}\,|x| + \epsilon\left[A\left(x^2-\frac{r^2}{2}\right)+B\right]\\
&\qquad+\epsilon^{3/2}\sqrt{2}\,|x|\left[A\left(\frac{1}{8}-\frac{x^2}{3}+\frac{r^2}{2}\right)-B\right]\\
&\qquad+\epsilon^{2}\left[A\left(-\frac{x^2}{4}+\frac{x^4}{6}-\frac{r^2x^2}{2}+\frac{r^2}{8}+\frac{r^4}{16}\right)+B\left(x^2-\frac{r^2}{2}\right)+C\right]\\
&\qquad+\epsilon^{5/2}\sqrt{2}\,|x|\left[A\left(-\frac{5}{384}+\frac{x^2}{8}+\frac{x^2r^2}{6}-\frac{3r^2}{16}-\frac{r^4}{16}-\frac{x^4}{30}\right)\right.\\
&\qquad\qquad\left.+B\left(\frac{1}{8}-\frac{x^2}{3}+\frac{r^2}{2}\right)-C\right]+O(\epsilon^3)\Bigg].
\end{aligned}
\tag{3.35}
$$

The expansion (3.35) of W in near-field coordinates contains integral powers of $\sqrt{\epsilon}$, whereas the expansion (3.34) in far-field coordinates contains only integral powers of ϵ. The powers of $\sqrt{\epsilon}$ arise from expanding the exponential in (3.34). It should be noted that an individual term of $O(\epsilon^n)$ in (3.34) contributes to all orders in (3.35). Consequently, although each term in (3.34) is the solution to a particular problem in the far-field, each term in (3.35) is not related in any simple way to a physical problem in the far field.

It is interesting that even in the far-field expression (3.34) the potential depends on ϵ in two different ways. In the near region of the far field ($x^* \rightarrow 0$) the potential can be described as a series which ascends in powers of $\sqrt{\epsilon}$, whereas in the far part of the far field, ($x^* \gtrsim 1$) the potential can be described as a series which ascends in powers of ϵ. This illustrates the fundamentally different behavior of the potential for small and large x.

4. NEAR-FIELD POTENTIAL

In the vicinity of the point source the potential is a rather complex function of position, and there is apparently no simple mathematical representation in terms of elementary functions, as there is in the far field. The potential has a singularity at the source point; the current diverges from this point, half the current going toward $x = +\infty$, and half toward $x = -\infty$. Close to the source, the lines of current flow are diverging outward, equally in all directions. Those lines which are directed toward the membrane must curve to avoid the membrane as, again, only a small fraction of the local current leaves the cylinder. As the current flows down the cylinder, the lines become predominantly in the axial direction, and the potential joins smoothly onto the far-field potential calculated in Section 3.

In terms of the asymptotic expansions representing the near and far fields, this behavior requires that the near-field expansion increase in powers of $\sqrt{\epsilon}$ so it can join to the expansion (3.35) of the far field. Furthermore, in accordance with the arguments in Section 2, which concluded that the O(1) term in the near field has a linear dependence on $|x|$ as $|x| \to \infty$, we see that the second term in (3.35) must be O(1) in order to match the near field. Consequently,

$$\zeta_0(\epsilon) = \frac{1}{\sqrt{\epsilon}}, \tag{4.1}$$

and the near-field expansion must be of the form

$$V(x, r, \theta; \epsilon) = \epsilon^{-1/2} V_0(x, r, \theta) + V_1(x, r, \theta) + \epsilon^{1/2} V_2(x, r, \theta) + \epsilon V_3(x, r, \theta) + \cdots \tag{4.2}$$

Substituting (4.2) in (1.10) and (1.11), and requiring the large-x behavior in each order to conform to (3.35), we obtain the following sequence of near-field problems,

$$\begin{cases} \nabla^2 V_0 = 0 \\ \dfrac{\partial V_0}{\partial r}(x, 1, \theta) = 0 \\ V_0(x, r, \theta) \to A \text{ as } |x| \to \infty, \end{cases} \tag{4.3}$$

$$\begin{cases} \nabla^2 V_1 = -\dfrac{1}{r}\delta(x)\delta(r-R)\delta(\theta) \\ \dfrac{\partial V_1}{\partial r}(x, 1, \theta) = 0 \\ V_1(x, r, \theta) \to -A\sqrt{2}\,|x| \text{ as } |x| \to \infty, \end{cases} \tag{4.4}$$

$$\begin{cases} \nabla^2 V_2 = 0 \\ \dfrac{\partial V_2}{\partial r}(x, 1, \theta) = -V_0(x, 1, \theta) \\ V_2(x, r, \theta) \to A(x^2 - \dfrac{r^2}{2}) + B \text{ as } |x| \to \infty, \end{cases} \tag{4.5}$$

$$\begin{cases} \nabla^2 V_3 = 0 \\ \dfrac{\partial V_3}{\partial r}(x, 1, \theta) = -V_1(x, 1, \theta) \\ V_3(x, r, \theta) \to \sqrt{2}\,|x| \left[A(\dfrac{1}{8} - \dfrac{x^2}{3} + \dfrac{r^2}{2}) - B\right] \text{ as } |x| \to \infty, \end{cases} \tag{4.6}$$

$$\begin{cases} \nabla^2 V_4 = 0 \\ \dfrac{\partial V_4}{\partial r}(x, 1, \theta) = -V_2(x, 1, \theta) \\ V_4(x, r, \theta) \to A(-\dfrac{x^2}{4} + \dfrac{x^4}{6} - \dfrac{r^2x^2}{2} + \dfrac{r^2}{8} + \dfrac{r^4}{16}) + B(x^2 - \dfrac{r^2}{2}) + C \text{ as } |x| \to \infty, \end{cases} \tag{4.7}$$

$$
\begin{cases}
\nabla^2 V_5 = 0 \\
\dfrac{\partial V_5}{\partial r}(x, 1, \theta) = - V_3(x, 1, \theta) \\
V_5(x, r, \theta) \rightarrow \sqrt{2}\, |x| \left[A\left(-\dfrac{5}{384} + \dfrac{x^2}{8} + \dfrac{x^2 r^2}{6} - \dfrac{3r^2}{16} - \dfrac{r^4}{16} - \dfrac{x^4}{30}\right) \right. \\
\qquad\qquad \left. + B\left(\dfrac{1}{8} - \dfrac{x^2}{3} + \dfrac{r^2}{2}\right) - C \right] \text{ as } |x| \rightarrow \infty,
\end{cases}
\tag{4.8}
$$

The delta-function source appears in the V_1 problem, consistent with the linear growth with x as $|x| \rightarrow \infty$. All other orders of the potential are source-free.

Each even(odd)-order problem (except for the first two) is coupled to the preceding even(odd)-order problem via the boundary condition on the $r = 1$ surface. The physical interpretation of this coupling is that the current crossing the membrane in the nth problem is proportional to the membrane potential in the (n-2)th problem. The even-order problems are coupled to the odd-order problems by their asymptotic behavior as $|x| \rightarrow \infty$, i.e., the constants A, B, C, ... appear in both even- and odd-order problems.

It should be noted that it is only from considerations of the behavior of the near-field potential at large $|x|$, required to match the behavior of the far-field potential at small x^*, that we conclude that the V_0, $V_2, \ldots$ terms are even necessary. The V_1, $V_3, \ldots$ terms alone are sufficient to satisfy (1.10) and (1.11) at small x. This is an extremely interesting result. The dominant term as $x \rightarrow 0$, V_0, can only be determined by considering the behavior of the potential as $|x| \rightarrow \infty$.

By direct substitution of the $|x| \rightarrow \infty$ asymptotic forms of V_0, V_2 and V_4 in the respective equations and boundary conditions (4.3), (4.5) and (4.7), it is seen that the asymptotic forms are actually solutions valid for all x, and by the uniqueness theorem for Laplace's equation, they are the only solutions. Thus

$$V_0 = A, \tag{4.9}$$

$$V_2 = A(x^2 - \frac{r^2}{2}) + B, \tag{4.10}$$

$$V_4 = A(-\frac{x^2}{4} + \frac{x^4}{6} - \frac{r^2x^2}{2} + \frac{r^2}{8} + \frac{r^4}{16}) + B(x^2 - \frac{r^2}{2}) + C. \tag{4.11}$$

Now we discuss the V_1 problem. Integrating (4.4) over the large volume of the cylinder between -x and x, $|x| \to \infty$, and using the divergence theorem, we obtain,

$$\lim_{|x|\to\infty} \int_0^{2\pi} d\theta \int_0^1 rdr \int_{-x}^{x} dx\, \nabla^2 V_1 = -1$$

$$= \lim_{|x|\to\infty} \int_0^{2\pi} d\theta \int_0^1 rdr \left[\frac{\partial V_1}{\partial x}(x, r, \theta) - \frac{\partial V_1}{\partial x}(-x, r, \theta)\right]$$

$$= -2\pi A\sqrt{2} \tag{4.12}$$

In accordance with the r = 1 boundary condition in (4.4) the integral over the surface of the cylinder is zero, leaving only the integral over the discs at ±x. The last equality follows after substituting the asymptotic behavior of V_1, as $|x| \to \infty$, obtained from (4.4). Solving for A,

$$A = \frac{\sqrt{2}}{4\pi}, \tag{4.13}$$

and hence

$$\lim_{|x|\to\infty} V_1(x, r, \theta) = -\frac{|x|}{2\pi} \tag{4.14}$$

It is now convenient to decompose the near-field potential V_1, into two terms,

$$V_1(x, r, \theta) = \Phi_1(x, r, \theta) - \frac{|x|}{2\pi}. \tag{4.15}$$

Substituting (4.15) in (4.4), we obtain the problem for Φ_1,

$$
\begin{cases}
\nabla^2 \Phi_1 = -\delta(x)\left[\frac{1}{r}\,\delta(r-R)\delta(\theta) - \frac{1}{\pi}\right] \\
\frac{\partial \Phi_1}{\partial r}(x, 1, \theta) = 0 \\
\Phi_1(\pm\infty, r, \theta) = 0
\end{cases}
\tag{4.16}
$$

The source term in (4.16) is the unit point source at (0, R, 0) plus the uniform disc sink in the x = 0 plane. The net current source for Φ_1 is zero, i.e., all the current which enters the cylinder at the point (0, R, 0) is removed uniformly in the cross section (0, r, θ). Unlike the problem for V_1, which contains unit current flowing outward as $|x| \to \infty$, the problem for Φ_1 contains no current flow as $|x| \to \infty$.

Taking the Fourier cosine transform of the equation in (4.16) observing that Φ_1 is an even function of x, we obtain

$$
\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial \theta^2} - k^2 \phi_1 = -\frac{1}{r}\,\delta(r-R)\delta(\theta) + \frac{1}{\pi},
\tag{4.17}
$$

where ϕ_1 is the Fourier transform of Φ_1, i.e.,

$$
\phi_1(k, r, \theta) = \int_{-\infty}^{\infty} \cos kx \; \Phi_1(x, r, \theta)dx,
\tag{4.18a}
$$

and

$$
\Phi_1(x, r, \theta) = \frac{1}{\pi}\int_0^{\infty} \cos kx \; \phi_1(k, r, \theta)dk.
\tag{4.18b}
$$

Multiplying (4.17) by $e^{-in\theta}$ and integrating over θ, we obtain

$$
\frac{1}{r}\frac{d}{dr} r \frac{d\psi_1^{(n)}}{dr} - \left(k^2 + \frac{n^2}{r^2}\right)\psi_1^{(n)} = -\frac{1}{r}\,\delta(r-R) + 2\delta_{0n},
\tag{4.19}
$$

where δ_{0n} is the Kronecker delta function and

$$\psi_1^{(n)}(k, r) = \int_0^{2\pi} e^{-in\theta} \phi_1(k, r, \theta) d\theta, \tag{4.20a}$$

and

$$\phi_1(k, r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \psi_1^{(n)}(k, r) e^{in\theta}. \tag{4.20b}$$

The solution to (4.19) may be written in the form

$$\psi_1^{(n)} = -\frac{2\delta_{0n}}{k^2} + \begin{cases} a_n I_n(kr), & 0 \leqq r \leqq R \\ b_n I_n(kr) + c_n K_n(kr), & R \leqq r \leqq 1, \end{cases} \tag{4.21}$$

where I_n and K_n are the modified Bessel functions.

Continuity of $\psi_1^{(n)}$ at $r = R$ implies that,

$$(a_n - b_n) I_n(kR) = c_n K_n(kR). \tag{4.22}$$

From (4.16), the boundary condition at $r = 1$ requires that,

$$b_n I_n'(k) + c_n K_n'(k) = 0. \tag{4.23}$$

Integrating (4.19) across the delta function at $r = R$, we obtain for the discontinuity in the derivative,

$$\frac{d\psi_1^{(n)}}{dr}(k, R^+) - \frac{d\psi_1^{(n)}}{dr}(k, R^-) = -\frac{1}{R},$$

which yields, when substituted in (4.21),

$$(b_n - a_n) I_n'(kR) + c_n K_n'(kR) = -\frac{1}{kR}. \tag{4.24}$$

Using (4.22) to eliminate $(a_n - b_n)$ from (4.24) yields

$$c_n = I_n(kR), \tag{4.25}$$

where we used the Wronskian,[11]

$$I_n(kR)K_n'(kR) - K_n(kR)I_n'(kR) = -\frac{1}{kR},$$

to obtain (4.25). Substituting (4.25) in (4.23), we find

$$b_n = - I_n(kR) \frac{K_n'(k)}{I_n'(k)} . \tag{4.26}$$

Substituting (4.25) and (4.26) in (4.22), we have

$$a_n = K_n(kR) - I_n(kR) \frac{K_n'(k)}{I_n'(k)} . \tag{4.27}$$

Using the expressions (4.25) - (4.27) in (4.21), we obtain

$$\psi_1^{(n)}(k, r) = - \frac{2\delta_{0n}}{k^2} - I_n(kR) \frac{K_n'(k)}{I_n'(k)} I_n(kr) + \begin{cases} K_n(kR)\, I_n(kr), & 0 \leqq r \leqq R \\ K_n(kr)\, I_n(kR), & R \leqq r \leqq 1 . \end{cases} \tag{4.28}$$

Taking the inverse transforms (4.18b) and (4.20b) of (4.28), yields

$$\begin{aligned} \Phi_1(x, r, \theta) &= \frac{1}{2\pi^2} \int_0^\infty dk \cos kx \sum_{n=-\infty}^{\infty} \psi_1^{(n)}(k, r)\, e^{in\theta} \\ &= \frac{1}{4\pi} (x^2 + r^2 + R^2 - 2rR \cos\theta)^{-\frac{1}{2}} \\ &\quad - \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty dk \cos kx \left[\frac{K_n'(k)}{I_n'(k)} I_n(kR) I_n(kr) + \frac{2\delta_{0n}}{k^2} \right] \end{aligned} \tag{4.29}$$

The first term in (4.29) is just the free-space potential of a point source at (0, R, 0). It is the double transform of the curly-bracketed term in (4.28) and is obtained using the known sum,[12]

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{in\theta} K_n(kR) I_n(kr) &= \sum_{n=-\infty}^{\infty} e^{in\theta} K_n(kr) I_n(kR) \\ &= K_0\left(k\sqrt{r^2 + R^2 - 2rR \cos\theta}\right) , \end{aligned}$$

and the integral[13]

$$\int_0^\infty dk \cos(kx) K_0(k\xi) = \frac{\pi}{2\sqrt{\xi^2+x^2}} \quad .$$

The potential $V_1(x, r, \theta)$ is obtained by substituting (4.29) in (4.15) and the result is

$$V_1(x, r, \theta) = -\frac{|x|}{2\pi} + \frac{1}{4\pi}(x^2+r^2+R^2-2rR\cos\theta)^{-\frac{1}{2}}$$
$$-\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk \cos kx \left[\frac{K_n'(k)}{I_n'(k)} I_n(kR) I_n(kr) + \frac{2\delta_{0n}}{k^2}\right]. \tag{4.30}$$

The integral over k in (4.30) can be replaced by an equivalent sum, by considering the integral in (4.30) as a portion of a contour integral. This is done in Appendix A and the result is

$$V_1(x, r, \theta) = -\frac{|x|}{2\pi} - \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{s=1}^{\infty} e^{-\lambda_{ns}x}\frac{J_n(\lambda_{ns}R)J_n(\lambda_{ns}r)}{\lambda_{ns}\left(\frac{n^2}{\lambda_{ns}^2}-1\right)J_n^2(\lambda_{ns})} \tag{4.31}$$

where λ_{ns} is the s-th zero of $J_n'(\lambda)$ excluding the one at $\lambda = 0$.

Using (4.31) it can be demonstrated that as $|x| \to \infty$, $V_1 \to -|x|/2\pi$ plus terms which are exponentially small in x, and hence the solution (4.30) or (4.31) is the required solution to (4.4).

In Appendix B, a second alternative to (4.30) is developed by removing from the integral a term which is the free-space potential of a uniform unit disc sink located in the x = 0 plane. The remaining integral then has continuous derivatives everywhere in the interior of the cylinder. The result is given in (B.5).

In Appendix D, a third alternative to (4.30) is derived which is useful for calculating the potential near the location of the source, (0, R, 0). For values $|x| \lesssim 0.2$ the convergence of (4.31) is too slow for doing numerical calculations and in fact at x = 0 it diverges. Equation (D. 14) has an adjustable parameter, β. It converges for $|x| < 2\pi/\beta$ and converges rapidly and hence is a very convenient formula for calculating V_1 when $|x| \lesssim 1/\beta$. Equations (4. 30) and (4. 31) are special cases of (D. 14), for $\beta = 0$ and $\beta = \pi/x$, respectively. Numerical calculations of V_1, using (4. 31) and (D. 14) will be included in a future report. The method developed in Appendix D for obtaining a rapidly converging series is of more general applicability and has been successfully applied to the Green's function of Laplace's equation for a Dirichlet boundary condition on the cylinder surface.

We now turn to the V_3 problem. Integrating the Laplacian in (4. 6) over a large cylinder extending from - x to x, and using the divergence theorem, we have

$$\lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^1 r dr \int_0^{2\pi} d\theta \, \nabla^2 V_3 = 0$$

$$= \lim_{|x|\to\infty} \left[\int_{-x}^{x} dx \int_0^{2\pi} d\theta \, \frac{\partial V_3}{\partial r}(x, 1, \theta) + \int_0^1 r dr \int_0^{2\pi} d\theta \left\{ \frac{\partial V_3}{\partial r}(x, r, \theta) - \frac{\partial V_3}{\partial x}(-x, r, \theta) \right\} \right] \tag{4.32}$$

Using the boundary condition in (4. 6) and (4. 15), (4. 18b) and (4. 20b) for V_1, the first integral in (4. 32) becomes,

$$- \lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^{2\pi} d\theta \, V_1(x, 1, \theta)$$

$$= - \lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^{2\pi} d\theta \left[- \frac{|x|}{2\pi} + \frac{1}{2\pi^2} \int_0^{\infty} dk \cos kx \sum_{n=-\infty}^{\infty} \psi_1^{(n)}(k, 1) e^{in\theta} \right]$$

$$= x^2 - \frac{1}{\pi} \int_{-\infty}^{\infty} dx \int_0^{\infty} dk \cos kx \, \psi_1^{(0)}(k, 1)$$

$$= x^2 - \psi_1^{(0)}(0, 1) \tag{4.33}$$

From (4.28), we obtain, with the aid of a known identity[14] and the power series expansion of $I_n(k)$,[15]

$$\psi_1^{(0)}(k,1) = -\frac{2}{k^2} + \frac{K_1(k)}{I_1(k)} I_0(kR)I_0(k) + K_0(k)I_0(kR)$$

$$= -\frac{2}{k^2} + \frac{K_1(k)I_0(k) + K_0(k)I_1(k)}{I_1(k)} I_0(kR)$$

$$= -\frac{2}{k^2} + \frac{I_0(kR)}{kI_1(k)}$$

$$= -\frac{2}{k^2} + \frac{1+(\frac{kR}{2})^2+\ldots}{\frac{k^2}{2}[1+\frac{1}{2}(\frac{k}{2})^2+\ldots]}$$

$$= \frac{1}{2}(R^2 - \frac{1}{2}) + O(k^2),$$

so that,

$$\psi_1^{(0)}(0,1) = \frac{1}{2}(R^2 - \frac{1}{2}). \tag{4.34}$$

Using the asymptotic form for large $|x|$ for V_3 from (4.6), the second integral in (4.32) becomes,

$$\int_0^1 r\,dr \int_0^{2\pi} d\theta\, 2\sqrt{2}\,\left[A(\frac{1}{8} - x^2 + \frac{r^2}{2}) - B\right] = 2\pi\sqrt{2}\,\left[A(\frac{3}{8} - x^2) - B\right]. \tag{4.35}$$

Combining (4.32) - (4.35) yields the equation

$$2\pi\sqrt{2}\,\left[A(\frac{3}{8} - x^2) - B\right] + x^2 - \frac{1}{2}(R^2 - \frac{1}{2}) = 0. \tag{4.36}$$

Substituting (4.13) for A in (4.36) and solving for B,

$$B = \frac{\sqrt{2}}{4\pi}(\frac{5}{8} - \frac{R^2}{2}). \tag{4.37}$$

As consequence of (4.37), W_1 and V_2 depend on R, the distance from the source to the axis of the cylinder, whereas lower-order terms do not.

Having evaluated A and B in (4.13) and (4.37), we have now obtained the near field up to terms of $O(\epsilon^{\frac{1}{2}})$, i.e., we obtained V_0, V_1, and V_2 as given by (4.9), (4.30) and (4.10), respectively. In addition we have also obtained the far field to $O(\epsilon^{\frac{1}{2}})$: W_0 and W_1 as given by (3.20) and (3.27) respectively. These terms represent that part of the potential which is numerically significant; all higher-order terms are too small to detect in a physiological measurement made anywhere in a cylindrical cell. Nevertheless, it is of some mathematical interest to carry out the calculation further in order to demonstrate that the process can be continued indefinitely, although it clearly soon becomes quite tedious. We continue as far as necessary to calculate the constant C, and will discuss the results at that point.

In order to obtain C, we must proceed in solving the problem for V_3. The method employed is identical to that applied to the V_1 problem, namely, a new potential, Φ_3, is defined which approaches zero for large $|x|$. Thus,

$$V_3(x, r, \theta) = \Phi_3(x, r, \theta) + \frac{|x|}{4\pi} (R^2+r^2-1-\frac{2}{3}x^2), \tag{4.38}$$

where we have used the expressions (4.13) and (4.37) for A and B in the asymptotic form of V_3 given in (4.6).

Substituting (4.38) and the definition of Φ_1, (4.15), in the problem for V_3, (4.6), yields the problem for Φ_3,

$$\begin{cases} \nabla^2 \Phi_3 = - \frac{1}{2\pi} (R^2+r^2-1)\delta(x) \\ \frac{\partial \Phi_3}{\partial r} = - \Phi_1 \\ \Phi_3(\pm\infty, r, \theta) = 0 \, . \end{cases} \tag{4.39}$$

The source term in (4.39) is a nonuniform distribution of current on the disc at x = 0, plus the current crossing the membrane given by the r = 1 boundary condition.

It can be easily verified that the algebraic term in (4.38) satisfies Laplace's equation for $|x| > 0$, as well as the $r = 1$ boundary condition in (4.6). It has a discontinuous derivative at $x = 0$, however, and so is not a solution to (4.6) at $x = 0$. The function Φ_3, which has a source at $x = 0$, must be added to the algebraic term to obtain a solution valid everywhere. The discontinuity in the derivative of Φ_3 will be just the negative of the discontinuity in the derivative of the algebraic term.

As in the Φ_1 problem, Φ_3 is an even function of x, and we define the Fourier cosine transform of Φ_3 by

$$\phi_3(k, r, \theta) = \int_{-\infty}^{\infty} \cos kx \ \Phi_3(x, r, \theta) dx, \tag{4.40a}$$

and

$$\Phi_3(x, r, \theta) = \frac{1}{\pi} \int_0^{\infty} \cos kx \ \phi_3(k, r, \theta) dk, \tag{4.40b}$$

so that the Fourier transform of the equation in (4.39) is

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_3}{\partial \theta^2} - k^2 \phi_3 = -\frac{1}{2\pi}(R^2 + r^2 - 1) . \tag{4.41}$$

Multiplying (4.41) by $e^{-in\theta}$, and defining $\psi_3^{(n)}(k, r)$ by

$$\psi_3^{(n)}(k, r) = \int_0^{2\pi} e^{-in\theta} \phi_3(k, r, \theta) d\theta , \tag{4.42a}$$

and

$$\phi_3(k, r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \psi_3^{(n)}(k, r) e^{in\theta}, \tag{4.42b}$$

the Fourier transform of (4.41) with respect to θ is,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi_3^{(n)}}{\partial r} - (k^2 + \frac{n^2}{r^2}) \psi_3^{(n)} = -(R^2 + r^2 - 1)\delta_{0n} . \tag{4.43}$$

The constant B was determined by considering the volume integral of the V_3 problem (4.6), and was seen to be related to the large-$|x|$ behavior of V_3 and V_1, and to $\psi_1^{(0)}(0, 1)$. In exactly the same way, by considering the volume integral of the V_5 problem (4.8), (without actually solving for V_5), we can determine C from our present knowledge of the large-$|x|$ behavior of V_5 and V_3, and of $\psi_3^{(0)}(0, 1)$. Having determined C, we will then have $V_4(x, r)$ and $W_2(x^*, r)$.

The volume integral of (4.8) is

$$\lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^1 r dr \int_0^{2\pi} d\theta \, \nabla^2 V_5 = 0$$

$$= \lim_{|x|\to\infty} \left[-\int_{-x}^{x} dx \int_0^{2\pi} d\theta \, V_3(x, 1, \theta) + 2\int_0^1 r dr \int_0^{2\pi} d\theta \, \frac{\partial V_5}{\partial r}(x, r, \theta) \right], \qquad (4.48)$$

where we have used the boundary condition in (4.8) to obtain the first integral on the right-hand side, and the evenness in x of V_5 to obtain the second.

The first integral in (4.48) can be evaluated by using (4.38) for V_3 and (4.40b) and (4.42b) for the inverse transforms of Φ_3,

$$- \lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^{2\pi} d\theta \, V_3(x, 1, \theta)$$

$$= - \lim_{|x|\to\infty} \int_{-x}^{x} dx \int_0^{2\pi} d\theta \left[\frac{|x|}{4\pi} (R^2 - \frac{2}{3} x^2) + \frac{1}{2\pi^2} \int_0^{\infty} dk \cos kx \sum_{n=-\infty}^{\infty} \psi_3^{(n)}(k, 1) e^{in\theta} \right]$$

$$= - \frac{x^2}{2} (R^2 - \frac{1}{3} x^2) - \psi_3^{(0)}(0, 1). \qquad (4.49)$$

We now evaluate $\psi_3^{(0)}(0, 1)$ by expanding (4.46) in a power series about k = 0. All the singularities in (4.46) at k=0 cancel, as they must if the integral in (4.47) is to converge. From (4.46),

$$\psi_3^{(0)}(k, 1) = \frac{1}{k^2}\left[R^2 + \frac{4}{k^2} - \frac{I_0(kR)I_0(k)}{I_1^{\,2}(k)}\right]$$

$$= \frac{1}{k^2}\left[R^2 + \frac{4}{k^2} - \frac{\{1+(\frac{kR}{2})^2+\frac{1}{4}(\frac{kR}{2})^4+\cdots\}\{1+(\frac{k}{2})^2+\frac{1}{4}(\frac{k}{2})^4+\cdots\}}{(\frac{k}{2})^2\{1+\frac{1}{2}(\frac{k}{2})^2+\frac{1}{12}(\frac{k}{2})^4+\ldots\}^2}\right]$$

$$= \frac{1}{16}\left(\frac{2}{3} - R^4\right) + \cdots \tag{4.50}$$

The second integral on the right hand side of (4.48) is

$$4\pi\int_0^1 rdr\,\frac{\partial}{\partial x}\left[\sqrt{2}\,x\left\{A\left(-\frac{5}{384}+\frac{x^2}{8}+\frac{x^2r^2}{6}-\frac{3r^2}{16}-\frac{r^4}{16}-\frac{x^4}{30}\right)+B\left(\frac{1}{8}-\frac{x^2}{3}+\frac{r^2}{2}\right)-C\right\}\right]$$

$$= 4\pi\,\frac{\partial}{\partial x}\left[\sqrt{2}\,x\left\{A\left(-\frac{49}{768}+\frac{5x^2}{48}-\frac{x^4}{60}\right)+B\left(\frac{3}{16}-\frac{x^2}{6}\right)-\frac{1}{2}C\right\}\right]$$

$$= 4\pi\,\sqrt{2}\left\{A\left(-\frac{49}{768}+\frac{5x^2}{16}-\frac{x^4}{12}\right)+B\left(\frac{3}{16}-\frac{x^2}{2}\right)-\frac{1}{2}C\right\}$$

$$= \frac{41}{384} - \frac{3R^2}{16} + \frac{x^2R^2}{2} - \frac{x^4}{6} - 2\pi\sqrt{2}\,C \tag{4.51}$$

Substituting (4.49) - (4.51) in (4.48), and solving for C,

$$C = \frac{\sqrt{2}}{64\pi}\left[\frac{25}{24} - 3R^2 + R^4\right]. \tag{4.52}$$

5. SUMMARY AND DISCUSSION OF RESULTS

We have now calculated all the coefficients appearing in the first three terms in the far-field expansion, and in the first five terms in the near-field expansion. This permits us to write expressions for the potential (far- or near-field) to $O(\epsilon^{3/2})$. Substituting the results for A, B and C from (4.13), (4.37) and (4.52) in the expressions for W_0, W_1, and W_2 given in (3.20), (3.27) and (3.33), and then substituting these plus ζ_0 from (4.1) in the expansion (3.34), we obtain the expression for the far-field potential,

$$W(x^*, r, \theta; \epsilon) = \epsilon^{-1/2} W_0(x^*, r) + \epsilon^{1/2} W_1(x^*, r) + \epsilon^{3/2} W_2(x^*, r) + \cdots$$

$$= \frac{\sqrt{2}}{4\pi} e^{-\sqrt{2}|x^*|} \Big[\epsilon^{-1/2} + \frac{1}{2}\epsilon^{1/2}\{\frac{5}{4} - (r^2 + R^2)\}$$

$$+ \frac{1}{16}\epsilon^{3/2}\{\frac{25}{24} - 3(r^2 + R^2) + r^4 + 4r^2R^2 + R^4\}$$

$$+ O(\epsilon^{5/2})\Big], \tag{5.1}$$

where the far field axial variable is

$$x^* = \sqrt{\epsilon}\, x(1 - \frac{1}{8}\epsilon + \frac{5}{384}\epsilon^2 - \cdots). \tag{3.32}$$

Similarly, if we substitute the results for A, B and C in the near-field expressions for V_0, V_2, and V_4 given by (4.9), (4.10) and (4.11), and then substitute these plus V_1 and V_3 given by (4.30) and (4.47) in (4.2), we obtain the expression for the near-field potential,

$$V(x,r,\theta;\epsilon) = \epsilon^{-1/2} V_0(x,r,\theta) + V_1(x,r,\theta) + \epsilon^{1/2} V_2(x,r,\theta) + \cdots$$

$$= \epsilon^{-1/2} \frac{\sqrt{2}}{4\pi} - \frac{|x|}{2\pi} + \frac{1}{4\pi}(x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

$$- \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty dk \cos kx \left\{ \frac{K_n'(k)}{I_n'(k)} I_n(kR) I_n(kr) + \frac{2\delta_{0n}}{k^2} \right\}$$

$$+ \epsilon^{1/2} \frac{\sqrt{2}}{8\pi} \left[\frac{5}{4} + 2x^2 - (r^2 + R^2) \right]$$

$$+ \epsilon \left[\frac{|x|}{4\pi} (r^2 + R^2 - \frac{2}{3} x^2 - 1) \right.$$

$$\left. - \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty dk \frac{\cos kx}{k^2} \left\{ \frac{I_n(kR) I_n(kr)}{[I_n'(k)^2]} - (r^2 + R^2 - 1 + \frac{4}{k^2}) \delta_{0n} \right\} \right]$$

$$+ \epsilon^{3/2} \frac{\sqrt{2}}{64\pi} \left[\frac{1}{24} (25 + 144 x^2 + 64 x^4) - (r^2 + R^2)(3 + 8x^2) \right.$$

$$\left. + r^4 + 4r^2R^2 + R^4 \right]$$

$$+ O(\epsilon^2). \tag{5.2}$$

The two k integrals in (5.2) can be replaced by the equivalent representations (4.31) or (D.14), and (C.9), respectively.

The leading terms in the far-field expansion (5.1), and in the near-field expansion (5.2), are each of order $\epsilon^{-1/2}$. In the near field, the leading term is a constant. Thus, near the point source, the interior of the cylinder is raised to a large, constant potential, relative to the zero potential at infinity. The physical basis for the large potential is that the membrane permits only a small fraction of the current to leave the cylinder per unit length. Consequently, most of the current flows a long distance before getting out, and a large potential drop is required to force this current down he cylinder. The existence of this large constant potential, and its magni- ude of $O(\epsilon^{-1/2})$, could only be deduced from considerations of the far field.

The leading term in the far field decays as $\exp(-\sqrt{2\epsilon}\ |x|)$. Consequently, to lowest order, $1/e$ of the current leaves the cylinder in a distance of $1/\sqrt{2\epsilon}$. The corresponding potential required to drive a current this distance is of $O(\epsilon^{-1/2})$, which is the physical basis for the order of the large potential in the near field. The precise numerical ratio of the leading terms in (5.1) and (5.2) was determined by requiring in the limit $|x| \to \infty$, $x^* \to 0$, that the two terms be identical to the lowest order in ϵ.[8] In the far far field, i.e., $x^* = x\sqrt{\epsilon}(1 - \frac{\epsilon}{8} + \ldots) \to \infty$, the potential is seen to approach zero exponentially.

The leading term in the far-field expansion (5.1) is independent of r and θ. Thus, to the lowest order the far-field current is distributed uniformly over the circular cross-section of the cylinder. The leading term in (5.1) is the known result of one-dimensional cable theory.[16] The high-order terms are all independent of the azimuthal angle θ. They do, however, depend on the radial coordinate r. The dependence is in the form of a polynomial in r^2, the degree of the polynomial increasing by one in each successive term. We also see that the higher-order terms also depend on R, the radial distance between the source and the axis of the cylinder. The potential is seen to be symmetric with respect to an interchange of r and R. This must be so because the potential is the Green's function (with source at $x = 0$, $\theta = 0$) for the cylindrical problem.[17]

Successive terms in the far-field expansion decrease in powers of ϵ, whereas in the near-field expansion they decrease in powers of $\sqrt{\epsilon}$.

The second, $O(1)$, term in the near-field expansion, as written in (5.2), contains three parts. It is the solution to the problem (4.4), in which no current crosses the membrane, i.e., $\partial V_1/\partial r = 0$ at $r = 1$. The first part of this term decreases linearly with increasing $|x|$. It corresponds to the potential required to drive a constant current parallel to the axis of the cylinder in the interior of the cylindrical cell. It is the appearance of this term in the expansion which led us to conclude that an expansion of the form (5.2) could not describe the potential for all x, since we could not satisfy the boundary condition at $|x| = \infty$ with such a term present.

The second part of the O(1) term is the free-space potential of a point source. It is the only singular part of the solution, accounting fully for the singularity at the location (0, R, 0) of the delta-function source. The third part is more complicated. When added to the first two parts, it satisfies the boundary condition at r = 1, and removes the discontinuity in the x-derivatives of the potential at x = 0, arising from the first part.

The third, $O(\epsilon^{1/2})$, term in the near-field expansion is a polynomial of second degree in x, r and R. In general, each term of $O(\epsilon^{(2n+1)/2})$, n an integer, are simply polynomials of degree 2n+2. The $O(\epsilon^{1/2})$ term was required as a consequence of the $O(\epsilon^{-1/2})$ term and the coupling between orders given in the sequence of problems (4.3) - (4.8), and matching to the far field.

The fourth, $O(\epsilon)$, term (specified by (4.6)), and subsequently all higher terms of $O(\epsilon^{n})$, contain a polynomial of degree 2n + 1, in $|x|$, r, and R and a more complicated infinite sum, infinite integral term. Higher order terms are determined by solving the appropriate differential equation (analagous to (4.8)) and boundary conditions.

The first two terms in the far field, W_0 and W_1, and the first three terms in the near field, V_0, V_1, and V_2, represent the physiologically significant part of the potential. Higher-order terms are too small to be detectable at any location in a cylindrical biological cell. The higher-order terms W_2, V_3 and V_4 are given to illustrate their interesting mathematical properties, and to provide a precise measure of the magnitude of the error introduced by using only the preceding terms.

A single expression which is uniformly valid in x can be written down. We saw in Section 4 that the polynomial parts of the near field are exactly equal to the respective terms in the expansion (3.35) of the far field in near-field coordinates. The two complicated infinite-sum and infinite-integral terms in the near field are exponentially small in the far field. Consequently, the potential everywhere can be obtained from the single representation,

$$V(x, r, \theta; \epsilon) = \frac{\sqrt{2}}{4\pi} e^{-\sqrt{2\epsilon}\,|x|\,(1-\frac{\epsilon}{8}+\frac{5\epsilon}{384}-O(\epsilon^2))} \epsilon^{-1/2}\Big[1+\frac{\epsilon}{2}\{\frac{5}{4}-(r^2+R^2)\}$$
$$+\frac{\epsilon^2}{16}\{\frac{25}{24}-3(r^2+R^2)+r^4+4r^2R^2+R^4\}+O(\epsilon^3)\Big]$$
$$+\frac{1}{4\pi}(x^2+r^2+R^2-2rR\cos\theta)^{-1/2}$$
$$-\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\cos kx\left[\frac{K_n'(k)}{I_n'(k)} I_n(kR)I_n(kr)+\frac{2\delta_{0n}}{k^2}\right.$$
$$\left.+\frac{\epsilon}{k^2}\left\{\frac{I_n(kR)I_n(kr)}{[I_n'(k)]^2}-(r^2+R^2-1+\frac{4}{k^2})\delta_{0n}\right\}+O(\epsilon^2)\right], \qquad (5.3)$$

which is asymptotic to (5.1) in the limit $\epsilon \to 0$ with $x\sqrt{\epsilon}$ held fixed, and to (5.2) in the limit $\epsilon \to 0$ with x held fixed. Again, we can replace the two integrals by (4.31) or (D.14), and (C.9). Equation (5.3) is more compact than (5.1) and (5.2), but the latter two have the advantage of clearly separating the terms according to their order in ϵ in the regions $x >> \epsilon^{-1/2}$ and $x << \epsilon^{-1/2}$ respectively, and each order of ϵ is related to a simple physical problem. This solution is closely related to the solution derived by Barcilon, Cole and Eisenberg[9] using another technique of singular perturbation theory, multiple scaling.

6. RELATION OF PERTURBATION EXPANSIONS TO EXACT SOLUTION

A. Exact Solution

The exact solution to the problem defined by (1.10) - (1.12) can be obtained directly in the form of an eigenfunction expansion. It is of some interest to show how our asymptotic expansions can be obtained directly from this eigenfunction solution. This serves as a check on the singular perturbation procedures used, and as a bonus, gives us a closed-form expression for the sum over ϵ in the far-field expansion (5.1).

The eigenfunction expansion for the solution to (1.10) - (1.12) is[10]

$$V(x, r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{s=1}^{\infty} \frac{\beta_{ns} e^{-\beta_{ns}|x|}}{\beta_{ns}^2 - n^2 + \epsilon^2} \frac{J_n(\beta_{ns} r) J_n(\beta_{ns} R)}{J_n^2(\beta_{ns})}$$

$$+ \frac{1}{2\pi} \frac{\beta_{00} e^{-\beta_{00}|x|}}{\beta_{00}^2 + \epsilon^2} \frac{J_n(\beta_{00} r) J_n(\beta_{00} R)}{J_n^2(\beta_{00})} , \qquad (6.1)$$

in which β_{ns} is a nonzero root of

$$\beta_{ns} J_n'(\beta_{ns}) = -\epsilon J_n(\beta_{ns}). \qquad (6.2)$$

The roots are arranged in ascending magnitude with $s = 1, 2, 3, \ldots$ for $n \neq 0$, and $s = 0, 1, 2, \ldots$ for $n = 0$. The $n = 0$ case is distinguished by the fact that in the $\epsilon \to 0$ limit, the smallest root, β_{00}, approaches zero. All other roots β_{ns} approach finite values. With this definition of s, we have, except for $n = s = 0$,

$$\lim_{\epsilon \to 0} \beta_{ns} = \lambda_{ns},$$

where λ_{ns} is an eigenvalue of the perfectly insulated cylinder problem, that is, the nonzero roots of $J_n'(\lambda_{ns}) = 0$, which appear in the eigenfunction expansions of $V_1(x, r, \theta)$ and $V_3(x, r, \theta)$ in (4.31) and (4.61), respectively.

We will now show that the $n = s = 0$ term in (6.1) is equivalent to the far-field expansion (5.1) and also gives rise to the algebraic terms in the near-field expansion (5.2), and that the double sum in (6.1) can be expanded in the $\epsilon \to 0$ limit to obtain the nonalgebraic part of the near-field expansion (5.2)

B. Far-field limit of exact solution

The far-field expansion (5.1) has a factor $e^{-\sqrt{2}\,\eta|x|}$. For compactness, we define

$$\beta(\epsilon) = \sqrt{2}\,\eta(\epsilon) \tag{6.3}$$

If we let

$$V(x, r) = e^{-\beta|x|}\,\Psi(r) \tag{6.4}$$

and substitute (6.4) in (1.10) and (1.11) for V, we obtain the equation

$$\frac{1}{r}\frac{d}{dr}\,r\,\frac{d\Psi}{dr} + \beta^2\Psi = 0 \tag{6.5}$$

and the boundary condition

$$\frac{d\Psi}{dr}(1) + \epsilon\Psi(1) = 0 \tag{6.6}$$

which must be satisfied by the r-dependent part of the far field, $\Psi(r)$. The solution to (6.5) is

$$\Psi(r) = \mu(\epsilon)\, J_0(\beta R)\, J_0(\beta r) \tag{6.7}$$

where we have utilized the requirement that Ψ be invariant to an interchange of r and R to obtain the R dependence. From (6.6), β must be a root of

$$\epsilon = \frac{\beta J_1(\beta)}{J_0(\beta)} \tag{6.8}$$

where we have used $J_1(\beta) = -J_0'(\beta)$. For the smallest root of (6.8), we see that $\beta \to 0$ in the limit of $\epsilon \to 0$. All other roots of (6.8) have finite limits. Thus, in the far field, which is defined by taking the limit of the potential as $\epsilon \to 0$ for a fixed value of $x\beta(\epsilon)$, the other roots all give exponentially small contributions to V in comparison to the contribution of the smallest root. This is also true of all the roots of (6.2) for $n \neq 0$. It explains why only a single value of β survives in the far-field expansion. Comparing (6.8) to (6.2), we see that $\beta = \beta_{00}$.

We now find an expansion for this root, β, useful for small ϵ. Using the power series expansion of the Bessel functions around $\beta = 0$[18] in (6.8), we find

$$\epsilon = \frac{1}{2}\beta^2 + \frac{1}{16}\beta^4 + \frac{1}{96}\beta^6 + \dots \tag{6.9}$$

From (6.9) we see that we have $\beta \to \sqrt{2\epsilon}$ in the limit $\epsilon \to 0$, and β can be represented by an expansion of the form

$$\beta = \sqrt{2\epsilon}\,(1 + a^{(1)}\epsilon + a^{(2)}\epsilon^2 + \cdots). \tag{6.10}$$

Substituting (6.10) in (6.9), collecting terms in each power of ϵ, we evaluate $a^{(1)}$, $a^{(2)}$, ... and thereby obtain the reversion of the series (6.9),

$$\beta = \sqrt{2\epsilon}\,(1 - \frac{1}{8}\epsilon + \frac{5}{384}\epsilon^2 + \dots), \tag{6.11}$$

which, as is no great surprise, is identical to the series for $\sqrt{2}\,x^*/x$ obtained from (3.32). Thus, in the far field, the x-dependence of the potential as given by (5.1) is identical to the x-dependence of the $n = s = 0$ term of (6.1). The remaining task is to show that the form of $\mu(\epsilon)$ required to make (6.7) equivalent to (5.1) is just the form prescribed by the $n = s = 0$ term in (6.2). That is, the expansion of

$$W(x^*, r; \epsilon) = \frac{1}{2\pi}\,\frac{\beta e^{-\beta|x|}}{\beta^2+\epsilon^2}\,\frac{J_0(\beta r)J_0(\beta R)}{J_0^2(\beta)}, \tag{6.12}$$

the $n = s = 0$ term of the exact eigenfunction expansion, in powers of ϵ, in the limit $\epsilon \to 0$, for fixed βx, is identical to (5.1). In anticipation of the result we have designated this term $W(x^*, r; \epsilon)$ in (6.12).

The procedure for showing that this is true is simply to substitute the power series (6.11) for β in (6.12), expand each term in (6.12) in a power series and perform all indicated multiplications and divisions. We find:

$$\frac{\beta}{\beta^2+\epsilon^2} = \frac{1}{\sqrt{2\epsilon}}\left(1-\frac{3}{8}\epsilon+\frac{25}{384}\epsilon^2-\ldots\right) \tag{6.13a}$$

$$J_0(\beta) = 1-\frac{\epsilon}{2}+\frac{3}{16}\epsilon^2-\ldots \tag{6.13b}$$

$$J_0(\beta r) = 1-\frac{\epsilon r}{2}+\frac{3}{16}\epsilon^2 r^2-\ldots \tag{6.13c}$$

$$J_0(\beta R) = 1-\frac{\epsilon R}{2}+\frac{3}{16}\epsilon^2 R^2-\ldots \tag{6.13d}$$

and, substituting (6.13a - d) (6.12) and doing the indicated operations in (6.12), we find that (6.12) does indeed yield (5.1).

Thus we have found that the far field may be represented either by (6.12) and (6.11), or by (5.1) and (3.32). The form of (6.12) is more compact, and demonstrates that the r (and R) dependence is given by a zero-order Bessel function, which is a consequence of the simple exponential dependence on x. For small ϵ, however, the expansion (5.1) is more useful, and demonstrates the form of the potential more clearly. That is, the potential is independent of r (and R) to lowest order in ϵ; the dependence on r (and R) appears only in higher-order terms.

C. Near-field limit of exact solution

We have just shown that the n = s = 0 term in (6.1) is equivalent to the far-field expansion (5.1). By taking the limit $\epsilon \to 0$ with x fixed, as was done in going from (3.34) to (3.35), we also find that the n = s = 0 term in (6.1) yields the algebraic terms in the near-field expansion (5.2).

We will now show that the $\epsilon \to 0$, x fixed limit of the double sum in (6.1) accounts correctly for the remainder of the near-field expansion (5.2). We will obtain the near-field expansion in its double-sum form of (4.31) and (4.61).

The procedure we use is to find the expansion in powers of ϵ which expresses β_{ns}, the eigenvalue of the exact problem, in terms of λ_{ns}, the eigenvalue of the insulated cylinder problem, to substitute this expansion in the double sum in (6.1), and then expand the double sum in powers of ϵ.

We write (except for the case n = s = 0 which has already been analyzed),

$$\beta_{ns} = \lambda_{ns} + \mu_{ns} \tag{6.14}$$

where we have $\mu_{ns} \to 0$ in the limit $\epsilon \to 0$. Writing the Taylor series for the Bessel functions in (6.2) around $\beta_{ns} = \lambda_{ns}$, where $J'_n(\lambda_{ns}) = 0$, we have

$$J_n(\beta_{ns}) = J_n(\lambda_{ns}) + \frac{1}{2}\mu_{ns}^2 J''_n(\lambda_{ns}) + \cdots \tag{6.15a}$$

$$J'_n(\beta_{ns}) = \mu_{ns} J''_n(\lambda_{ns}) + \frac{1}{2}\mu_{ns}^2 J'''_n(\lambda_{ns}) + \cdots \tag{6.15b}$$

Using Bessel's equation,

$$J''_n(\lambda) = -\frac{1}{\lambda} J'_n(\lambda) + \left(\frac{n^2}{\lambda^2} - 1\right) J_n(\lambda)$$

and its first derivative,

$$J'''_n(\lambda) = -\frac{1}{\lambda} J''_n(\lambda) + \left(\frac{n^2+1}{\lambda^2} - 1\right) J'_n(\lambda) - \frac{1}{\lambda} J''_n(\lambda)$$

we have, letting $\lambda = \lambda_{ns}$, and $J'_n(\lambda_{ns}) = 0$,

$$J''_n(\lambda_{ns}) = \left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J_n(\lambda_{ns}) \tag{6.16a}$$

$$J'''_n(\lambda_{ns}) = -\frac{1}{\lambda_{ns}}\left(\frac{3n^2}{\lambda_{ns}^2} - 1\right) J_n(\lambda_{ns}) \tag{6.16b}$$

Substituting (6.16a, b) in (6.15a, b) we obtain

$$J_n(\beta_{ns}) = J_n(\lambda_{ns})\left[1 + \left(\frac{n^2}{\lambda_{ns}^2} - 1\right)\frac{\mu_{ns}^2}{2} - \frac{1}{\lambda_{ns}}\left(\frac{3n^2}{\lambda_{ns}^2} - 1\right)\frac{\mu_{ns}^3}{6} + \cdots\right] \tag{6.17a}$$

$$J_n'(\beta_{ns}) = J_n(\lambda_{ns})\,\mu_{ns}\left[\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) - \frac{1}{\lambda_{ns}}\left(\frac{3n^2}{\lambda_{ns}^2} - 1\right)\frac{\mu_{ns}}{2} + \cdots\right] \tag{6.17b}$$

which, when substituted in (6.2) yield the equation

$$(\lambda_{ns} + \mu_{ns})\mu_{ns}\left[\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) - \frac{1}{\lambda_{ns}}\left(\frac{3n^2}{\lambda_{ns}^2} - 1\right)\frac{\mu_{ns}}{2} + \cdots\right] \tag{6.18}$$

$$= -\epsilon\left[1 + \left(\frac{n^2}{\lambda_{ns}^2} - 1\right)\frac{\mu_{ns}^2}{2} + \cdots\right]$$

Letting

$$\mu_{ns} = a_{ns}^{(1)}\epsilon + a_{ns}^{(2)}\epsilon^2 + \cdots \tag{6.19}$$

in (6.18), we obtain

$$\left(\frac{n^2}{\lambda_{ns}^2} - 1\right)\lambda_{ns}a_{ns}^{(1)}\epsilon + \left[\lambda_{ns}\,a_{ns}^{(2)}\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) - \frac{a_{ns}^{(1)2}}{2}\left(\frac{n^2}{\lambda_{ns}^2} + 1\right)\right]\epsilon^2 + O(\epsilon^3)$$

$$= -\epsilon + O(\epsilon^3) \tag{6.20}$$

Equating coefficients of each power of ϵ,

$$a_{ns}^{(1)} = \frac{\lambda_{ns}}{\lambda_{ns}^2 - n^2}, \tag{6.21a}$$

$$a_{ns}^{(2)} = -\frac{\lambda_{ns}}{2}\,\frac{\lambda_{ns}^2 + n^2}{(\lambda_{ns}^2 - n^2)^3}, \tag{6.21b}$$

so that, substituting (6.21a, b) in (6.19) and (6.14), yields the relations between the eigenvalues of the exact problem and the eigenvalues of the insulated cylinder,

$$\beta_{ns} = \lambda_{ns}\left[1 + \frac{\epsilon}{\lambda_{ns}^2 - n^2} - \frac{(\lambda_{ns}^2 + n^2)\epsilon^2}{2(\lambda_{ns}^2 - n^2)^3} + \cdots\right] \tag{6.22}$$

Substituting (6.22) in the double sum in (6.1), we obtain, finally,

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{s=1}^{\infty}\frac{\beta_{ns}e^{-\beta_{ns}|x|}}{\beta_{ns}^2-n^2+\epsilon^2}\;\frac{J_n(\beta_{ns}r)\,J_n(\beta_{ns}R)}{J_n^2(\beta_{ns})}$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{s=1}^{\infty}\frac{\lambda_{ns}}{\lambda_{ns}^2-n^2}\left[1-\epsilon\,\frac{\lambda_{ns}^2+n^2}{(\lambda_{ns}^2-n^2)^2}+\cdots\right]e^{-\lambda_{ns}|x|\left(1+\frac{\epsilon}{\lambda_{ns}^2-n^2}+\cdots\right)}$$

$$\cdot\left[\frac{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)}{J_n^2(\lambda_{ns})}+\frac{\epsilon\lambda_{ns}}{\lambda_{ns}^2-n^2}\;\frac{\frac{d}{d\lambda_{ns}}\{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)\}}{J_n^2(\lambda_{ns})}\right]$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{s=1}^{\infty}\frac{\lambda_{ns}}{\lambda_{ns}^2-n^2}\,e^{-\lambda_{ns}|x|}\;\frac{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)}{J_n^2(\lambda_{ns})}\left[1-\epsilon\,\frac{\lambda_{ns}}{\lambda_{ns}^2-n^2}\right.$$

$$\left.\cdot\left\{\frac{\lambda_{ns}^2+n^2}{\lambda_{ns}(\lambda_{ns}^2-n^2)}+|x|-\frac{\frac{d}{d\lambda_{ns}}\{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)\}}{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)}\right\}\right], \qquad (6.23)$$

which is identical to the two double sums in V_1 and V_3 appearing in (4.31) and (4.61).

APPENDIX A

The integral over k in (4.30) can be replaced by an equivalent sum, by considering the contour integral

$$\oint_C dz\, e^{iz|x|} \left[\frac{K_n'(z)}{I_n'(z)} I_n(zR) I_n(zr) + \frac{2\delta_{0n}}{z^2}\right] = 0 \qquad (A.1)$$

where C is the contour in the $z = k + i\lambda$ plane shown in Figure 2. The contour integral is zero because all the singularities of the integrand, which occur at the zeroes of $I_n'(i\lambda) = - i^{-n-1} J_n'(-\lambda)$ along the imaginary axis are outside the contour. Furthermore the integral along the circular arc vanishes as the radius becomes infinite. Therefore, the integral in (4.30) may be replaced by the principal value of an integral along the imaginary axis plus the sum of πi times the residues at the poles of the integrand along the imaginary axis;

$$\mathrm{Re} \int_0^\infty dk\, e^{ik|x|} \left[\frac{K_n'(k)}{I_n'(k)} I_n(kR) I_n(kr) + \frac{2\delta_{0n}}{k^2}\right]$$

$$= \mathrm{Re} \int_0^\infty i d\lambda\, e^{-\lambda|x|} \left[\frac{\frac{d}{d\lambda}\left\{\frac{\pi}{2}\, i^{n+1} [J_n(-\lambda) + iY_n(-\lambda)]\right\}}{\frac{d}{d\lambda}\left\{i^{-n}\, J_n(-\lambda)\right\}}\, i^{-2n} J_n(-\lambda r) J_n(-\lambda R) - \frac{2\delta_{0n}}{\lambda^2}\right]$$

$$+ \mathrm{Re}\, \pi i \sum_{s=1}^{\infty} \mathrm{Res} \left[e^{-\lambda|x|} \frac{\frac{d}{d\lambda}\left\{\frac{\pi}{2} i^{n+1} [J_n(-\lambda) + iY_n(-\lambda)]\right\}}{\frac{d}{d\lambda}\left\{i^{-n} J_n(-\lambda)\right\}}\, i^{-2n} J_n(-\lambda r) J_n(-\lambda R)\right]_{\lambda=\lambda_{ns}} \qquad (A.2)$$

where λ_{ns} is the s-th zero of $J_n'(\lambda)$, excluding the zero which occurs at $\lambda = 0$ when $n \neq 0$.

Noting that $\mathrm{Re}\{iY_n(-\lambda)\} = - 2\, J_n(-\lambda)$[19] in the integral in (A.2) and that $J_n'(-\lambda_{ns}) = 0$ in the sum, (A.2) becomes

$$\frac{\pi}{2} \int_0^\infty d\lambda e^{-\lambda|x|}\, J_n(\lambda r) J_n(\lambda R) - \mathrm{Re}\, \frac{i\pi^2}{2} \sum_{s=1}^{\infty} \mathrm{Res} \left[e^{-\lambda|x|}\, \frac{Y_n'(-\lambda)}{J_n'(-\lambda)}\, J_n(\lambda r) J_n(\lambda R)\right]_{\lambda=\lambda_{ns}} \qquad (A.3)$$

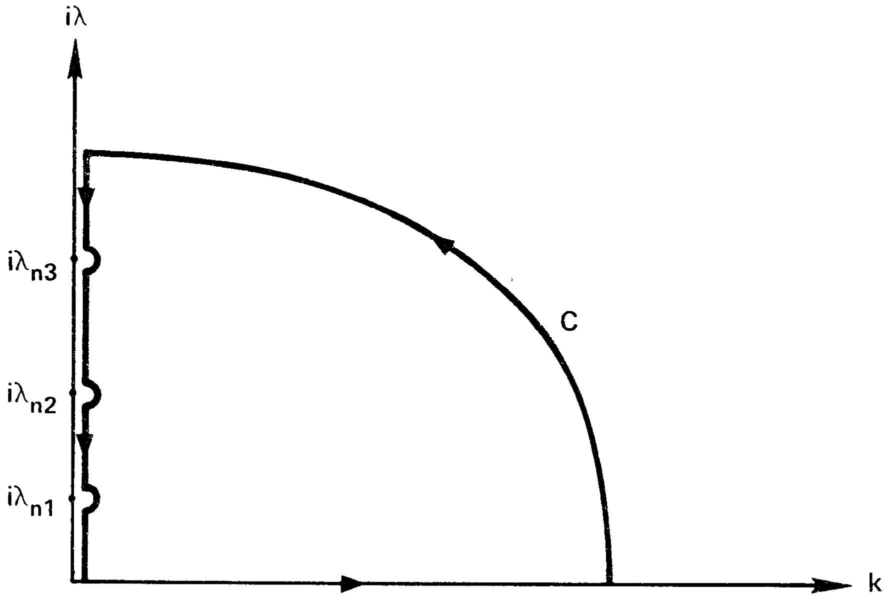

Figure 2. Contour for Converting k-Integral in (4.30) to a Sum over λ_{ns}.

The Taylor series for $J_n'(\lambda)$ around $\lambda = \lambda_{ns}$ is, noting again that $J_n'(\lambda_{ns}) = 0$,

$$J_n'(\lambda) = (\lambda - \lambda_{ns})\, J_n''(\lambda_{ns}) + \cdots$$

$$= (\lambda - \lambda_{ns})\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J_n(\lambda_{ns}) + \cdots \tag{A. 4}$$

and from the Wronskian[20] of $J_n(\lambda)$ and $Y_n(\lambda)$ we obtain

$$Y_n'(\lambda_{ns}) = \frac{2}{\pi\lambda_{ns} J_n(\lambda_{ns})} \tag{A. 5}$$

To evaluate the residues in (A. 3) we multiply by $(i\lambda - i\lambda_{ns})$, substitute (A. 4) and (A. 5) in (A. 3), and let $\lambda = \lambda_{ns}$. The result is that (A. 3) becomes

$$\frac{\pi}{2}\int_0^\infty d\lambda\, e^{-\lambda|x|}\, J_n(\lambda r) J_n(\lambda R) + \pi \sum_{s=1}^{\infty} \frac{e^{-\lambda_{ns}|x|}\, J_n(\lambda_{ns} r) J_n(\lambda_{ns} R)}{\lambda_{ns}\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J_n^2(\lambda_{ns})} \tag{A. 6}$$

which may now be substituted for the integral in (4. 30). The first term in (A. 6) leads to

$$-\frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta}\, \frac{\pi}{2}\int_0^\infty d\lambda\, e^{-\lambda|x|}\, J_n(\lambda r) J_n(\lambda R)$$

$$= -\frac{1}{4\pi}\int_0^\infty d\lambda\, e^{-\lambda|x|} \sum_{n=-\infty}^{\infty} e^{in\theta}\, J_n(\lambda r) J_n(\lambda R)$$

$$= -\frac{1}{4\pi}\int_0^\infty d\lambda\, e^{-\lambda|x|}\, J_0\!\left(\lambda\sqrt{r^2 + R^2 - 2rR\cos\theta}\right)$$

$$= -\frac{1}{4\pi}(x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

where we have used a known summation formula[21] and a tabulated integral.[22] Equation (A. 7) exactly cancels the second term in (4. 30). Substituting (A. 6) in (4. 30), utilizing (A. 7), therefore results in (4. 31).

APPENDIX B

In this appendix, a second alternate form is developed for (4.30). The linear term, $-|x|/2\pi$, in the expressions for V_1, has a discontinuous first derivative at $x = 0$. Since $\partial V_1/\partial x$ must be continuous at $x = 0$ [except at the point $(0, R, 0)$], the integral in (4.30), or the sum in (4.31), must have a discontinuous derivative of equal magnitude and opposite sign. By extracting the free-space potential of a uniform disc source at $x = 0$, we obtain an integral which has continuous derivatives everywhere inside the cylinder.

By adding and substracting the same quantity from the integrand in (4.30) we obtain

$$-\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx\left[\frac{K_n'(k)}{I_n'(k)}\,I_n(kr)I_n(kR)+\frac{2\delta_{0n}}{k^2}\right]$$

$$= -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx\left[\frac{K_n'(k)}{I_n'(k)}\,I_n(kr)I_n(kR)+\frac{2\delta_{0n}}{k}\,K_1(k)I_0(kr)\right.$$
$$\left.-\frac{2\delta_{0n}}{k}\left\{K_1(k)I_0(kr)-\frac{1}{k}\right\}\right] \tag{B.1}$$

The integral in curly brackets can be treated as was the integral in (A.2), using the same contour as in Figure 2, except that in the present case there are no poles on the imaginary axis. The integral of the expression in curly brackets in (B.1) is

$$-2\delta_{0n}\int_0^\infty dk\,\cos kx\left\{\frac{K_1(k)}{k}\,I_0(kr)-\frac{1}{k^2}\right\}$$

$$= -2\delta_{0n}\,\mathrm{Re}\int_0^\infty dk\,e^{ik|x|}\left\{\frac{K_1(k)}{k}\,I_0(kr)-\frac{1}{k^2}\right\}$$

$$= -2\delta_{0n}\,\mathrm{Re}\int_0^\infty i d\lambda\, e^{-\lambda|x|}\left\{\frac{-1}{i\lambda}\frac{\pi}{2}[J_1(-\lambda)+iY_1(-\lambda)]J_0(-\lambda r)+\frac{1}{\lambda^2}\right\}$$

$$= \pi\delta_{0n}\int_0^\infty d\lambda\, e^{-\lambda|x|}\,\frac{J_1(\lambda)}{\lambda}\,J_0(\lambda r) \qquad \text{(B. 2)}$$

where we use Re $iY_1(-\lambda) = -2J_1(-\lambda)$ to obtain the last equality.[19]

Noting that[23]

$$\int_0^1 J_0(\lambda R) R dR = \frac{J_1(\lambda)}{\lambda}, \qquad \text{(B. 3)}$$

and performing some further manipulations, (B. 2) becomes,

$$\pi\delta_{0n}\int_0^\infty d\lambda\, e^{-\lambda|x|}\int_0^1 RdR\; J_0(\lambda R)J_0(\lambda r)$$

$$= \pi\delta_{0n}\frac{1}{2\pi}\int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} e^{im\theta}\int_0^\infty d\lambda\, e^{-\lambda|x|}\int_0^1 RdR\; J_m(\lambda R)J_m(\lambda r)$$

$$= \frac{\delta_{0n}}{2}\int_0^{2\pi} d\theta\int_0^1 RdR\int_0^\infty d\lambda\, e^{-\lambda|x|}\sum_{m=-\infty}^{\infty} e^{im\theta}J_m(\lambda r)J_m(\lambda R)$$

$$= \frac{\delta_{0n}}{2}\int_0^{2\pi} d\theta\int_0^1 RdR\int_0^\infty d\lambda\, e^{-\lambda|x|}\; J_0(\lambda\sqrt{r^2+R^2-2rR\cos\theta})$$

$$= \frac{\delta_{0n}}{2}\int_0^{2\pi} d\theta\int_0^1 RdR\,[x^2+r^2+R^2-2rR\cos\theta]^{-1/2} \qquad \text{(B. 4)}$$

where two previously used formulas[21,22] were used to obtain the last two equalities. Multiplying (B. 4) by the factor $-1/2\pi^2$ appearing in (B. 1) yields the negative of the integral over the disc at $x = 0$, of the free-space Green's function, $(4\pi)^{-1}[x^2+r^2+R^2-2rR\cos\theta]^{-1/2}$, divided by the area of the disc, π.

The expression (4.30) for V_1 may therefore be written as

$$V_1(x, r, \theta) = -\frac{|x|}{2\pi} + \frac{1}{4\pi}(x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

$$-\frac{1}{4\pi^2}\int_0^{2\pi} d\theta \int_0^1 R\,dR\,(x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

$$-\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\cos kx\left[\frac{I_n(kR)}{I_n'(k)} - \frac{2\delta_{0n}}{k}\right]K_n'(k)I_n(k) \tag{B.5}$$

which decomposes the potential into a linear term, the free-space potential of a unit point source, the free-space potential of a unit disc sink, and an infinite sum of an infinite integral which has continuous derivatives everywhere in the interior of the cylinder. The distinction between (B.5) and (4.30) is only in the $n = 0$ term.

APPENDIX C

It is possible to obtain an alternate representation (analogous to (4. 31) for V_1) for V_3, in terms of a double sum of Bessel functions. The k integral in (4. 47) can be expressed as an integral over the range $-\infty < k < \infty$:

$$\int_0^\infty dk \frac{\cos kx}{k^2}\left[\frac{I_n(kR)I_n(kr)}{[I_n'(k)]^2} - (R^2 + r^2 - 1 + \frac{4}{k^2})\delta_{0n}\right]$$

$$= \frac{1}{2}\int_{-\infty}^\infty dk \frac{e^{ikx}}{k^2}\left[\frac{I_n(kR)I_n(kr)}{[I_n'(k)]^2} - (R^2 + r^2 + \frac{4}{k^2} - 1)\delta_{0n}\right] \qquad (C.\ 1)$$

The integral in (C. 1) can be considered a portion of a contour integral along the real axis in the complex $z = k + i\lambda$ plane. Closing the contour along the large semicircle in the upper-half plane adds nothing to (C. 1) since the integrand vanishes exponentially on the semicircle. The integral can therefore be replaced by $2\pi i$ times the sum of the residues at the second-order poles of the integrand, which occur at the zeros of $I_n'(z)$ along the imaginary axis, where

$$I_n'(i\lambda) = -\, i^{-n-1} J_n'(-\lambda) = -\,(-i)^{-n-1} J_n'(\lambda) = 0\,. \qquad (C.\ 2)$$

As in (4. 31), the roots of (C. 2) are denoted λ_{ns}, where s runs from 1 to ∞, excluding the zeros at $\lambda = 0$. To evaluate the residues we must determine

$$\mathrm{Res}\left[\frac{e^{iz|x|}}{z^2}\;\frac{I_n(zr)I_n(zR)}{I_n'^2(z)}\right]_{z = i\lambda_{ns}}$$

$$= \left[\frac{d}{d(i\lambda)}\;\frac{(i\lambda - i\lambda_{ns})^2 e^{-\lambda|x|}}{-\lambda^2}\;\frac{I_n(i\lambda r)I_n(i\lambda R)}{I_n'^2(i\lambda)}\right]_{\lambda=\lambda_{ns}}$$

$$= \left[i\frac{d}{d\lambda}\;\frac{(\lambda-\lambda_{ns})^2 e^{-\lambda|x|}\, J_n(\lambda r)J_n(\lambda R)}{\lambda^2 J_n'^2(\lambda)}\right]_{\lambda = \lambda_{ns}} \qquad (C.\ 3)$$

Writing the Taylor series for $J_n'(\lambda)$ about $\lambda = \lambda_{ns}$, we have

$$J_n'(\lambda) = J_n''(\lambda_{ns})(\lambda-\lambda_{ns}) + \frac{1}{2} J_n'''(\lambda_{ns})(\lambda-\lambda_{ns})^2 + \cdots \tag{C. 4}$$

From Bessel's equation,

$$J_n''(\lambda) + \frac{1}{\lambda} J_n'(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n(\lambda) = 0,$$

and the vanishing of $J_n'(\lambda_{ns})$, we have

$$J_n''(\lambda_{ns}) = \left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J_n(\lambda_{ns}), \tag{C. 5}$$

and differentiating Bessel's equation and setting $\lambda = \lambda_{ns}$,

$$J_n'''(\lambda_{ns}) = \frac{1}{\lambda_{ns}} \left(1 - \frac{3n^2}{\lambda_{ns}^2}\right) J_n(\lambda_{ns}) \tag{C. 6}$$

Substituting (C. 5) and (C. 6) in (C. 4):

$$J_n'(\lambda) = (\lambda-\lambda_{ns}) J_n(\lambda_{ns}) \left[\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) + \left(1 - \frac{3n^2}{\lambda_{ns}^2}\right) \frac{\lambda-\lambda_{ns}}{2\lambda_{ns}} + \cdots\right] \tag{C. 7}$$

Finally, using (C. 7) in (C. 3):

$$\operatorname{Res}\left[\frac{e^{iz|x|}}{z^2} \frac{I_n(zr) I_n(zR)}{I_n'^2(z)}\right]_{z = i\lambda_{ns}}$$

$$= \left[i \frac{\partial}{\partial\lambda} \frac{e^{-\lambda|x|} J_n(\lambda r) J_n(\lambda R)}{\lambda^2 J_n^2(\lambda_{ns}) \left[\frac{(n^2-\lambda_{ns}^2)^2}{\lambda_{ns}^4} + \frac{(n^2-\lambda_{ns}^2)(\lambda_{ns}^2-3n^2)(\lambda-\lambda_{ns})}{\lambda_{ns}^5}\right]}\right]_{\lambda = \lambda_{ns}}$$

$$= -\frac{i\lambda_{ns} J_n(\lambda_{ns} r) J_n(\lambda_{ns} R) e^{-\lambda_{ns}|x|}}{J_n^2(\lambda_{ns})(n^2-\lambda_{ns}^2)^2} \left(2 + \lambda_{ns}|x| - \frac{3n^2-\lambda_{ns}^2}{n^2-\lambda_{ns}^2}\right)$$

$$+ \frac{i\lambda_{ns}^2 \frac{\partial}{\partial\lambda_{ns}} \{J_n(\lambda_{ns} r) J_n(\lambda_{ns} R)\}}{J_n^2(\lambda_{ns})(n^2-\lambda_{ns}^2)^2}$$

$$= \frac{i\lambda_{ns}}{(n^2-\lambda_{ns}^2)^2} \frac{e^{-\lambda_{ns}|x|}}{J_n^{\;2}(\lambda_{ns})} \Bigg[\left(\frac{n^2+\lambda_{ns}^2}{n^2-\lambda_{ns}^2} - \lambda_{ns}|x| \right) J_n(\lambda_{ns}r) J_n(\lambda_{ns}R)$$

$$+ \lambda_{ns} \frac{\partial}{\partial \lambda_{ns}} \left(J_n(\lambda_{ns}r) J_n(\lambda_{ns}R) \right) \Bigg] \qquad \text{(C. 8)}$$

Substituting $2\pi i$ times (C. 8) for the integral in (C. 1), and then substituting the result in (4. 47) yields the alternate expression,

$$V_3(x,\ r,\ \theta) = \frac{|x|}{4\pi} \left(R^2 + r^2 - 1 - \frac{2}{3} x^2 \right)$$

$$- \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{s=1}^{\infty} \frac{\lambda_{ns}^2 e^{-\lambda_{ns}|x|}}{(n^2-\lambda_{ns}^2)^2 J_n^{\;2}(\lambda_{ns})}$$

$$\cdot \Bigg[\left\{ \frac{(n^2+\lambda_{ns}^2)}{\lambda_{ns}(n^2-\lambda_{ns}^2)} - |x| \right\} J_n(\lambda_{ns}r)\ J_n(\lambda_{ns}R)$$

$$+ \frac{\partial}{\partial \lambda_{ns}} \left\{ J_n(\lambda_{ns}r)\ J_n(\lambda_{ns}R) \right\} \Bigg] \qquad \text{(C. 9)}$$

for the $O(\epsilon)$ term in the near field.

APPENDIX D

When $|x| << 1$, the sum in (4. 30) converges very slowly, and actually diverges for $x = 0$. It is only a convenient representation for calculating the potential for $x \gtrsim 1$. For $|x| << 1$ it would therefore be preferable to do a numerical evaluation of the integral in (4. 31) or (B. 5) to determine the potential. However, there is a superior method for calculating the potential in this region near the source. We now develop a formula for the potential, which contains a free parameter for adjusting the rate of convergence. The method for obtaining this formula, which starts from the integral representation of the potential (4. 30), is an extension of a method used previously[24] for a similar integral which did not contain a branch point.

We consider the contour integral

$$\frac{1}{2\pi i} \int_C dz \cos zx \left[\frac{K_n'(z)}{I_n'(z)} I_n(zr) I_n(zR) + \frac{2\delta_{0n}}{z^2} \right] \frac{z}{(z^2-t^2) \cos\left(\frac{\pi z}{\beta}\right)} \qquad (D.1)$$

in which β and t are real parameters, $\beta > 0$ and $\cos(\pi t/\beta) \neq 0$, so that the poles introduced are distinct. The integration path is shown in Figure 3. The branch cut is taken between the origin and infinity somewhere in the left-half plane. If $|x| < \pi/\beta$, the integrand along the semicircular arc is exponentially small when the radius increases to infinity. There are no singularities within the contour so the integral (D. 1) is zero. Therefore we know that the sum of the principal value of the integral (D. 1) up the imaginary axis, the residues on the real axis, and one-half times the residues on the imaginary axis is zero.

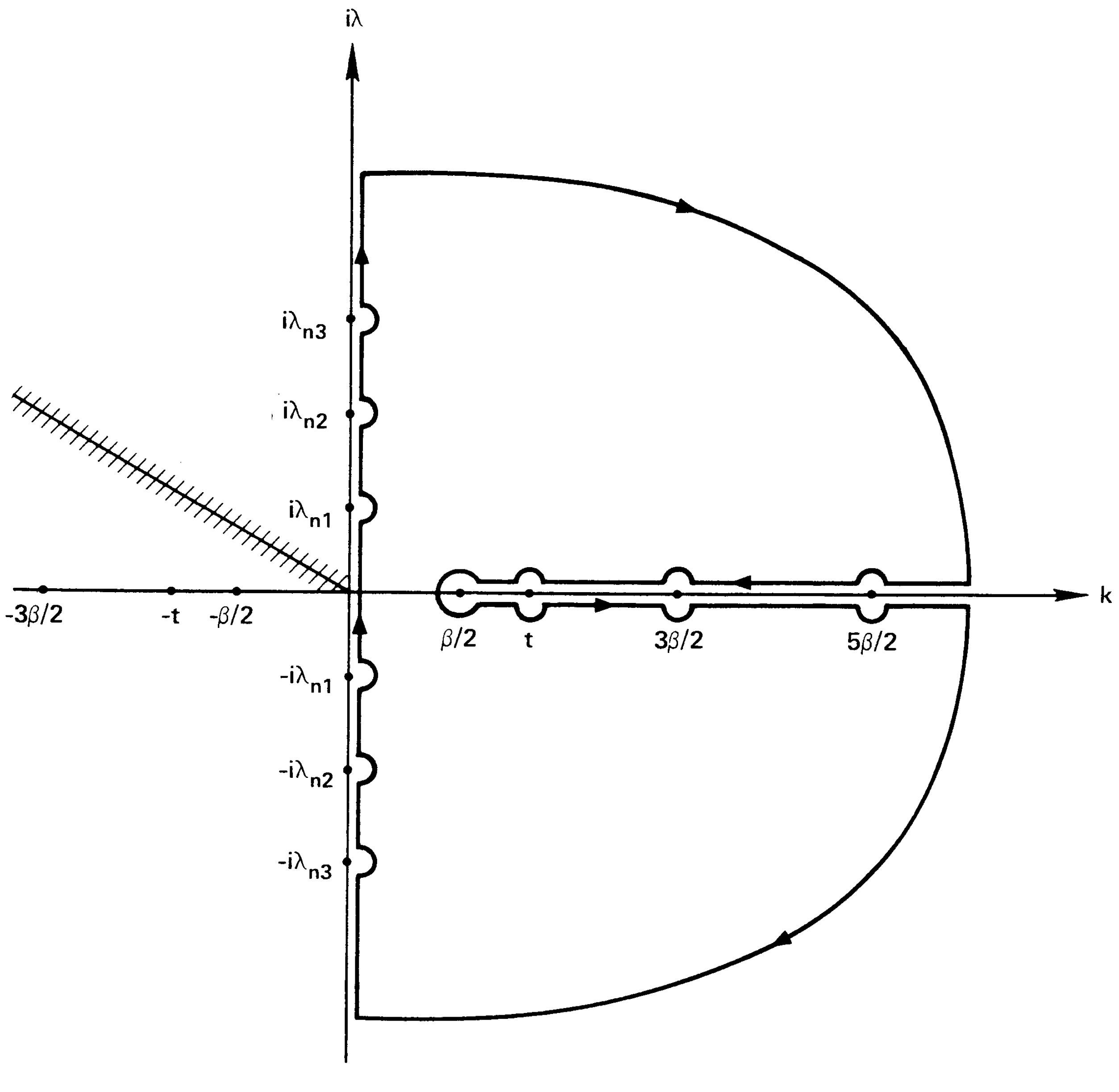

Figure 3. Contour for (D.1).

First we consider the integral on the imaginary axis. We separate it into a piece over $-\infty < \lambda \leq 0$, for which we take $z = |\lambda| e^{-i\pi/2}$, and a piece over the range $0 \leq \lambda < \infty$ for which we take $z = |\lambda| e^{i\pi/2}$. The integral up the imaginary axis is thus:

$$\frac{1}{2\pi i}\int_{-\infty}^{0} i d\lambda \cosh \lambda x \left[\frac{K'_n(e^{-i\pi/2}|\lambda|)}{I'_n(i\lambda)} I_n(i\lambda r) I_n(i\lambda R) - \frac{2\delta_{0n}}{\lambda^2} \right] \frac{i\lambda}{(-\lambda^2 - t^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$+\frac{1}{2\pi i}\int_{0}^{\infty} i d\lambda \cosh \lambda x \left[\frac{K'_n(e^{i\pi/2}|\lambda|)}{I'_n(i\lambda)} I_n(i\lambda r) I_n(i\lambda R) - \frac{2\delta_{0n}}{\lambda^2} \right] \frac{i\lambda}{(-\lambda^2 - t^2)\cosh\frac{\pi\lambda}{\beta}}$$

$$= \frac{1}{2\pi}\int_0^{\infty} d\lambda \cosh \lambda x \left[\frac{K'_n(e^{-i\pi/2}\lambda)}{I'_n(-i\lambda)} I_n(-i\lambda r) I_n(-i\lambda R) - \frac{K'_n(e^{i\pi/2}\lambda)}{I'_n(i\lambda)} I_n(i\lambda r) I_n(i\lambda R) \right] \frac{i\lambda}{(\lambda^2 + t^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)} \tag{D. 2}$$

where we have used $\int_{-\infty}^{0} d\lambda\, f(\lambda) = \int_0^{\infty} d\lambda\, f(-\lambda)$ and have been careful to keep track of the phase of the argument of K_n, which has a branch point at $z = 0$. Using the identities[25]

$$I_n(z) = i^{-n} J_n(iz)$$

$$K_n(z) = \frac{\pi}{2} i^{n+1} \left[J_n(iz) + i\, Y_n(e^{i\pi/2} z) \right]$$

$$J_n(-\lambda) = (-1)^n J_n(\lambda)$$

$$Y_n(\lambda e^{i\pi}) = (-1)^n \left[Y_n(\lambda) + 2i\, J_n(\lambda) \right],$$

(D. 2) becomes

$$\frac{1}{2\pi}\int_0^\infty d\lambda\,\cosh(\lambda x)\left(\frac{i\pi}{2}\right)\left[\frac{J_n'(\lambda)+iY_n'(\lambda)}{J_n'(\lambda)}-\frac{J_n'(e^{i\pi}\lambda)+iY_n'(e^{i\pi}\lambda)}{J_n'(e^{i\pi}\lambda)}\right]$$

$$\cdot\frac{i\lambda\, J_n(\lambda r)\, J_n(\lambda R)}{(\lambda^2+t^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$= -\frac{1}{4}\int_0^\infty d\lambda\,\cosh\lambda x\left[\frac{J'_n(\lambda)+Y'_n(\lambda)}{J'_n(\lambda)}-\frac{-J'_n(\lambda)+iY'_n(\lambda)}{J'_n(\lambda)}\right]$$

$$\cdot\frac{\lambda J_n(\lambda r)\, J_n(\lambda R)}{(\lambda^2+t^2)\cosh(\frac{\pi\lambda}{\beta})}$$

$$= -\frac{1}{2}\int_0^\infty d\lambda\,\cosh\lambda x\quad\frac{\lambda J_n(\lambda r)\, J_n(\lambda R)}{(\lambda^2+t^2)\cosh(\frac{\pi\lambda}{\beta})}\tag{D. 3}$$

The residue at z = t is

$$\left[\frac{K'_n(t)}{I'_n(t)}I_n(tr)I_n(tR)+\frac{2\delta_{0n}}{t^2}\right]\frac{\cos tx}{2\cos(\frac{\pi t}{\beta})}\tag{D. 4}$$

The residues at $z = \pm\, i\lambda_{ns}$ are evaluated in the same way as were the residues leading to (A.6), except for the additional factor

$$\frac{\pm\, i\lambda_{ns}}{(\lambda^2_{ns}+t^2)\cosh\left(\frac{\pi\lambda_{ns}}{\beta}\right)}$$

and a factor $\cosh\lambda_{ns}x$ instead of $e^{-\lambda_{ns}|x|}$. The sum of the two residues at $z = \pm\, i\lambda_{ns}$ are

$$(z - i\lambda_{ns}) \frac{K'_n(z)}{I'_n(z)} I_n(zr)\, I_n(zR) \left.\frac{z \cos (zx)}{(z^2-t^2) \cos\left(\frac{\pi z}{\beta}\right)}\right|_{z = e^{i\pi/2}\lambda_{ns}}$$

$$+ (z + i\lambda_{ns}) \frac{K'_n(z)}{I'_n(z)} I_n(zr)\, I_n(zR) \left.\frac{z \cos (zx)}{(z^2-t^2)\cos\left(\frac{\pi\lambda}{\beta}\right)}\right|_{z = e^{-i\pi/2}\lambda_{ns}}$$

$$= (\lambda-\lambda_{ns}) \frac{i\pi}{2} \frac{J'_n(-\lambda) + iY'_n(e^{i\pi}\lambda)}{J'_n(-\lambda)} J_n(-\lambda r)\, J_n(-\lambda R) \left.\frac{\lambda \cosh \lambda x}{(\lambda^2+t^2) \cosh\left(\frac{\pi\lambda}{\beta}\right)}\right|_{\lambda=\lambda_{ns}}$$

$$+(-\lambda+\lambda_{ns}) \frac{i\pi}{2} \frac{J'_n(\lambda) + iY'_n(\lambda)}{J'_n(\lambda)} J_n(\lambda r)\, J_n(\lambda R) \left.\frac{-\lambda \cosh \lambda x}{(\lambda^2+t^2) \cosh\left(\frac{\pi\lambda}{\beta}\right)}\right|_{\lambda=\lambda_{ns}}$$

$$= \frac{i\pi}{2} \frac{(\lambda-\lambda_{ns})\lambda \cosh \lambda x}{(\lambda^2+t^2) \cosh\left(\frac{\pi\lambda}{\beta}\right)} J_n(\lambda r)\, J_n(\lambda R) \left[\frac{\frac{d}{d\lambda}\left\{J_n(-\lambda) + iY_n(e^{i\pi}\lambda)\right\}}{\frac{d}{d\lambda} J_n(-\lambda)} + \left.\frac{\frac{d}{d\lambda}\left\{J_n(\lambda) + iY_n(\lambda)\right\}}{\frac{d}{d\lambda} J_n(\lambda)}\right]\right|_{\lambda=\lambda_{ns}}$$

Using the same relations as we used in (D. 3), the bracketed expression becomes

$$\frac{\frac{d}{d\lambda}\left\{-J_n(\lambda) + iY_n(\lambda)\right\}}{\frac{d}{d\lambda} J_n(\lambda)} + \frac{\frac{d}{d\lambda}\left\{J_n(\lambda) + iY_n(\lambda)\right\}}{\frac{d}{d\lambda} J_n(\lambda)} = \frac{2i\, Y'_n(\lambda)}{J'_n(\lambda)}$$

Using (A. 4) and (A. 5) this becomes, as $\lambda \to \lambda_{ns}$

$$\frac{2i\, Y'_n(\lambda)}{J'_n(\lambda)} \to \frac{4i}{\pi \lambda_{ns}(\lambda-\lambda_{ns})\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J^2_n(\lambda_{ns})}$$

so that the sum of the two residues at $z = \pm\, i\lambda_{ns}$ is

$$\frac{-2\cosh(\lambda_{ns}x)\, J_n(\lambda_{ns} r)\, J_n(\lambda_{ns}R)}{(\lambda_{ns}^2+t^2)\cos\left(\frac{\pi\lambda_{ns}}{\beta}\right)\left(\frac{n^2}{\lambda_{ns}^2}-1\right) J_n^2(\lambda_{ns})} \tag{D.5}$$

The residue at $z = (\nu+1/2)\beta$ is

$$-\frac{\beta}{\pi}(-1)^{\nu}\left[\frac{K'_n[(\nu+\frac{1}{2})\beta]}{I'_n[(\nu+\frac{1}{2})\beta]} I_n[(\nu+\tfrac{1}{2})\beta r]\, I_n[(\nu+\tfrac{1}{2})\beta R] + \frac{2\delta_{0n}}{(\nu+\frac{1}{2})^2\beta^2}\right]$$

$$\cdot\ \frac{\cos[(\nu+\frac{1}{2})\beta x]\,(\nu+\frac{1}{2})\beta}{(\nu+\frac{1}{2})^2\beta-t^2} \tag{D.6}$$

Equating the negative of the integral up the imaginary axis, (D.3), to the sum of the residue at $z = t$, (D.4); one-half of the residues at $z = \pm i\lambda_{ns}$, (D.5), and the residues at $z = (\nu+1/2)\beta$, (D.6), we obtain

$$\frac{1}{2}\int_0^\infty d\lambda\, \cosh\lambda x\, \frac{\lambda J_n(\lambda r)\, J_n(\lambda R)}{(\lambda^2+t^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$= \left[\frac{K'_n(t)}{I'_n(t)} I_n(tr)\, I_n(tR) + \frac{2\delta_{0n}}{t^2}\right]\frac{\cos tx}{2\cos\left(\frac{\pi t}{\beta}\right)}$$

$$-\sum_{s=1}^{\infty}\frac{\cosh(\lambda_{ns}x)\, J_n(\lambda_{ns}r)\, J_n(\lambda_{ns}R)}{(\lambda_{ns}^2+t^2)\cosh\left(\frac{\pi\lambda_{ns}}{\beta}\right)\left(\frac{n^2}{\lambda_{ns}^2}-1\right)J_n^2(\lambda_{ns})}$$

$$-\frac{\beta}{\pi}\sum_{\nu=0}^{\infty}(-1)^{\nu}\left[\frac{K'_n[(\nu+\frac{1}{2})\beta]}{I'_n[(\nu+\frac{1}{2})\beta]} I_n[(\nu+\tfrac{1}{2})\beta r]\, I_n[(\nu+\tfrac{1}{2})\beta R] + \frac{2\delta_{0n}}{(\nu+\frac{1}{2})^2\beta^2}\right]$$

$$\cdot\ \frac{\cos[(\nu+\frac{1}{2})\beta x]\,(\nu+\frac{1}{2})\beta}{(\nu+\frac{1}{2})^2\beta^2-t^2} \tag{D.7}$$

The left-hand side of (D. 7) is analytic everywhere in the finite right-half complex t-plane. So far we have shown that it equals the right-hand side everywhere in this region, except perhaps at the points $t = (\mu + 1/2)\beta$; $\mu = 0, 1, 2, \ldots$ At $t = (\mu + 1/2)\beta$, however, the residue of the first term on the right-hand side is minus the residue of the $\nu = \mu$ term in the last sum and hence the right-hand side is also analytic everywhere in the right-half t-plane. By analytic continuation, (D. 7) must therefore be valid even when $t = (\mu + 1/2)\beta$.

If we multiply (D. 7) by $2\cos(\pi t/\beta)$ and integrate over t from 0 to ∞, the first term on the right is just the integral we wish to evaluate. Using the formulas[26]

$$\int_0^\infty \frac{\cos(\pi t/\beta)}{\lambda^2 + t^2}\, dt = \frac{\pi}{2\lambda} e^{-\pi\lambda/\beta}; \quad \int_0^\infty \frac{\cos \pi t/\beta}{(\nu + \frac{1}{2})^2 \beta^2 - t^2}\, dt = \frac{(-1)^\nu \pi}{2(\nu + \frac{1}{2})\beta} \tag{D. 8}$$

yields

$$\begin{aligned}
\int_0^\infty dt \cos(tx) \left[\frac{K_n'(t)}{I_n'(t)} I_n(tr) I_n(tR) + \frac{2\delta_{0n}}{t^2} \right] &= \frac{\pi}{2} \int_0^\infty d\lambda \cosh(\lambda x) \frac{e^{-\lambda\pi/\beta}}{\cosh(\pi\lambda/\beta)} J_n(\lambda r) J_n(\lambda R) \\
&+ \pi \sum_{s=1}^\infty \frac{J_n(\lambda_{ns} r) J_n(\lambda_{ns} R)}{\lambda_{ns}\left(\frac{n^2}{\lambda_{ns}^2} - 1\right) J_n^2(\lambda_{ns})} \frac{e^{-\pi\lambda_{ns}/\beta} \cosh(\lambda_{ns} x)}{\cosh(\pi\lambda_{ns}/\beta)} \\
&+ \beta \sum_{\nu=0}^\infty \cos[(\nu + \tfrac{1}{2})\beta x] \left\{ \frac{K_n'[(\nu + \frac{1}{2})\beta]}{I_n'[(\nu + \frac{1}{2})\beta]} I_n[(\nu + \tfrac{1}{2})\beta r] I_n[(\nu + \tfrac{1}{2})\beta R] + \frac{2\delta_{0n}}{(\nu + \frac{1}{2})^2 \beta^2} \right\}
\end{aligned} \tag{D. 9}$$

To obtain the potential $V_1(x, r, \theta)$ we multiply (D. 9) by $(2\pi)^{-2}\, e^{in\theta}$, sum over n and substitute in (4. 30). First, let us consider the result of doing this to the first term on the right-hand side of (D. 9). We have

$$-\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{\infty} d\lambda \cosh \lambda x \; \frac{e^{-\pi\lambda/\beta}}{\cosh(\pi\lambda/\beta)}\, J_n(\lambda r) J_n(\lambda R) \tag{D. 10}$$

Noting that

$$\frac{e^{-\pi\lambda/\beta}}{2\cosh(\pi\lambda/\beta)} = \frac{1}{1+e^{2\pi\lambda/\beta}} = 1 - \frac{1}{1+e^{-2\pi\lambda/\beta}}$$

$$= 1 - \sum_{m=0}^{\infty} (-1)^m\, e^{-2\pi m\lambda/\beta} = - \sum_{m=1}^{\infty} (-1)^m\, e^{-2\pi m\lambda/\beta}$$

(D. 10) becomes

$$\frac{1}{4\pi} \sum_{m=1}^{\infty} (-1)^m \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{\infty} d\lambda \left[e^{-\lambda(x+2\pi m/\beta)} + e^{+\lambda(x-2\pi m/\beta)} \right] J_n(\lambda r) J_n(\lambda R)$$

$$= \frac{1}{4\pi} \left(\sum_{m=-\infty}^{-1} + \sum_{m=1}^{\infty} \right) (-1)^m \left[\left(x - \frac{2\pi m}{\beta}\right)^2 + r^2 + R^2 - 2rR\cos\theta \right]^{-1/2}, \tag{D. 11}$$

where we have used (A. 7) to obtain the right-hand side of (D. 11).

Substituting (D. 9) in (4. 30), using the results (D. 10) and (D. 11), we obtain

$$V_1(x,r,\theta) = -\frac{|x|}{2\pi} + \frac{1}{4\pi}\sum_{m=-\infty}^{\infty}(-1)^m\left[\left(x-\frac{2\pi m}{\beta}\right)^2 + r^2 + R^2 - 2rR\cos\theta\right]^{-1/2}$$

$$-\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{s=1}^{\infty}\frac{2\cosh(\lambda_{ns}x)}{1+e^{2\pi\lambda_{ns}/\beta}}\;\frac{J_n(\lambda_{ns}r)J_n(\lambda_{ns}R)}{\lambda_{ns}\left(\frac{n^2}{\lambda_{ns}^2}-1\right)J_n^2(\lambda_{ns})}$$

$$-\frac{\beta}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{\nu=0}^{\infty}\cos[(\nu+\tfrac{1}{2})\beta x]$$

$$\cdot\left\{\frac{K'_n[(\nu+1/2)\beta]}{I'_n[(\nu+1/2)\beta]}\, I_n[(\nu+\tfrac{1}{2})\beta r]\, I_n[(\nu+\tfrac{1}{2})\beta R] + \frac{2\delta_{0n}}{(\nu+\frac{1}{2})^2\beta^2}\right\}. \qquad \text{(D. 12)}$$

(D. 12) is an alternate to formulas (4. 30) and (4. 31), which is much more suitable for calculating V_1 near $x = 0$. The sum over ν converges for all x; the sum over m converges everywhere except the points $(2\pi m/\beta, R, 0)$, when one term is infinite; the sum over s converges for $|x| < 2\pi/\beta$. Thus (D. 12) converges over twice the range of convergence of (D. 1), and although its proof was limited to $|x| < \pi/\beta$, by analytic continuation (D. 12) is valid for $|x| < 2\pi/\beta$. Because the parameter β is arbitrary, we can choose it to make the sum over s converge rapidly in the vicinity of $x = 0$. The rate of convergence and the value of each sum in (D. 12) depend on β, but the value of (D. 12) is, of course, independent of β.

Similar to what was found by Bouwkamp and de Bruijn,[24] our sum over ν can be considered as a rectangular approximation to the integral in (4. 30), with spacing β. The sum over s (again similar to theirs) and the sum over m excluding the m = 0 term (which has no counterpart in their analysis) then represent the correction to this approximation. The sum over m is the potential of an infinite number of "image" point sources located at $x = \pm 2\pi m/\beta$, with alternating sign. These are the images obtained by reflecting the source at $x = 0$ in the planes $x = \pm \pi/\beta$, then reflecting the two images in these two planes, and so on.

In the limit $\beta \to 0$, the sum over ν becomes the integral in (4.30). In this limit, the sum over s vanishes because of the $e^{2\pi \lambda_{ns}/\beta}$ in the denominator. In addition, the $m \neq 0$ terms in the sum over m vanish as the images retreat to infinity, leaving just the required point source at $x = 0$.

When $\beta = \pi/|x|$, the sum over s becomes identical to the sum in (4.31) since in this case

$$\frac{2 \cosh(\lambda_{ns} x)}{1 + e^{2\pi \lambda_{ns}/\beta}} = e^{-\lambda_{ns}|x|} .$$

The sum over ν vanishes because of the factor $\cos[(\nu + 1/2)\pi]$ and the sum over m becomes

$$\sum_{m=-\infty}^{\infty} (-1)^m [x^2(1-2m)^2 + r^2 + R^2 - 2rR \cos\theta]^{-1/2}$$

which vanishes because the $m = 1, 2, 3, \ldots$ terms cancel the $m = 0, -1, -2, \ldots$ terms, respectively.

Noting that the sum over ν of the algebraic term in the curly brackets in (D.12) is just the Fourier series representation of a periodic sawtooth wave:

$$\sum_{\nu=0}^{\infty} \frac{\cos[(\nu + 1/2)\beta x]}{(\nu + 1/2)^2} = \frac{\pi^2}{2}\left(1 - \frac{\beta|x|}{\pi}\right), \quad |x| \le \frac{2\pi}{\beta} , \qquad \text{(D. 13)}$$

we can simplify (D.12) somewhat to

$$
\begin{aligned}
V(x, r, \theta) = & - \frac{1}{2\beta} + \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} (-1)^m \left[\left(x - \frac{2\pi m}{\beta}\right)^2 + r^2 + R^2 - 2rR\cos\theta \right]^{-1/2} \\
& - \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{s=1}^{\infty} \frac{\cosh(\lambda_{ns} x)}{1+e^{2\pi \lambda_{ns}/\beta}} \frac{\lambda_{ns} J_n(\lambda_{ns} r) J_n(\lambda_{ns} R)}{(n^2 - \lambda_{ns}^2) J_n^2(\lambda_{ns})} \\
& - \frac{\beta}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{\nu=0}^{\infty} \cos[(\nu+\tfrac{1}{2})\beta x] \frac{K_n'[(\nu+\frac{1}{2})\beta]}{I_n'[(\nu+\frac{1}{2})\beta]} I_n[(\nu+\tfrac{1}{2})\beta r] I_n[(\nu+\tfrac{1}{2})\beta R]
\end{aligned}
\tag{D.14}
$$

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ABSTRACT

The potential is determined everywhere inside an infinitely long cylindrical cell, for a point source of current at an arbitrary location in the cell interior. The mathematical techniques used are applicable to a wide variety of physical problems with mixed boundary conditions in cylindrical geometry in which a small parameter appears in the boundary condition. The model for the cell consists of a cylinder of radius a, with an interior of conductivity σ_i, surrounded by a thin membrane of conductivity σ_m and thickness δ. The cell is bathed in a highly conducting external medium which maintains the outer membrane surface at zero potential. The problem is solved by obtaining asymptotic expansions for the potential in terms of the small parameter $\epsilon = \sigma_m a/\sigma_i \delta$. One (inner or near-field) expansion is valid in a region including the point source, and a second (outer or far-field) expansion is valid in a region away from the source. Using the singular perturbation technique of matching, the near-field and far-field expansions are made to coincide in an intermediate overlap region. A relatively simple form is obtained for the far-field expansion, consisting of the known result of one-dimensional cable theory plus correction terms. The near-field expansion is more complicated. However, its leading term is shown to be merely a large constant potential proportioned to $1/\sqrt{\epsilon}$. The higher-order terms are obtained as eigenfunction expansions, double infinite sums of Bessel functions, or alternatively as single infinite sums of infinite integrals of modified Bessel functions. In the latter case the singularity, which is present at the location of the point source, appears separately in a simple algebraic form. It is also shown that these expansions can be obtained directly from the exact solution of the problem, but this procedure does not yield the physical insight into the behavior for small ϵ as does the

singular perturbation analysis. The double sum eigenfunction expansion is useful for computing the potential except when the axial distance from the source is small. In this case an alternative, rapidly converging series representation is obtained consisting of a rectangular approximation to the above integral, a double sum closely related to the above double sum, and a sum of the free-space potentials of an infinite number of alternating image sinks and sources located within the cylinder at uniform axial separation.

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