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A POINT SOURCE IN A CYLINDRICAL CELL: POTENTIAL FOR A STEP-FUNCTION OF CURRENT INSIDE AN INFINITE CYLINDRICAL CELL IN A MEDIUM OF FINITE CONDUCTIVITY

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ABSTRACT

The potential is found for all time, everywhere inside an infinitely long cylindrical cell and in the external bathing medium for the case of a point source of current switched on abruptly at $t = 0$. The solution is expressed as a Fourier series in the azimuthal coordinate, and a Fourier integral (of functions of modified Bessel functions of the radial coordinate) in the longitudinal coordinate. The cell is modeled by an infinitely long cylinder of radius a and conductivity σ_i, surrounded by a membrane of thickness δ, conductivity σ_m and surface capacitance C_m, bathed in a medium of conductivity σ_o. For the physiologically interesting case of $\varepsilon = \sigma_m a/\sigma_i \delta << 1$, asymptotic expansions are obtained for the two special cases of $t = \infty$ and $\sigma_o = \infty$. The expansions are simplified by introducing synthetic independent variables. In the steady state, $t = \infty$ case, a uniform expansion is obtained inside the cell consisting of Fourier-series-integral terms identical to earlier results for the steady state, perfectly conducting exterior medium problem ($\sigma_o = \infty$, $t = \infty$), and exponential integral terms which are new. Outside the cell a similar expansion is obtained for radial distances much less than $a\varepsilon^{-1/2}$. The relation of the expansion to a singular perturbation analysis is studied. The results are generalized to the sinusoidal steady state. Using the same synthetic longitudinal coordinate and a synthetic time variable which depends on spatial variables, time and ε, an asymptotic expansion is obtained for times much longer than $C_m\ a/\sigma_i$ and a perfectly conducting exterior ($\sigma_o = \infty$). The expansion contains complementary error functions and the first term, in the $\varepsilon \to 0$ limit, reduces to the result of classical one-dimensional cable theory. Formulas are developed giving the initial jump and time-rate-of-change of the potential, useful for times much shorter than $C_m\ a/\sigma_i$. For the $t = 0^+$ potential, which is the solution of a Dirichlet problem, a rapidly converging summation representation is found for calculating the potential near the source-point singularity. It is shown that the full time-dependent potential problem simplifies considerably for the case of equal inside and outside conductivities.

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I. INTRODUCTION

Analysis of the natural electrical activity of biological cells and tissues requires measurement of linear electrical properties, particularly the properties of membranes. The membrane of a cell is a structure of very high impedance since its evolutionary significance is to define the cell, to isolate the cell interior from the extracellular space, and to protect the life of the cell from external disturbance. In order to study the electrical properties of cells it is best to apply current so that it all must cross the structure with properties of greatest interest, usually one of the membranes of the cell. It is best then to apply current inside the cell so that it must flow across the membrane to an electrode outside the cell. Micropipettes filled with conductive salt solution can be inserted into cells to allow the application of such current. These microelectrodes have tip diameters very much smaller than the size of the cells and can therefore be represented for most purposes as point sources. Much of our recent work has been devoted to an analysis of the potential induced by current flow from such a microelectrode inserted into a cell.[1,2] This problem also specifies the fundamental mathematical solution for the geometry and boundary conditions (the Green's function) and so the solution to the problem can be used to generate the solution to problems containing sources with other spatial distributions.

In order to analyze this experimental situation and to determine the Green's function for the problem, we construct a mathematical and physical model for the cell in which the interior is represented as an homogeneous isotropic resistive material, and a boundary condition is written which is appropriate for a thin structure with both resistive and capacitive properties. A complete derivation of the model is given in Reference 1. The boundary condition arises in many problems describing membranes — not just the electrical problems we discuss, but also in diffusion, water flow, and other problems — and so is named the "membrane boundary condition". The boundary condition states that the normal component of the current density which flows up to one side of the membrane is equal to the normal component of the current density which flows from the other side of the membrane and is equal to the current which crosses through the interior of the membrane. The membrane current is modelled by a capacitive current in parallel with a resistive current.[1] The boundary condition written in dimensionless form contains a sma parameter, since the membrane conductance is small; the small parameter ε contains the ratio of the conductivity of the membrane to that of the cell interior, and the dimensions of the cell and membrane.

In Section II we obtain the exact solution of the problem of a unit point source of current switched on abruptly at $t = 0$, at an arbitrary point inside an infinitely long cylindrical cell surrounded by a medium of finite conductivity. The solution is in terms of Fourier series and transforms. Solving the problem this way is straightforward, but leads to an intractable representation which permits little physical understanding and is difficult if not impractical to compute. We exploit the small parameter ε of the boundary condition to construct simpler representations, in the form of asymptotic expansions in that parameter. These expansions can be constructed by the techniques of singular perturbation theory or by direct expansion of the exact solution. Singular perturbation techniques have the advantage that they provide considerable physical insight; they are, however, often laborious to apply and in some cases it is not clear how they can be applied at all. Expansion of the exact solution by direct methods is often complicated, particularly if the resulting expansion is nonuniform; that is, if the expansion is only valid for certain domains of the spatial or temporal variables. In the case of nonuniform expansions, it is a difficult and sometimes hopeless task to determine the region of validity of the expansion. It is possible, however, in the present case to modify the direct expansion method by applying a common trick of singular perturbation theory, which is to handle nonuniform expansions by introducing synthetic independent variables. These synthetic variables are functions of both spatial and/or temporal coordinates as well as the small parameter of the problem. We use this trick by assuming an appropriate form of the synthetic variables, with free parameters, and rewriting the exact solution in terms of these variables. An asymptotic expansion is then obtained for the rewritten form of the exact solution, and with a proper choice of the free parameters, the resulting expansion is uniform in the region of interest. Each synthetic variable describes a natural physical coordinate of the problem in some region, and expansions written in terms of the appropriate synthetic variable are uniform in this region.

In Section III we apply this procedure to the exact solution of the steady state problem. In Section V we generalize it to the sinusoidal state. In Section VI we do it for the transient problem, with the simplification of infinite conductivity of the medium surrounding the cell.

It is possible to develop physical and physiological insight into the meaning of each term of the asymptotic expansion: physical problems (equation, boundary conditions and initial condition) can be constructed which specify each term and which have obvious physiological meaning. In this way one can use physical reasoning to guess the properties of related situations which have not yet been analyzed. In Section IV we set up the sequence of singular perturbation problems corresponding to the expansion obtained in Section III, taking advantage of our a priori knowledge of the expansion. The actual form of the expansion (that is, the particular sequence of functions of ε) and the form of the synthetic variables are not easy to understand physically, and it may be necessary to construct a self-contained singular perturbation treatment to provide this physical insight.

In this report we consider a number of situations of some complexity which have resisted analysis up to now. We consider the cylindrical cell, taking into account the resistive properties of the extracellular space and the capacitive properties of the cell membrane. It proves possible to analyze the effects of the resistive properties of the extracellular space in the steady-state and to analyze the time dependence of the potential when the extracellular resistance is neglected, but it has not been possible to include both effects simultaneously. Nonetheless, we are able to derive physiologically useful results, showing the range of validity of earlier more approximate treatments, and predicting a number of experimentally observable phenomena. It does appear likely that an asymptotic expansion including time dependence can be constructed for the experimentally accessible special case of equal inside and outside resistivities, but we present only the exact solution for this case in Section VII.

Some interesting problems remain to be explored. Mathematically, it would be worthwhile to try a formal singular perturbation analysis of the problem, assuming no prior knowledge of the exact solution and its asymptotic behavior. The method we use in Section VI (and also in Reference 2) to make weakly convergent sums rapidly convergent seems to have promise as a general technique; we hope its basis and applicability will be studied. Interesting physiological problems include the treatment of other cell geometries, for example, the thin plane cell which might represent the properties of syncitial tissues like epithelia or heart or smooth muscle. Furthermore, it would be most useful to analyze the effect of current flow in the extracellular space on the transmembrane potential, for a number of configurations of the extracellular space. This problem might give insight into how the central nervous system can function with each cell electrically isolated from its neighbor even though the cells are so close together.

II. MATHEMATICAL PROBLEM AND ITS GENERAL SOLUTION USING INTEGRAL TRANSFORMS

The derivation of the equation and boundary conditions governing the electrical potential for a point source inside a cell has been given in an earlier report.[1] The interior of the cell is assumed to be electrically homogeneous and isotropic, with conductivity σ_i. It is bounded by a membrane with conductivity σ_m, and immersed in a bath of infinite extent with conductivity σ_o. The earlier report[1] considered the case of a finite-sized cell, and in particular a spherical cell with a point source switched on abruptly at $t = 0$.

A subsequent report[2] considered the case of an infinitely long cylindrical cell, but only in the special case of a perfectly conducting external medium and a resistive, but not capacitive, membrane. An asymptotic representation of the potential inside the cell was found using singular perturbation theory, which led directly to an expansion of the potential in powers of the small parameter $\varepsilon = \sigma_m a/\sigma_i \delta$ (a = cylinder radius, δ = membrane thickness), and also by expanding the exact solution itself in powers of ε.

In the present report, we study the potential in an infinitely long cylindrical cell and consider the more general case, allowing time dependence of the potential as well as finite external conductivity. In the general case, it seems more convenient to first obtain the exact solution for arbitrary ε and then expand the result in powers of ε. Here we go directly from an integral representation of the potential to the asymptotic expansion, whereas in the earlier report[2] we went from an infinite summation representation to the asymptotic expansion.

The problem for the potential, V, due to a point source turned on at $t = 0$, inside an infinitely long cylindrical cell immersed in a conducting medium of infinite extent, is specified by the following Poisson equation, boundary conditions and initial conditions:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial^2 V}{\partial x^2}=-\frac{1}{r}\,\delta(x)\delta(r-R)\delta(\theta)u(t) \tag{2.1}$$

$$\frac{1}{\varepsilon}\frac{\partial V^-}{\partial r}=\frac{1}{\varepsilon\alpha}\frac{\partial V^+}{\partial r}=V^+-V^-+\frac{\partial V^+}{\partial t}-\frac{\partial V^-}{\partial t} \tag{2.2}$$

$$V(x,r,\theta,t)=0 \text{ at } x=\pm\infty \text{ or } r=\infty \tag{2.3}$$

$$V(x,1^+,\theta,0^+)=V(x,1^-,\theta,0^+) \tag{2.4}$$

The spatial variables x and r are made dimensionless by dividing the physical variables x', r' by the radius, a, of the cylinder (i.e., $x = x'/a$, $r = r'/a$). The coordinate system is shown in Figure 1. The point source of current located at (0,R,0), where $R < 1$, is represented by the three-dimensional delta function in (2.1), and is turned on abruptly at $t = 0$, represented by the unit step function u(t) in (2.1). As shown in Reference 1, if the units of u(t) are amperes, then the scaling for V is $V = a\sigma_i V'$, where V' is the potential in volts.

The first equality in the boundary condition (2.2) expresses continuity of the normal component of the current density crossing the membrane (more precisely, a^2 times the current density). The superscripts $^-$ and $^+$ represent the conditions just inside ($r = 1^-$), and just outside ($r = 1^+$) the membrane, respectively. The ratio of interior to exterior conductivity is denoted by $\alpha = \sigma_i/\sigma_o$. The second equality in (2.2) relates the current crossing the membrane to the electrical properties of the membrane, which is assumed to have a capacitance C_m per unit area in addition to the conductivity σ_m. The time variable is dimensionless and is related to the real time t' by $t = (\sigma_m/C_m\delta)t'$. For a derivation of (2.1) and (2.2) the reader is referred to an earlier report.[1]

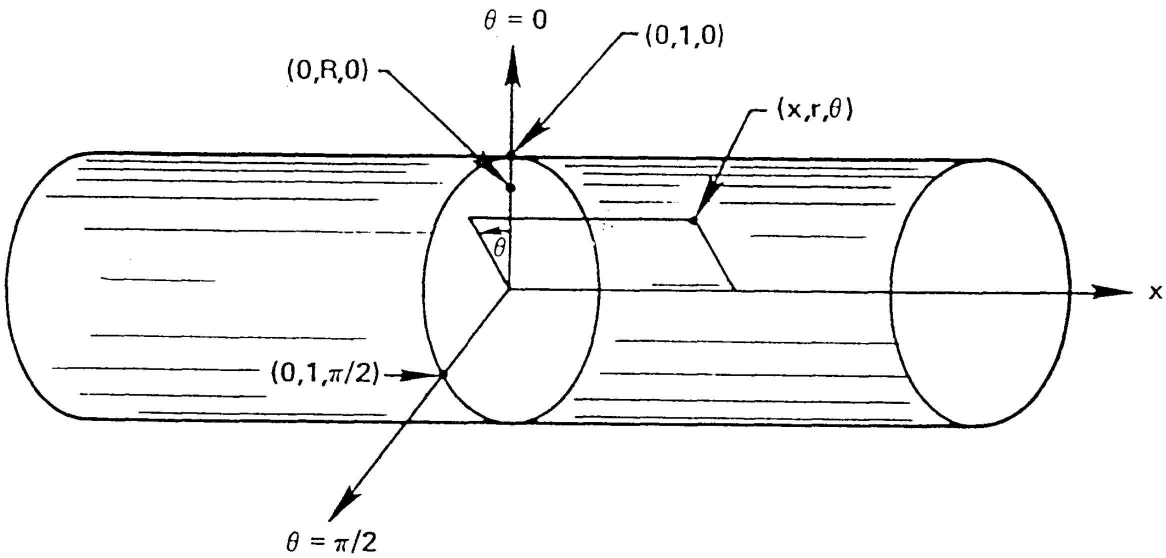

Figure 1. Coordinate System for Cylindrical Cell.

The boundary condition (2.3) assumes the zero reference for the potential is at infinity. The initial condition (2.4) requires a finite time for the membrane capacitance to accumulate charge, that is, (2.4) implies that a charge and hence a potential difference $V^+ - V^-$ cannot appear across the capacitance instantaneously since an infinite current density at $t = 0$, would be required. Such infinite currents are not possible because of the finite conductivities of the interior and exterior media.

Observing that $V(x,r,\theta,t)$ must be an even function of x, we can solve for the potential by first taking the Fourier cosine transform in x of Equation (2.1). Defining the Fourier cosine transform by

$$\phi(k,r,\theta,t) = \int_{-\infty}^{\infty} \cos kx \; V(x,r,\theta,t)\, dx \tag{2.5}$$

the Fourier transform of (2.1) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\phi^2} - k^2\phi = -\frac{1}{r}\,\delta(r-R)\delta(\theta) \tag{2.6}$$

Letting

$$\psi_n(k,r,t) = \int_0^{2\pi} e^{-in\theta}\, \phi(k,r,\theta,t)\, d\theta \tag{2.7}$$

equation (2.6) may be transformed to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_n}{\partial r}\right) - \left(k^2 + \frac{n^2}{r^2}\right)\psi_n = -\frac{1}{r}\,\delta(r-R)\; u(t) \tag{2.8}$$

This is a doubly transformed representation of our original Poisson equation. Using the transforms (2.5) and (2.7) on the boundary conditions (2.2) and (2.3) and on the initial condition (2.4) for $V(x,r,\theta,t)$, the corresponding conditions on $\psi_n(k,r,t)$ are

$$\frac{1}{\varepsilon}\frac{\partial\psi_n^-}{\partial r} = \frac{1}{\varepsilon\alpha}\frac{\partial\psi_n^+}{\partial r} = \psi_n^+ - \psi_n^- + \frac{\partial\psi_n^+}{\partial t} - \frac{\partial\psi_n^-}{\partial t} \tag{2.9}$$

$$\psi_n(k,r,t) = 0 \text{ at } r = \infty \tag{2.10}$$

$$\psi_n^+ = \psi_n^- \quad \text{at } t = 0^+ \tag{2.11}$$

Integrating (2.8) across the delta function, from $r = R^-$ to $r = R^+$, shows that there is a discontinuity in the derivative of ψ_n given by

$$\frac{\partial\psi_n(k,R^+,t)}{\partial r} - \frac{\partial\psi_n(k,R^-,t)}{\partial r} = -\frac{u(t)}{R} \tag{2.12}$$

The inhomogeneous equation (2.8) may be replaced by the corresponding homogeneous equation plus the jump condition (2.12) on the r-derivative at $r = R$.

In each of the three regions $0 < r < R$, $R < r < 1$ and $1 < r < \infty$, the right hand side of (2.8) is zero. The solution to this homogeneous equation in each region is a linear combination of the modified Bessel functions $I_n(kr)$ and $K_n(kr)$. The solution which is finite at $r = 0$ and which satisfies the boundary condition (2.10) at $r = \infty$ is of the form

$$\psi_n(k,r,t) = \begin{cases} a_n(t)\ I_n(kr) & 0<r<R \\ b_n(t)I_n(kr) + c_n(t)K_n(kr) & R<r<1 \\ d_n(t)\ K_n(kr) & 1<r<\infty \end{cases} \tag{2.13}$$

If the first derivative has the finite discontinuity (2.12) at $r = R$, ψ_n itself must be continuous. Thus, from (2.13),

$$\left[a_n(t) - b_n(t)\right] I_n(kR) = c_n(t)\ K_n(kR) \tag{2.14}$$

Substituting the form (2.13) in the jump condition (2.12), we obtain

$$\left[b_n(t) - a_n(t)\right] I_n'(kR) + c_n(t)\ K_n'(kR) = -\frac{u(t)}{kR} \tag{2.15}$$

and substituting (2.13) in the $r = 1$ boundary condition (2.9), we obtain the two equations

$$\frac{k}{\varepsilon}\left[b_n(t)I_n'(k) + c_n(t)K_n'(k)\right] = \frac{k}{\varepsilon\alpha} d_n(t)\ K_n'(k)$$

$$= (d_n+\dot{d}_n)\ K_n(k) - (c_n+\dot{c}_n)\ K_n(k) - (b_n+\dot{b}_n)\ I_n(k) \tag{2.16}$$

where a dot denotes differentiation with respect to time.

Equations (2.14) - (2.16) are four equations which may be solved to obtain the functional form of $a_n(t)$, $b_n(t)$, $c_n(t)$ and $d_n(t)$. If we solve (2.14) for $a_n - b_n$, substitute the result in (2.15) and solve for c_n, using the Wronskian

$$I_n(kR)\ K_n'(kR) - K_n(kR)\ I_n'(kR) = -\frac{1}{kR},$$

we obtain

$$c_n(t) = u(t)\ I_n(kR) \tag{2.17}$$

and consequently

$$a_n(t) = b_n(t) - u(t)\ K_n(kR) \tag{2.18}$$

Substituting the result (2.17) for c_n in the first of Equations (2.16) we obtain an expression for d_n in terms of b_n,

$$d_n(t) = \alpha b_n(t)\ \frac{I_n'(k)}{K_n'(k)} + \alpha u(t)\ I_n(kR) \tag{2.19}$$

Substituting (2.17, (2.18) and (2.19) in (2.13), we obtain an expression for the double transform of the potential in terms of only a single unknown function $b_n(t)$.

$$\psi_n(k,r,t) = \begin{cases} b_n(t)\ I_n(kR) + u(t) \begin{cases} I_n(kr)\ K_n(kR), & 0<r<R \\ I_n(kR)\ K_n(kr), & R<r<1 \end{cases} \\ \alpha b_n(t)\ \frac{I_n'(k)}{K_n'(k)}\ K_n(kr) + \alpha u(t)\ I_n(kR)\ K_n(kr), & 1<r<\infty. \end{cases} \tag{2.20}$$

In order to obtain the potential $V(x,r,\theta,t)$ we now need to determine the functional form of $b_n(t)$ and then take the inverse transforms

$$\phi(k,r,\theta,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \psi_n(k,r,t)\, e^{in\theta} \tag{2.21}$$

and

$$V(x,r,\theta,t) = \frac{1}{\pi}\int_0^{\infty} \cos kx\, \phi(k,r,\theta,t)\, dk \tag{2.22}$$

of (2.20).

Using (2.17) and (2.19) in the second of Equations (2.16) leads, after some algebraic manipulation, to the following differential equation for b_n.

$$\dot{b}_n(t) + \left[\frac{\frac{k}{\varepsilon} I_n'(k)K_n'(k)}{I_n(k)K_n'(k) - \alpha I_n'(k)K_n(k)} + 1\right] b_n(t)$$

$$= \frac{I_n(kR)K_n'(k)}{I_n(k)K_n'(k) - \alpha I_n'(k)K_n(k)}\left[\left\{(\alpha-1)K_n(k) - \frac{k}{\varepsilon}K_n'(k)\right\}u(t) + (\alpha-1)K_n(k)\delta(t)\right] \tag{2.23}$$

Equation (2.23) is of the form

$$\dot{b}_n(t) + p\, b_n(t) = q\, u(t) + s\, \delta(t) \tag{2.24}$$

for which the solution is

$$b_n(t) = u(t)\left[\frac{q}{p} + \left(s - \frac{q}{p}\right) e^{-pt}\right] \tag{2.25}$$

Using the appropriate form of p,q, and s which appear in (2.23), in the solution (2.25), we find after more algebraic manipulation and use of the Wronskian of I_n and K_n,

$$b_n(t) = \frac{u(t)I_n(kR)K_n'(k)}{kI_n'(k)K_n'(k) + \varepsilon I_n(k)K_n'(k) - \varepsilon\alpha I_n'(k)K_n(k)}$$

$$\cdot\left[\varepsilon(\alpha-1)K_n(k)-kK_n'(k) - \frac{K_n'(k)\, e^{-\left\{\frac{kI_n'(k)K_n'(k)}{I_n(k)K_n'(k)-\alpha I_n'(k)K_n(k)} + \varepsilon\right\}\frac{t}{\varepsilon}}}{I_n(k)K_n'(k) - \alpha I_n'(k)K_n(k)}\right]. \tag{2.26}$$

Noting that[3,4]

$$\frac{1}{2\pi^2}\int_0^\infty dk \cos kx \sum_{n=-\infty}^{\infty} e^{in\theta} \begin{cases} I_n(kr)K_n(kR), & 0<r<R \\ I_n(kR)K_n(kr), & R<r<1 \end{cases}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk \cos kx\, K_0\left(k\sqrt{r^2+R^2-2rR\cos\theta}\right)$$

$$= \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2}, \tag{2.27}$$

we can take the inverse transforms (2.21) and (2.22) of (2.20), with $b_n(t)$ given by (2.26).

Inside the cell, for $0 < r < 1$, $t > 0$, the potential is

$$V(x,r,\theta,t) = \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2}$$

$$+ \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk \cos kx \frac{I_n(kr)I_n(kR)K_n'(k)}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n}$$

$$\cdot\left[\varepsilon(\alpha-1)K_n-kK_n' - \frac{K_n'}{I_nK_n'-\alpha I_n'K_n}\, e^{-\left(\frac{kI_n'K_n'}{I_nK_n'-\alpha I_n'K_n} + \varepsilon\right)\frac{t}{\varepsilon}}\right] \tag{2.28}$$

where for brevity we have deleted the argument of I_n and K_n whenever it is k.

For the outside region, $1 < r < \infty$, $t > 0$, we substitute (2.26) in (2.20), and do the following manipulations for the time independent part of ψ_n:

$$\alpha b_n(\infty)\frac{I'_n}{K'_n}K_n(kr) + \alpha I_n(kR)\,K_n(kr)$$

$$= \alpha I_n(kR)\,K_n(kr)\left\{\frac{I'_n\left[\varepsilon(\alpha-1)K_n - kK'_n\right]}{kI'_nK'_n+\varepsilon I_nK'_n-\varepsilon\alpha I'_nK_n} + 1\right\}$$

$$= \alpha I_n(kR)\,K_n(kr)\cdot\frac{-\varepsilon K_nI'_n + \varepsilon I_nK'_n}{kI'_nK'_n+\varepsilon I_nK'_n-\varepsilon\alpha I'_nK_n}$$

$$= \frac{-\frac{\varepsilon\alpha}{k}\,I_n(kR)K_n(kr)}{kI'_nK'_n+\varepsilon I_nK'_n-\varepsilon\alpha I'_nK_n}$$

The inverse transforms (2.21) and (2.22) of (2.20), with $b_n(t)$ given by (2.26) yields, for the outside region, $1 < r < \infty$, $t > 0$,

$$V(x,r,\theta,t) = \frac{\alpha}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty}dk\;\cos kx\;\frac{K_n(kr)I_n(kR)}{kI'_nK'_n+\varepsilon I_nK'_n-\varepsilon\alpha I'_nK_n}$$
$$\left[-\frac{\varepsilon}{k} - \frac{I'_nK'_n}{I_nK'_n-\alpha I'_nK_n}\,e^{-\left(\frac{kI'_nK'_n}{I_nK'_n-\alpha I'_nK_n}+\varepsilon\right)\frac{t}{\varepsilon}}\right] \tag{2.29}$$

Equations (2.28) and (2.29) express the general solution for the potential inside and outside the cylindrical cell, respectively, for a point source of current inside the cell. In most of the remainder of this report, we will consider the special case of small ε, which is the case of greatest physiological interest. In this case, great simplifications of Equations (2.28) and (2.29) are obtained by studying the asymptotic behavior of these integrals in the $\varepsilon \to 0$ limit.

III. STEADY STATE: ASYMPTOTIC EXPANSION FOR HIGH MEMBRANE RESISTANCE

In this section, we investigate the steady state behavior of the potential for small ε. Letting $t = \infty$ in the formulas (2.28) and (2.29) for the inside and outside potentials, respectively, we obtain for $0 < r < 1$,

$$V(x,r,\theta) = \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2} + \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx\; I_n(kr)\, I_n(kR) \cdot \frac{\left[\varepsilon(\alpha-1)K_n(k) - kK_n'(k)\right] K_n'(k)}{kI_n'(k)K_n'(k) + \varepsilon I_n(k)K_n'(k) - \varepsilon\alpha I_n'(k)K_n(k)} \tag{3.1}$$

and for $1 < r < \infty$

$$V(x,r,\theta) = -\frac{\varepsilon\alpha}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx \cdot \frac{1}{k} \cdot \frac{K_n(kr)I_n(kR)}{kI_n'K_n'+\varepsilon I_nK_n'-\varepsilon\alpha I_n'K_n} \tag{3.2}$$

where we have defined the steady state potential by $V(x,r,\theta) = V(x,r,\theta,\infty)$.

A. Potential Inside the Cell

First, we will consider the inside potential, (3.1). It consists of an algebraic term which is just the free-space potential of a unit point source at (0,R,0), and a complicated term which is an infinite sum of integrals (Fourier transforms). We would like now to study the small ε behavior of (3.1) by expanding the integrals in powers of ε, and interchanging the order of integration and summation over powers of ε. This results in an asymptotic representation of (3.1) in the $\varepsilon \to 0$ limit. The procedure is straightforward for the integrands with $n \neq 0$. In the $n = 0$ integrand, however, the expansion in powers of ε diverges at the lower limit of integration, $k = 0$. Hence we must treat this case with special care.

We start by expanding the fraction appearing in the integrand in (3.1) in powers of ε, treating ε as a small quantity. We find

$$\frac{[\varepsilon(\alpha-1)K_n - kK_n']K_n'}{kI_n'K_n' + \varepsilon I_n K_n' - \varepsilon\alpha I_n' K_n} = -\frac{K_n'}{I_n'} \cdot \frac{1 - \dfrac{\varepsilon(\alpha-1)}{k} \cdot \dfrac{K_n}{K_n'}}{1 + \dfrac{\varepsilon}{k}\left(\dfrac{I_n}{I_n'} - \alpha\dfrac{K_n}{K_n'}\right)}$$

$$= -\frac{K_n'}{I_n'} \cdot \left[1 + \frac{\dfrac{\varepsilon}{k}\left(\dfrac{K_n}{K_n'} - \dfrac{I_n}{I_n'}\right)}{1 + \dfrac{\varepsilon}{k}\left(\dfrac{I_n}{I_n'} - \alpha\dfrac{K_n}{K_n'}\right)}\right]$$

$$= -\frac{K_n'}{I_n'} \cdot \left[1 + \frac{\varepsilon}{k^2 I_n' K_n'} \cdot \frac{1}{1 + \dfrac{\varepsilon}{k}\left(\dfrac{I_n}{I_n'} - \alpha\dfrac{K_n}{K_n'}\right)}\right]$$

$$= -\frac{K_n'}{I_n'} \cdot \left[1 + \frac{\varepsilon}{k^2 I_n' K_n'} \cdot \left\{1 - \frac{\varepsilon}{k}\left(\frac{I_n}{I_n'} - \alpha\frac{K_n}{K_n'}\right) + \dots\right\}\right] \tag{3.3}$$

The third equality in (3.3) follows using the Wronskian of $I_n(k)$ and $K_n(k)$.

For sufficiently small ε, and $\alpha \sim 1$, the expansion (3.3) converges for $k \neq 0$. If $k \to 0$ and $n \neq 0$, we have for the expansion of the function in the integrand in (3.1) whose inverse Fourier cosine transform we must calculate,

$$-I_n(kr)I_n(kR)\frac{K_n'}{I_n'} \cdot \left[1 + \frac{\varepsilon}{k^2 I_n' K_n'}\left\{1 - \frac{\varepsilon}{k}\left(\frac{I_n}{I_n'} - \alpha\frac{K_n}{K_n'}\right) + \dots\right\}\right]$$

$$\xrightarrow[k\to 0]{} \frac{1}{2n}\left[1 - \frac{2\varepsilon}{n}\left\{1 - \varepsilon\left(1 + \frac{\alpha}{n}\right) + \dots\right\}\right] \tag{3.4a}$$

which converges absolutely if $\varepsilon < (1 + \alpha/n)^{-1}$, so that the expansion (3.3) may be used even when $k = 0$ if $n \neq 0$. On the other hand, if $k \to 0$ and $n = 0$, we have

$$-I_0(kr)I_0(kR)\,\frac{K_0'}{I_0'}\left[1 + \frac{\varepsilon}{k^2 I_0' K_0'}\left\{1 - \frac{\varepsilon}{k}\left(\frac{I_0}{I_0'} - \alpha\,\frac{K_0}{K_0'}\right) + \ldots\right\}\right]$$

$$\xrightarrow[k\to 0]{}\ \frac{2}{k^2}\left[1 - \frac{2\varepsilon}{k^2}\left\{1 - \frac{2\varepsilon}{k^2} + \ldots\right\}\right] \tag{3.4b}$$

which diverges at $k = 0$. The expansion (3.3) thus may not be used when $n = 0$. We can, however, get around this difficulty by determining an explicit expression for the remainder, for any finite number of terms in (3.3). We will now do this for the first two terms in the expansion (3.3). We begin by separating the fraction in the integrand of (3.1) into its $\varepsilon \to 0$ limit and the remainder:

$$\frac{\left[\varepsilon(\alpha-1)K_n - kK_n'\right]K_n'}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n} = -\frac{K_n'}{I_n'} + \frac{K_n'}{I_n'}\cdot\left[1 + \frac{\varepsilon(\alpha-1)K_nI_n' - kK_n'I_n'}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n}\right]$$

$$= -\frac{K_n'}{I_n'} + \frac{K_n'}{I_n'}\cdot\frac{-\varepsilon(K_nI_n' - I_nK_n')}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n}$$

$$= -\frac{K_n'}{I_n'} - \frac{\varepsilon}{k}\cdot\frac{K_n'}{I_n'}\cdot\frac{1}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n} \tag{3.5}$$

We can continue this process, and now isolate the $O(\varepsilon)$ term as well. To do this, we write

$$\frac{1}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n} = \frac{1}{kI_n'K_n'} + \frac{1}{kI_n'K_n'}\cdot\left[\frac{kI_n'K_n'}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n} - 1\right]$$

$$= \frac{1}{kI_n'K_n'} + \frac{1}{kI_n'K_n'}\cdot\frac{\alpha I_n'K_n - I_nK_n'}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n}$$

and substitute in (3.5) to obtain an $O(1)$ term, an $O(\varepsilon)$ term, and the remainder:

$$\frac{[\varepsilon(\alpha-1)K_n - kK_n']K_n'}{kI_n'K_n' + \varepsilon I_nK_n' - \varepsilon\alpha I_n'K_n} = -\frac{K_n'}{I_n'} - \frac{\varepsilon}{k^2I_n'^2} - \frac{\varepsilon^2}{k^2I_n'^2} \cdot \frac{1}{\dfrac{kI_n'K_n'}{\alpha I_n'K_n - I_nK_n'} - \varepsilon} \tag{3.6}$$

Note that the first two terms in (3.6) are identical to the first two terms in (3.3).

We could obviously continue this process indefinitely and thus obtain an asymptotic expansion for V to any desired order in ε, but we will stop at this point and be content with an expansion correct to $O(\varepsilon)$.

Substituting (3.6) in (3.1) we obtain an expression for the potential inside the cylinder,

$$V(x,r,\theta) = \frac{1}{4\pi}(x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

$$- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk \cos kx\, I_n(kr)I_n(kR)$$

$$\cdot\left[\frac{K_n'}{I_n'} + \frac{\varepsilon}{k^2I_n'^2} + \frac{\varepsilon^2}{k^2I_n'^2}\cdot\frac{1}{\dfrac{kI_n'K_n'}{\alpha I_n'K_n - I_nK_n'} - \varepsilon}\right] \tag{3.7}$$

It is not permissible to discard the last term in the square bracket in (3.7), which at first glance appears to be of $O(\varepsilon^2)$, because when $n = 0$ its contribution to the integral is of lower order. We knew in advance, from the divergence of (3.3) at $n = 0$, $k = 0$, that this must occur since the first two terms [$O(1)$ and $O(\varepsilon)$] in this case do not correctly represent the integrand in (3.1).

We now derive this point directly from (3.7). Note that as $k \to 0$,

$$I_n'(k)K_n'(k) \sim \begin{cases} -\frac{n}{2}\, k^{-2} & \text{if } n \neq 0 \\ -\frac{1}{2} & \text{if } n = 0 \end{cases}$$

$$I_n'(k)K_n(k) \sim \begin{cases} \frac{1}{2}\, k^{-1} & \text{if } n \neq 0 \\ -\frac{1}{2}\, k \log k & \text{if } n = 0 \end{cases}$$

$$I_n(k)K_n'(k) \sim \begin{cases} -\frac{1}{2}\, k^{-1} & \text{if } n \neq 0 \\ -k^{-1} & \text{if } n = 0 \end{cases}$$

Consequently we have for the limiting behavior of the function whose inverse Fourier cosine transform is to be calculated in (3.7), for $n \neq 0$, $k \to 0$,

$$I_n(kr)I_n(kR) \cdot \left[\frac{K_n'}{I_n'} + \frac{\varepsilon}{k^2 I_n'^2} + \frac{\varepsilon^2}{k^2 I_n'^2} \cdot \frac{1}{\dfrac{kI_n'K_n'}{\alpha I_n'K_n - I_nK_n'} - \varepsilon} \right]$$

$$\xrightarrow[k\to 0]{} r^n R^n \cdot \left[-\frac{1}{2n} + \frac{\varepsilon}{4n^2} - \frac{\varepsilon^2}{4n^2} \cdot \frac{1+\alpha}{n + \varepsilon(1+\alpha)} \right]$$

The last term is $O(\varepsilon^2)$ for $n \neq 0$, $k \to 0$. Since $kI_n'K_n' \cdot (\alpha I_n'K_n - I_nK_n')^{-1}$ is also nonzero for any $k > 0$, the third term is $O(\varepsilon^2)$ for all k and may be omitted in the $O(\varepsilon)$ calculation of the potential.

The corresponding term for $n = 0$, however, cannot be omitted. It is

$$I_0(kr)I_0(kR) \left[-\frac{K_1}{I_1} + \frac{\varepsilon}{k^2 I_1^2} - \frac{\varepsilon^2 I_0}{k^2 I_1^2} \cdot \frac{1}{\dfrac{kI_1}{1 + \alpha \dfrac{I_1K_0}{I_0K_1}} + \varepsilon I_0} \right]$$

in which each term diverges as $k \to 0$:

$$\frac{K_1(k)}{I_1(k)} \xrightarrow[k\to 0]{} -\frac{2}{k^2}$$

$$\frac{\varepsilon}{k^2 I_1^2(k)} \xrightarrow[k\to 0]{} \frac{4\varepsilon}{k^4}$$

$$\frac{\varepsilon^2 I_0}{k^2 I_1^2} \cdot \frac{1}{\dfrac{kI_1}{1+\alpha\dfrac{I_1K_0}{I_0K_1}} + \varepsilon I_0} \xrightarrow[k\to 0]{} \frac{8\varepsilon^2}{k^4(k^2+2\varepsilon)}$$

Retaining terms up to and including $O(\varepsilon)$, the inside potential (3.7) becomes.

$$\begin{aligned} V(x,r,\theta) &= \frac{1}{4\pi}(x^2+r^2+R^2-2rR\cos\theta)^{-1/2} \\ &\quad - \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx\; I_n(kr)I_n(kR) \\ &\quad \cdot \left[\frac{K_n'}{I_n'} + \frac{\varepsilon}{k^2 I_n'^2} - \frac{\varepsilon^2 I_0}{k^2 I_1^2}\cdot\frac{\delta_{n0}}{\dfrac{kI_1}{1+\alpha\dfrac{I_1K_0}{I_0K_1}} + \varepsilon I_0}\right] \end{aligned} \tag{3.8}$$

The first two terms in square brackets in the integrand are independent of α and are the same terms found earlier[2] for the $\alpha = 0$ case. We would like to evaluate the integral of the third term in the $\varepsilon \to 0$ limit, but we cannot, of course, separate it from the first two terms and do the integrals separately because of the singularities in each of the three terms when $n = 0$, $k = 0$. We can, however, remove the singularities by subtracting the divergent small-k behavior from the first two terms and adding it back to the third term. Doing this, (3.8) becomes

$$
\begin{aligned}
V(x,r,\theta) &= \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2} \\
&- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx\left[\frac{K_n'}{I_n'}\, I_n(kr) I_n(kR) + \frac{2\delta_{0n}}{k^2}\right. \\
&+ \left.\frac{\varepsilon}{k^2}\left\{\frac{I_n(kr)I_n(kR)}{I_n'^{\,2}} + \left(1 - r^2 - R^2 - \frac{4}{k^2}\right)\delta_{0n}\right\}\right] \\
&+ \frac{1}{2\pi^2}\int_0^\infty dk\,\cos kx\left[\frac{\varepsilon^2 I_0}{k^2 I_1^2}\cdot\frac{I_0(kr)I_0(kR)}{\dfrac{kI_1}{1+\alpha\dfrac{I_1K_0}{I_0K_1}} + \varepsilon I_0} + \frac{2}{k^2} + \frac{\varepsilon}{k^2}\left(1-r^2-R^2-\frac{4}{k^2}\right)\right] \qquad (3.9) \\
&+ \cdots
\end{aligned}
$$

and all integrands in (3.9) are well behaved at $k = 0$.

The first two terms in (3.9), the inverse distance term and the infinite sum, appeared in identical form in Equations (5.2) and (5.3) of our earlier report[2] for the $\alpha = 0$ case. In that report it was demonstrated that for large x, the sum of these two terms was exponentially small. The rest of Equation (5.3), which survives for fixed $\sqrt{\varepsilon}\,x$ in the limit $\varepsilon \to 0$, was called the "far-field potential," and designated by $W(x^*,r)$, where $x^* = \sqrt{\varepsilon}\,x\,(1 - \varepsilon/8 + \ldots)$ is the far field longitudinal variable in the $\alpha = 0$ case. The last integral in (3.9) will similarly be the only survivor in this limit, and is therefore the generalization to arbitrary α of the earlier far-field result. Designating the last term in (3.9) by $U(x,r)$, we have

$$U(x,r) = \frac{1}{2\pi^2}\int_0^\infty dk\ \cos kx \left[\frac{\varepsilon^2 I_0}{k^2 I_1^2}\cdot\frac{I_0(kr)I_0(kR)}{\dfrac{kI_1}{1+\alpha\dfrac{I_1K_0}{I_0K_1}}+\varepsilon I_0} + \frac{2}{k^2} + \frac{\varepsilon}{k^2}\left(1-r^2-R^2-\frac{4}{k^2}\right)\right] \tag{3.10a}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\ \cos kx \left[\frac{4\varepsilon^2}{k^4}\cdot\frac{1+\frac{k^2}{4}(r^2+R^2)+\ldots}{\frac{k^2}{2}\left(1+\frac{k^2}{8}+\frac{\alpha k^2}{2}\left[\gamma+\log\frac{k}{2}\right]\right)+\varepsilon\left(1+\frac{k^2}{4}\right)+\ldots} + \frac{2}{k^2}+\frac{\varepsilon}{k^2}\left(1-r^2-R^2-\frac{4}{k^2}\right)\right] \tag{3.10b}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\ \cos kx \left[\frac{1+\varepsilon\left\{\frac{3}{4}-\frac{r^2+R^2}{2}-\alpha\left(\gamma+\log\frac{k}{2}\right)\right\}+\frac{k^2}{2}\left\{\frac{1}{4}+\alpha\left(\gamma+\log\frac{k}{2}\right)\right\}+\ldots}{\varepsilon\left(1+\frac{k^2}{4}\right)+\frac{k^2}{2}\left\{1+\frac{k^2}{8}+\alpha\frac{k^2}{2}\left(\gamma+\log\frac{k}{2}\right)\right\}+\ldots}\right] \tag{3.10c}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\ \cos kx \left[\frac{2+\varepsilon\left(1-r^2-R^2\right)}{k^2+2\varepsilon} + \frac{\varepsilon^2\left[1-4\alpha\left(\gamma+\log\frac{k}{2}\right)\right]}{\left(k^2+2\varepsilon\right)^2} + \ldots\right] \tag{3.10d}$$

In (3.10b) we have replaced the modified Bessel functions by their expansions around $k = 0$; γ is Euler's constant; in (3.10c) we have combined the three terms over a single denominator, and the asymptotic result of (3.10d) follows since in the $\varepsilon \to 0$ limit the integral comes increasingly from the vicinity of $k = 0$ and hence k^2 may be treated as a small quantity.

That (3.10d) is the asymptotic representation of (3.10c) in the $\varepsilon \to 0$ limit can be seen most easily by making the change of variable $k^2 = \varepsilon\,\zeta^2$, so that (3.10c) becomes

$$\frac{1}{2\pi^2\sqrt{\varepsilon}}\int_0^\infty d\zeta\,\cos(\zeta\sqrt{\varepsilon}\,x)\left[\frac{1+\varepsilon\left\{\frac{3}{4}-\frac{r^2+R^2}{2}-\alpha\left(\gamma+\frac{1}{2}\log\varepsilon+\log\frac{\zeta}{2}\right)\left(1-\frac{\zeta^2}{2}\right)+\frac{\zeta^2}{8}\right\}+O(\varepsilon^2)}{1+\frac{\zeta^2}{2}+\frac{\varepsilon\zeta^2}{4}\left\{1+\frac{\zeta^2}{4}+\alpha\zeta^2\left(\gamma+\frac{1}{2}\log\varepsilon+\log\frac{\zeta}{2}\right)\right\}+O(\varepsilon^2)}\right]$$

Expanding the expression in square brackets in ascending orders of ε and converting back to the original variable k, we obtain (3.10d).

As $k \to 0$, the denominator in (3.10c) approaches ε; it is $O(\varepsilon)$ if $k \lesssim \varepsilon^{1/2}$. Consequently, the integrand is large, $O(\varepsilon^{-1})$, if $k \lesssim \varepsilon^{1/2}$. In the $\varepsilon \to 0$ limit, the integrand becomes increasingly large in the vicinity of $k = 0$, but this behavior is confined to an increasingly narrow range of k. As a result the major contribution to the integral itself, in the $\varepsilon \to 0$ limit, comes from a range of k between zero and a small value which is $O(\varepsilon^{1/2})$. Replacing the expression in square brackets in (3.10a) by its expansion around $k = 0$ therefore leads to an asymptotic expansion for the integral. This is Laplace's method for obtaining an asymptotic representation of an integral.[5] From this discussion, it is clear that in making the expansion, k and $\varepsilon^{1/2}$ are to be treated as small quantities of the same order of smallness.

It is now possible to evaluate the integral in (3.10), using tabulated integrals. The result is

$$U(x,r) = \frac{2+\varepsilon(1-r^2-R^2)}{4\pi\sqrt{2\varepsilon}}\, e^{-x\sqrt{2\varepsilon}} + \frac{\alpha\sqrt{2\varepsilon}}{8\pi}\left[e^{-x\sqrt{2\varepsilon}}\left\{\left(\frac{1}{4\alpha} - \gamma - \frac{1}{2}\log\frac{\varepsilon}{2}\right)\left(1 + x\sqrt{2\varepsilon}\right) + 1\right\}\right.$$

$$\left. - \frac{1}{2}\left\{e^{x\sqrt{2\varepsilon}}\left(1-x\sqrt{2\varepsilon}\right)\mathrm{Ei}\left(-x\sqrt{2\varepsilon}\right) - e^{-x\sqrt{2\varepsilon}}\left(1+x\sqrt{2\varepsilon}\right)\mathrm{Ei}\left(x\sqrt{2\varepsilon}\right)\right\}\right] \qquad (3.11)$$

To obtain (3.11) we have used[6]

$$\int_0^\infty \frac{\cos bk}{\beta^2+k^2}\, dk = \frac{\pi}{2\beta}\, e^{-b\beta}$$

to obtain the first term and[7]

$$\int_0^\infty \log ak \cos bk \frac{\beta dk}{\beta^2+k^2} = \frac{\pi}{2}\, e^{-b\beta} \log a\beta + \frac{\pi}{4}\left[e^{b\beta}\,\mathrm{Ei}(-b\beta) - e^{-b\beta}\,\mathrm{Ei}(b\beta)\right]$$

to find the second term, where $\mathrm{Ei}(\xi)$ is the exponential integral. Differentiating β^{-1} times the last integral with respect to β, we obtain

$$\frac{d}{d\beta}\int_0^\infty \log ak \cos bk \frac{dk}{\beta^2+k^2} = -2\beta\int_0^\infty \log ak \cos bk \frac{dk}{\left(k^2+\beta^2\right)^2}$$

$$= \frac{d}{d\beta}\,\frac{1}{\beta}\left[\frac{\pi}{2}\, e^{-b\beta}\log a\beta + \frac{\pi}{4}\left\{e^{b\beta}\,\mathrm{Ei}(-b\beta) - e^{-b\beta}\,\mathrm{Ei}(b\beta)\right\}\right]$$

$$= -\frac{1}{\beta^2}\left[\frac{\pi}{2}\, e^{-b\beta}\log a\beta + \frac{\pi}{4}\left\{e^{b\beta}\,\mathrm{Ei}(-b\beta) - e^{-b\beta}\,\mathrm{Ei}(b\beta)\right\}\right]$$

$$+ \frac{1}{\beta}\left[\frac{\pi}{2}\left(-b\log a\beta + \frac{1}{\beta}\right)e^{-b\beta} + \frac{\pi}{4}\, b\left\{e^{b\beta}\,\mathrm{Ei}(-b\beta) + e^{-b\beta}\,\mathrm{Ei}(b\beta)\right\}\right]$$

so that

$$\int_0^\infty \log ak \cos bk \frac{dk}{\left(k^2+\beta^2\right)^2} = \frac{\pi}{4\beta^3}\left[e^{-b\beta}\left\{\log a\beta\,(1+b\beta) - 1\right\}\right.$$

$$\left. + \frac{1}{2}\left\{e^{b\beta}(1-b\beta)\,\mathrm{Ei}(-b\beta) - e^{-b\beta}(1+b\beta)\,\mathrm{Ei}(b\beta)\right\}\right]$$

Letting

$$\log a = -\frac{1}{4\alpha} + \gamma - \log 2$$

$$b = x$$

$$\beta^2 = 2\varepsilon$$

we obtain the last term in (3.11).

Regrouping the terms in (3.11) according to their order in ε, we have

$$U\left(\frac{\tilde{x}}{\sqrt{2\varepsilon}}, r\right) = \frac{1}{2\pi\sqrt{2\varepsilon}} e^{-\tilde{x}}$$

$$+ \frac{\sqrt{2\varepsilon}}{8\pi}\left[e^{-\tilde{x}}\left\{\frac{5}{4} - r^2 - R^2 - \alpha(\gamma-1) - \frac{\alpha}{2}\log\frac{\varepsilon}{2} + \tilde{x}\left(\frac{1}{4} - \alpha\gamma - \frac{\alpha}{2}\log\frac{\varepsilon}{2}\right)\right\}\right.$$

$$\left. - \frac{\alpha}{2} e^{\tilde{x}}(1-\tilde{x})\,\mathrm{Ei}(-\tilde{x}) + \frac{\alpha}{2} e^{-\tilde{x}}(1+\tilde{x})\,\mathrm{Ei}(\tilde{x})\right] + \ldots \qquad (3.12)$$

where we have defined a far field coordinate by $\tilde{x} = \sqrt{2\varepsilon}\, x$. It may be seen that for large $\tilde{x}$, (i.e., if the product $\tilde{x} \cdot \varepsilon$ is not small) the expansion (3.12) is <u>not</u> asymptotic because the second term $[O(\varepsilon^{1/2})]$ is <u>not</u> small compared to the first term $[O(\varepsilon^{-1/2})]$. This nonuniformity for large $\tilde{x}$ can be removed by making a change of variable to

$$X = \tilde{x}\,(1 + \eta(\varepsilon) + \ldots)$$

$$\tilde{x} = X\,(1 - \eta(\varepsilon) + \ldots) \qquad (3.13)$$

where η will be chosen to make (3.12) uniform in X. Expanding the exponential integrals in a Taylor series around $\pm$ X, we have

$$\mathrm{Ei}(\pm\tilde{x}) = \mathrm{Ei}\big(\pm X(1-\eta+\ldots)\big) = \mathrm{Ei}(\pm X) - \eta e^{\pm X} + \ldots \qquad (3.14)$$

and doing the same for the exponentials in (3.12), we have

$$e^{\pm\tilde{x}} = e^{\pm X(1-\eta+\ldots)} = e^{\pm X}(1 \pm X\eta + \ldots) \qquad (3.15)$$

Substituting (3.14) and (3.15) in (3.12), retaining terms up to and including $O(\varepsilon^{1/2})$, and defining the far field potential W in terms of the far field axial variable X we obtain

$$U(\varepsilon^{-1/2}\, X(1-\eta+\ldots),\ r)$$

$$\equiv W(X,r) = \frac{1}{2\pi\sqrt{2\varepsilon}}\ e^{-X}\left(1+X\eta\right) + \frac{\sqrt{2\varepsilon}}{8\pi}\left[e^{-X}\left\{\frac{5}{4} - r^2 - R^2 - \alpha(\gamma-1) - \frac{\alpha}{2}\log\frac{\varepsilon}{2}\right.\right.$$

$$\left. + X\left(\frac{1}{4} - \alpha\gamma - \frac{\alpha}{2}\log\frac{\varepsilon}{2}\right)\right\}$$

$$- \frac{\alpha}{2}\, e^{X}(1-X)\ \mathrm{Ei}(-X)$$

$$\left. + \frac{\alpha}{2}\, e^{-X}(1+X)\ \mathrm{Ei}(X)\right] \qquad (3.16)$$

where we have used the first two terms of (3.15) in the $O(\varepsilon^{-1/2})$ term of (3.12) but only require the leading term of (3.14) and (3.15) in the $O(\varepsilon^{1/2})$ term of (3.12). In order to make the coefficient of $X\,e^{-X}$ vanish in (3.16), we choose

$$\eta(\varepsilon) = -\varepsilon\left(\frac{1}{8} - \frac{\alpha\gamma}{2} - \frac{\alpha}{4}\log\frac{\varepsilon}{2}\right) + \ldots$$

and, substituting in (3.13), X is given by

$$X = \sqrt{2\varepsilon}\ x\left[1 - \varepsilon\left(\frac{1}{8} - \frac{\alpha\gamma}{2} - \frac{\alpha}{4}\log\frac{\varepsilon}{2}\right) + \ldots\right] \qquad (3.17)$$

and consequently W becomes

$$W = \frac{1}{2\pi\sqrt{2\varepsilon}}\ e^{-X} + \frac{\sqrt{2\varepsilon}}{8\pi}\left[e^{-X}\left\{\frac{5}{4} - r^2 - R^2 - \alpha(\gamma-1) - \frac{\alpha}{2}\log\frac{\varepsilon}{2}\right\}\right.$$

$$\left. - \frac{\alpha}{2}\, e^{X}(1-X)\ \mathrm{Ei}(-X) + \frac{\alpha}{2}\, e^{-X}(1+X)\ \mathrm{Ei}(X)\right]$$

$$+ \ldots \qquad (3.18)$$

If we let $\alpha = 0$ in (3.17) we obtain $X = \sqrt{2\varepsilon}\ x\ (1 - \varepsilon/8 + \ldots) = \sqrt{2}\ x^*$, where x^* is the far field variable used in the report[2] discussing the $\alpha = 0$ case. Our present variable X incorporates the $\sqrt{2}$ factor and therefore simplifies the formula (3.18) for the potential. Comparing with formula (5.1) of Reference 2, we see that (3.17) and (3.18) reduce to that formula when $\alpha = 0$. A second, and perhaps more important reason for redefining the far field variable is that $\sqrt{2\varepsilon}\,x$, the leading term in X, is a variable often used in the physiological literature. Thus X is the direct generalization of the common physiological variable to higher orders of ε.

Using the asymptotic formula for the exponential integral

$$\mathrm{Ei}(X) \sim \frac{e^X}{X}\left(1 + \frac{1}{X} + \frac{2!}{X^2} + \frac{3!}{X^3} + \ldots\right)$$

the combination of Ei (X) and Ei (-X) in (3.18) is, for large X:

$$e^X\,(1-X)\,\mathrm{Ei}(-X) - e^{-X}\,(1+X)\,\mathrm{Ei}(X) \sim -\frac{2}{X}\left(2 + \frac{2!+3!}{X^2} + \frac{4!+5!}{X^4} + \ldots\right) \qquad (3.19)$$

Thus, for sufficiently large X, the far field potential W will be dominated by the exponential integral expression in the $O(\varepsilon^{1/2})$ term, rather than by the leading $O(\varepsilon^{-1/2})$ term. This is not a nonuniformity in X, in the same sense that the form (3.12) is nonuniform in $\tilde{x}$. It merely means that the far field W is composed of two parts: an ordered sequence of exponential terms, and an ordered sequence of exponential integral terms. The former begin with $O(\varepsilon^{-1/2})$, the latter with $O(\varepsilon^{1/2})$. If we continued the expansion (3.18) to higher order, we would expect the exponential integral terms to be of increasing order in ε, uniformly in X. Thus, each of the two ordered sequences is uniform in X but one depends on X exponentially, the other as an exponential integral.

For X of O(1), the far field decays exponentially. However, if

$$\frac{X}{\alpha}\,e^{-X} << \varepsilon \qquad (3.20)$$

the exponential integral terms dominate and according to (3.19) the decay becomes algebraic, inversely proportional to the axial distance variable X.

In a typical physiological case, say $\varepsilon = 10^{-3}$, $\alpha = 0.3$, the two sides of (3.20) are equal for $X \cong 10$. For $X = 5$ the left hand side is 100 times the right hand side. At this point the potential (3.18) has decreased to 7×10^{-3} times its $X = 0$ maximum. Consequently, at any position where the exponential integral is of significant magnitude relative to the exponential, the total potential is so small as to be of no experimental significance, and we conclude that the exponential integral terms are of no practical importance in the interior of the cell for typical experimental situations. We will see below that the situation is somewhat different outside the cell, where there is no $O(\varepsilon^{-1/2})$ term, and the exponential integral can be the dominant term. In addition, in Section V we will discuss the sinusoidal steady state, in which case a complex "effective" ε can be defined which is a function of frequency, and whose magnitude can be much greater than 10^{-3}. In this case the exponential integral can be important inside the cell.

Using the definition (3.16) of W and (3.10) of U, we can substitute the expression (3.18) for W in (3.9) to obtain an asymptotic representation of the potential which is valid for all x. It is

$$\begin{aligned}
V(x,r,\theta) &= \frac{1}{2\pi\sqrt{2\varepsilon}}\, e^{-X} + \frac{1}{4\pi}\,(x^2+r^2+R^2-2rR\cos\theta)^{-1/2} \\
&\quad - \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx \left[\frac{K_n'}{I_n'}\, I_n(kr) I_n(kR) + \frac{2\delta_{0n}}{k^2}\right] \\
&\quad - \frac{\alpha\sqrt{2\varepsilon}\,\log\varepsilon}{16\pi}\, e^{-X} \\
&\quad + \frac{\sqrt{2\varepsilon}}{8\pi}\left[e^{-X}\left\{\frac{5}{4} - r^2 - R^2 - \alpha(\gamma-1) + \frac{\alpha}{2}\log 2\right\} - \frac{\alpha}{2}\, e^{X}(1-X)\,\mathrm{Ei}(-X)\right. \\
&\quad \left. + \frac{\alpha}{2}\, e^{-X}(1+X)\,\mathrm{Ei}(X)\right] \\
&\quad - \frac{\varepsilon}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\frac{\cos kx}{k^2}\left[\frac{I_n(kr) I_n(kR)}{I_n'^{\,2}} + \left(1-r^2-R^2-\frac{4}{k^2}\right)\delta_{0n}\right] + \cdots
\end{aligned} \tag{3.21}$$

with X related to x by (3.17). In (3.21) we have arranged the terms in ascending order of ε [viz., $O(\varepsilon^{-1/2})$, $O(1)$, $O(\varepsilon^{1/2} \log \varepsilon)$, $O(\varepsilon^{1/2})$, $O(\varepsilon)$,...]. This is the generalization to arbitrary α of Equation (5.3) of Reference 2. It is noteworthy that the leading, $O(\varepsilon^{-1/2})$, term, as well as each term whose order is an integer power of ε, are independent of α. There is no effect of the external medium conductivity until the third term in the expansion.

The two terms in (3.21) which are infinite sums of inverse Fourier transforms are identical to the corresponding terms in Equation (5.3) of Reference 2. The sum in the O(1) terms can be computed most efficiently for $x > 0.5$ by converting the integral to the sum given in Equation (4.31) of Reference 2, or, for $x < 0.5$, to the more complicated sums given in Equation (D.14) of Reference 2. The sum in the $O(\varepsilon)$ term can be computed, for any x, by converting the integrals into the sums given in Equation (C.9) of Reference 2.

We would now like to obtain an asymptotic representation of the inside potential which is useful in the "near field." More precisely, we seek an expansion arranged in ascending orders of ε when x is held fixed as $\varepsilon \to 0$ [rather than holding X fixed as $\varepsilon \to 0$, as was done in (3.18)]. Using (3.17) to express X, the far field variable, in terms of x, the near field variable, the Taylor expansion of the exponential yields

$$e^{-X} = 1 - \sqrt{2\varepsilon}\, x \left[1 - \varepsilon\left(\frac{1}{8} - \frac{\alpha\gamma}{2} - \frac{\alpha}{4} \log \frac{\varepsilon}{2}\right)\right] + \varepsilon x^2 + (2\varepsilon)^{3/2} \frac{x^3}{6} + O(\varepsilon^2 \log \varepsilon)$$

and the Taylor expansion of the exponential integral yields

$$e^{-X}(1+X)\,\mathrm{Ei}(X) - e^{X}(1-X)\,\mathrm{Ei}(-X) = 2\sqrt{2\varepsilon}\, x + O(\varepsilon^{3/2} \log \varepsilon).$$

Substituting these expansions in (3.21) and arranging the result in ascending orders of ε, the near field expansion of the inside potential is

$$V(x,r,\theta) = \frac{1}{2\pi\sqrt{2\varepsilon}}$$

$$- \frac{x}{2\pi} + \frac{1}{4\pi} (x^2 + r^2 + R^2 - 2rR\cos\theta)^{-1/2}$$

$$- \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{\infty} dk \cos kx \cdot \left[\frac{K_n'}{I_n'} I_n(kr) I_n(kR) + \frac{2\delta_{0n}}{k^2} \right]$$

$$- \frac{\alpha\sqrt{2\varepsilon} \log \varepsilon}{16\pi}$$

$$+ \frac{\sqrt{2\varepsilon}}{8\pi} \left\{ \frac{5}{4} - r^2 - R^2 + 2x^2 - \alpha(\gamma - 1 - \frac{1}{2}\log 2) \right\}$$

$$- \frac{\varepsilon x}{4\pi} \left(1 - r^2 - R^2 + \frac{2}{3} x^2 \right) - \frac{\varepsilon}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta}$$

$$\cdot \int_0^{\infty} dk \cdot \frac{\cos kx}{k^2} \cdot \left[\frac{I_n(kr) I_n(kR)}{I_n'^2} + \left(1 - r^2 - R^2 - \frac{4}{k^2} \right) \delta_{0n} \right]$$

$$+ \ldots \qquad (3.22)$$

Equation (3.22) is the generalization of Equation (5.2) of Reference 2 to arbitrary α. As in that case, it is not uniform in x, but fails to be asymptotic for large x.

B. Potential Outside the Cell

We now return to Equation (3.2) for the outside potential and proceed to obtain an asymptotic expansion in powers of ε, using the same approach as was just applied to Equation (3.1). Separating the integrand into its $\varepsilon \to 0$ limit plus the remainder [see the equation preceding (3.6)], we obtain from (3.2), for the potential in the region $1 < r < \infty$,

$$V(x,r,\theta) = - \frac{\varepsilon\alpha}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{\infty} \frac{dk}{k^2} \cos kx \cdot \frac{K_n(kr) I_n(kR)}{I_n'(k) K_n'(k)} \cdot \left[1 + \frac{\varepsilon}{\frac{k I_n' K_n'}{\alpha I_n' K_n - I_n K_n'} - \varepsilon} \right] \qquad (3.23)$$

As we have found above when considering the corresponding terms in (3.6) - (3.8), the second term in the square brackets in the (3.23) integrand is $O(\varepsilon)$ unless $n = 0$, $k^2 \lesssim \varepsilon$, in which case it is $O(1)$ and must be included in our calculation of V. For $n = 0$, the integrand in (3.23) is

$$-\frac{\cos kx}{k^2}\cdot\frac{K_0(kr)I_0(kR)}{I_1(k)K_1(k)}\cdot\left[1-\frac{\varepsilon}{\dfrac{kI_1K_1}{\alpha I_1K_0+I_0K_1}+\varepsilon}\right]$$

$$=-\frac{\cos kx}{k^2}\cdot\frac{K_0(kr)I_0(kR)}{I_1(k)K_1(k)}\cdot\left[1-\frac{\varepsilon I_0}{\varepsilon I_0+\dfrac{kI_1}{1+\alpha\dfrac{I_1K_0}{I_0K_1}}}\right]$$

$$=-\frac{\cos kx}{k^2}\left[\frac{K_0(kr)I_0(kR)}{I_1(k)K_1(k)}+\frac{2\varepsilon\left(\gamma+\log\frac{kr}{2}\right)}{\varepsilon+\frac{k^2}{2}}+\ldots\right] \tag{3.24}$$

where we have expanded the Bessel functions around $k = 0$ and deleted terms of higher order than $O(\varepsilon)$ or $O(k^2)$ to obtain the last term in the square brackets. Note that since there is an overall factor of ε multiplying (3.2), we need to retain one less order of ε or k^2 than we did for the corresponding step for dealing with the potential inside the cell. Substituting (3.24) in (3.23), we obtain

$$V(x,r,\theta) = -\frac{\varepsilon\alpha}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty \frac{dk}{k^2}\cos kx\left[\frac{K_n(kr)I_n(kR)}{K_n'(k)I_n'(k)}-\frac{2\varepsilon\left(\gamma+\log\frac{kr}{2}\right)\delta_{n0}}{\frac{k^2}{2}+\varepsilon}+\ldots\right] \tag{3.25}$$

Once again, we wish to separate this integral into two parts but cannot without first removing the singularities at $k = 0$. Doing this, we obtain

$$V(x,r,\theta) = -\frac{\varepsilon\alpha}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty \frac{dk}{k^2} \cos kx \left[\frac{K_n(kr)I_n(kR)}{I_n'(k)K_n'(k)} - 2\left(\gamma + \log \frac{kr}{2}\right) \delta_{n0} \right]$$

$$- \frac{\varepsilon\alpha}{\pi^2} \int_0^\infty dk \cos kx \frac{\gamma + \log \frac{kr}{2}}{k^2 + 2\varepsilon} + \ldots \tag{3.26}$$

With this separation, the first term is clearly $O(\varepsilon)$ since all ε-dependence has been removed from the integrand. The second term, however, is $O(\varepsilon^{1/2})$ as will be shown below. The term log $(kr/2)$, which originates from the expansion of K_0 (kr) in (3.24) restricts the validity of our asymptotic expansion to values of r which are not too large. In order for the first term of the expansion to approximate K_0 (kr) adequately, the product $kr/2$ must be small. Since the last integral in (3.26) originates mostly from the range $0 < k \lesssim \varepsilon^{1/2}$ (the integrand being negligible for $k >> \varepsilon^{1/2}$), if r satisfies the condition

$$1 \leq r << \varepsilon^{-1/2},$$

$kr/2$ will be small for values of k contributing significantly to the integrand. Our asymptotic expansion will only be valid for this restricted range of r, which is adequate for essentially all experimental situations. Letting

$$\log a = \gamma + \log \frac{r}{2}$$

$$2\varepsilon = \beta^2$$

in the tabulated integral[7] used to obtain (3.11), the last integral in (3.26) can be evaluated:

$$\int_0^\infty dk \cos kx \cdot \frac{\gamma + \log \frac{kr}{2}}{k^2 + 2\varepsilon}$$

$$= \frac{\pi}{4\sqrt{2\varepsilon}} \left[2e^{-\sqrt{2\varepsilon}\,x} \left(\gamma + \log \sqrt{\frac{\varepsilon}{2}}\, r\right) + e^{\sqrt{2\varepsilon}\,x}\, \mathrm{Ei}\left(-\sqrt{2\varepsilon}\,x\right) - e^{-\sqrt{2\varepsilon}\,x}\, \mathrm{Ei}\left(\sqrt{2\varepsilon}\,x\right) \right] \tag{3.27}$$

Substituting (3.27) in (3.26), replacing $\sqrt{2\varepsilon}\,x$ with X, which as a consequence of (3.14), (3.15), and (3.17) is correct to $O(\varepsilon)$, we obtain for the outside potential the asymptotic representation uniform in x, for the radial variable in the range $1 \leq r << \varepsilon^{-1/2}$,

$$V(x,r,\theta) = -\frac{\alpha}{4\pi}\sqrt{\frac{\varepsilon}{2}}\left[2e^{-X}\left(\gamma + \log\sqrt{\frac{\varepsilon}{2}}\,r\right) + e^{X}Ei(-X) - e^{-X}Ei(X)\right]$$
$$-\frac{\varepsilon\alpha}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty}\frac{dk}{k^2}\cos kx\left[\frac{K_n(kr)I_n(kR)}{I_n'(k)K_n'(k)} - 2\left(\gamma + \log\frac{kr}{2}\right)\delta_{n0}\right]$$
$$+ \ldots \tag{3.28}$$

Because of the absence from (3.28) of a term of $O(\varepsilon^{-1/2})$, as was present in (3.18) for the inside potential, the exponential integral is relatively more important in the expansion of the outside potential. In the first line of (3.28), the exponential integral terms dominate the exponential for values of X greater than about 1.5 if $\varepsilon \cong 10^{-3}$. It should be remembered that since the $O(\varepsilon^{-1/2})$ and $O(1)$ terms are missing from (3.28), the outside potential is much smaller than the inside potential. Compared to the <u>inside</u> potential, the exponential integral terms would again be less important.

Substituting $\sqrt{2\varepsilon}\,x$ for X in (3.28) and expanding in the limit $\varepsilon \to 0$, we have

$$e^{-X} = 1 - \sqrt{2\varepsilon}\,x + \ldots$$

$$e^{X}\,Ei(-X) - e^{-X}\,Ei(X) = 2X\log X - 2(1-\gamma)\,X + \ldots$$
$$= \sqrt{2\varepsilon}\,x\log\varepsilon + 2\sqrt{2\varepsilon}\,x\,\left(\log x + \frac{1}{2}\log 2 + 1 - \gamma\right) + \ldots$$

which, when substituted in (3.28) leads to the near field expansion of the outside potential,

$$
\begin{aligned}
V(x,r,\theta) = &- \frac{\alpha}{8\pi}\sqrt{2\varepsilon}\log\varepsilon \\
&- \frac{\alpha}{4\pi}\sqrt{2\varepsilon}\left(\gamma + \frac{1}{2}\log\frac{r^2}{2}\right) \\
&- \frac{\alpha x\varepsilon}{2\pi}\left(1 - 2\gamma + \log\frac{2x}{r}\right) - \frac{\alpha\varepsilon}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk \cdot \frac{\cos kx}{k^2} \\
&\qquad\qquad \cdot\left[\frac{K_n(kr)I_n(kR)}{I_n'(k)K_n'(k)} - 2\left(\gamma + \log\frac{kr}{2}\right)\delta_{n0}\right] \\
&+ \ldots
\end{aligned}
\tag{3.29}
$$

IV. RELATION OF THE STEADY STATE EXPANSION TO A SINGULAR PERTURBATION ANALYSIS

A. Far Field Potential

It is interesting to see how the present problem might be approached using singular perturbation theory. The problem specified by (2.1) - (2.4) can be decomposed into a sequence of problems each with physical meaning, corresponding to a single term in the expansion of the potential in orders of ε. It will then be seen that each term in the expansion obtained in Section III is the solution to one of these physical problems.

We consider the far field limit first. The singular perturbation analysis begins by expanding the far field potential in orders of ε, the particular orders of ε being chosen here because they appear in the exact solution. In an *a priori* analysis using singular perturbation techniques, it would be necessary to show that this is the appropriate expansion and to consider the physical origin of the log ε terms. Using the knowledge we have already gained from the exact solution, however, we can write the expansion:

$$W(X,r) = \varepsilon^{-1/2} W_0(X,r) + \varepsilon^{1/2} \log\varepsilon\, W_{1/2}(X,r) + \varepsilon^{1/2} W_1(X,r) + \varepsilon^{3/2} \log\varepsilon\, W_{3/2}(X,r) + \varepsilon^{3/2} W_2(X,r) + \ldots \tag{4.1}$$

The terms with integer subscripts correspond to the notation introduced in Reference 2 for the far field potential. The intermediate terms containing a log ε were not present in the $\alpha = 0$ case of Reference 2 and we use the half-odd integer subscript to indicate between which two integer terms the term falls. The far field coordinate X is related to the coordinate x by the transformation

$$X = \eta(\varepsilon)x$$

where $\eta(\varepsilon)$ is given below in (4.8).

The Equation (2.1) and the boundary condition (2.2) can then be broken down into the sequence of equations and boundary conditions

$$\begin{cases} \nabla_t^2 \, W_0 = 0 \\[2ex] \dfrac{\partial W_0^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial W_0^+}{\partial r} = 0 \end{cases} \tag{4.3}$$

$$\begin{cases} \nabla_t^2 \, W_{1/2} = 0 \\[2ex] \dfrac{\partial W_{1/2}^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial W_{1/2}^+}{\partial r} = 0 \end{cases} \tag{4.4}$$

$$\begin{cases} \nabla_t^2 \, W_1 = -2 \dfrac{\partial^2 W_0}{\partial X^2} \\[2ex] \dfrac{\partial W_1^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial W_1^+}{\partial r} = W_0^+ - W_0^- \end{cases} \tag{4.5}$$

$$\begin{cases} \nabla_t^2 \, W_{3/2} = -4\alpha_{1/2} \dfrac{\partial^2 W_0}{\partial X^2} - 2 \dfrac{\partial^2 W_{1/2}}{\partial X^2} \\[2ex] \partial W_{3/2}^- = \dfrac{1}{\alpha} \dfrac{\partial W_{3/2}^+}{\partial r} = W_{1/2}^+ - W_{1/2}^- \end{cases} \tag{4.6}$$

$$\begin{cases} \nabla_t^2 \, W_2 = -\, 2 \dfrac{\partial^2 W_1}{\partial X^2} - 4\alpha_1 \dfrac{\partial^2 W_0}{\partial X^2} \\[2ex] \dfrac{\partial W_2^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial W_2^+}{\partial r} = W_1^+ - W_1^- \end{cases} \tag{4.7}$$

where ∇_t^2 denotes the transverse Laplacian and $\eta(\varepsilon)$ in (4.2) has been taken equal to

$$\eta(\varepsilon) = \sqrt{2\varepsilon}\left(1 + \alpha_{1/2}\, \varepsilon \log \varepsilon + \alpha_1 \varepsilon + \ldots\right). \tag{4.8}$$

If the solutions (3.18) and (3.28) are used for the far field potential inside and outside the cell, respectively, and are decomposed in the manner indicated in (4.1), the result inside the cell is:

$$W_0 = \frac{\sqrt{2}}{4\pi} e^{-X} \tag{4.9}$$

$$W_{1/2} = -\frac{\alpha\sqrt{2}}{16\pi} e^{-X} \tag{4.10}$$

$$W_1 = \frac{\sqrt{2}}{8\pi}\left[\left\{\frac{5}{4} - r^2 - R^2 - \alpha\left(\gamma - 1 - \frac{1}{2}\log 2\right)\right\} e^{-X} - \frac{\alpha}{2} e^{X} (1-X)\ \mathrm{Ei}(-X) + \frac{\alpha}{2} e^{-X} (1+X)\ \mathrm{Ei}(X)\right] \tag{4.11}$$

and, outside the cell:

$$W_0 = 0 \tag{4.12}$$

$$W_{1/2} = -\frac{\alpha\sqrt{2}}{8\pi} e^{-X}$$

$$W_1 = -\frac{\alpha\sqrt{2}}{8\pi}\left[\left(2\gamma - \log 2 + 2\log r\right) e^{-X} + e^{X}\ \mathrm{Ei}(-X) - e^{-X}\ \mathrm{Ei}(X)\right] \tag{4.14}$$

By direct substitution of (4.9) - (4.14), it may be verified that these expressions are solutions of the problems (4.3) - (4.5). In order to verify that (4.6) and (4.7) are satisfied [with the coefficients $\alpha_{1/2}$ and α_1 given in (3.17)], one would have to obtain $W_{3/2}$ and W_2.

In a self-contained singular perturbation analysis, the far field potentials (4.9) - (4.14) could be found only after matching to the near field and repeated use of the divergence theorem as was done in Reference 2. Here we are just verifying that the solutions obtained in Section III satisfy the singular perturbation problems.

B. Physical Interpretation of the Far Field

The boundary conditions in (4.3) and (4.4) are homogeneous Neumann conditions on both sides of the membrane. Consequently to $O(\varepsilon^{-1/2})$ and $O(\varepsilon^{1/2} \log \varepsilon)$, no current crosses the membrane. The solutions (4.9), (4.10), (4.12) and (4.13) for W_0 and $W_{1/2}$ indicate current flow only in the longitudinal direction. The potential W_0 inside the cell has simple exponential decay in the longitudinal direction. This is the classical result of one-dimensional cable theory, except for the higher order corrections in the spatial variable X. To lowest order in ε, however, X is equal to $\sqrt{2\varepsilon}\, x$, and to this order, W_0 is precisely the classical result.

To lowest order, $O(\varepsilon^{-1/2})$, the outside potential W_0, is zero so that the inside of the cell is raised to a large potential relative to the outside. The potential difference across the membrane in the $O(\varepsilon^{-1/2})$ term is the driving force, $W_0^+ - W_0^-$ which forces a current to flow across the membrane in the problem (4.5) for the $O(\varepsilon^{1/2})$ term.

Substituting the potential W_0 from (4.9) and (4.12) in the boundary condition (4.5) results in the boundary condition for W_1,

$$\frac{\partial W_1^-}{\partial r} = \frac{1}{\alpha} \frac{\partial W_1^+}{\partial r} = - \frac{\sqrt{2}}{4\pi}\, e^{-X}$$

In order to satisfy this boundary condition, which requires current flow across the membrane, an r-dependence appears in the potential W_1 given by (4.11) and (4.14).

Continuing the process further, it is seen that for a higher order component of the potential, W_ν, the driving force is the transmembrane potential, $W_{\nu-1}^+ - W_{\nu-1}^-$, two orders earlier.

In the far field potential expansion (4.1) and (4.9) - (4.14), the leading term, W_0, is independent of α. The α dependence first appears in the second, $W_{1/2}$, term. The transmembrane potential $W^+_{1/2} - W^-_{1/2}$, also has an α dependence which first appears in this order.

It should be noticed that there is a jump by a factor of 2 in the value of $W_{1/2}$ upon crossing the membrane. This factor of 2 might seem to be an error and so we have carefully checked this result physically and mathematically. There seems to be no physical reason why this jump should occur; however, there is also no physical reason why $W_{1/2}$ should be continuous. Indeed, if $W_{1/2}$ were continuous, there would be no membrane current in the $W_{3/2}$ problem, which seems unlikely.

The mathematical derivation of the $W_{1/2}$ term shows the origin of the jump in $W_{1/2}$ at the membrane. Although $W_{1/2}$ appears separately in (4.13), it originates from the same term as does W_1 [see steps leading to (3.28)], and W_1 in (4.14) could not be changed by a factor of 2 without violating the boundary condition of continuity of current in (4.5). Finally, if we examine the other terms we see that the jumps in the values of W_0 or W_1 across the membrane are much more significant than the jump in $W_{1/2}$. If we are content with the jump in the W_0 and W_1, terms, perhaps we should also be content with the jump in $W_{1/2}$.

C. Near Field Potential

We now consider the singular perturbation analysis of the near field. We will show that the individual terms in the expansions of the potential in the near field, (3.22) and (3.29), satisfy the equations and boundary conditions in a sequence of near field problems.

Expanding the near field in powers of ε, we write

$$V(x,r,\theta) = \varepsilon^{-1/2} V_0(x,r,\theta) + V_1(x,r,\theta) + \varepsilon^{1/2} \log \varepsilon \, V_{3/2}(x,r,\theta) + \varepsilon^{1/2} V_2(x,r,\theta) + \varepsilon V_3(x,r,\theta) + \ldots \tag{4.15}$$

Once again, the orders of ε which appear in the expansion (4.15) are motivated by our prior knowledge of the expansion of the exact solution.

The sequence of equations and boundary conditions is

$$\begin{cases} \nabla^2 V_0 = 0 \\ \dfrac{\partial V_0^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial V_0^+}{\partial r} = 0 \\ V_0 \to \dfrac{1}{2\pi\sqrt{2\varepsilon}} \text{ as } x \to \infty, \ r < 1 \\ V_0 \to 0 \text{ as } x \to \infty, \ r > 1 \end{cases} \tag{4.16}$$

$$\begin{cases} \nabla^2 V_1 = -\dfrac{1}{r} \delta(x) \, \delta(r-R) \, \delta(\theta) \\ \dfrac{\partial V_1^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial V_1^+}{\partial r} = 0 \\ V_1 \to -\dfrac{x}{2\pi} \text{ as } x \to \infty, \ r < 1 \\ V_1 \to 0 \text{ as } x \to \infty, \ r > 1 \end{cases} \tag{4.17}$$

$$\begin{cases} \nabla^2 V_{3/2} = 0 \\ \dfrac{\partial V_{3/2}^-}{\partial r} = \dfrac{1}{\alpha} \dfrac{\partial V_{3/2}^+}{\partial r} = 0 \\ V_{3/2} \to -\dfrac{\alpha\sqrt{2}}{16\pi} \text{ as } x \to \infty, \ r < 1 \\ V_{3/2} \to -\dfrac{\alpha\sqrt{2}}{8\pi} \text{ as } x \to \infty, \ r > 1 \end{cases} \tag{4.18}$$

$$\begin{cases} \nabla^2 v_2 = 0 \\ \dfrac{\partial v_2^-}{\partial r} = \dfrac{1}{\alpha}\dfrac{\partial v_2^+}{\partial r} = v_0^+ - v_0^- \\ v_2 \to \dfrac{\sqrt{2}}{8\pi}\left(\dfrac{5}{4} - r^2 - R^2 + 2x^2 - \alpha[\gamma - 1 - \frac{1}{2}\log 2]\right) \text{ as } x \to \infty,\ r < 1 \\ v_2 \to -\dfrac{\alpha\sqrt{2}}{4\pi}\left(\gamma + \dfrac{1}{2}\log\dfrac{r^2}{2}\right) \text{ as } x \to \infty,\ r > 1 \end{cases} \tag{4.19}$$

$$\begin{cases} \nabla^2 v_3 = 0 \\ \dfrac{\partial v_3^-}{\partial r} = \dfrac{1}{\alpha}\dfrac{\partial v_3^+}{\partial r} = v_1^+ - v_1^- \\ v_3 \to \dfrac{x}{4\pi}\left(r^2 + R^2 - 1 - \dfrac{2}{3}x^2\right) \text{ as } x \to \infty,\ r < 1 \\ v_3 \to \dfrac{\alpha x}{2\pi}\left(2\gamma - 1 - \log\dfrac{2x}{r}\right) \text{ as } x \to \infty,\ r > 1 \end{cases} \tag{4.20}$$

The boundary conditions at $x \to \infty$ in (4.16) - (4.20) are made to conform to the known solutions (3.22) and (3.29), or, in a self-contained singular perturbation analysis, would be found by matching the near field to the far field (4.9)- (4.14).

Decomposing the near field potential (3.22) and (3.29) according to the prescription of (4.15) yields, <u>inside</u> the cell:

$$v_0 = \frac{\sqrt{2}}{4\pi} \tag{4.21}$$

$$v_1 = -\frac{x}{2\pi} + \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2} - \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk\,\cos kx \cdot \left[\frac{K_n'}{I_n'}I_n(kr)I_n(kR) + \frac{2\delta_{0n}}{k^2}\right] \tag{4.22}$$

$$v_{3/2} = -\frac{\alpha\sqrt{2}}{16\pi} \tag{4.23}$$

$$V_2 = \frac{\sqrt{2}}{8\pi}\left[\frac{5}{4} - r^2 - R^2 + 2x^2 - \alpha\left(\gamma - 1 - \frac{1}{2}\log 2\right)\right] \tag{4.24}$$

$$V_3 = -\frac{x}{4\pi}\left(1 - r^2 - R^2 + \frac{2}{3}x^2\right) - \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk \cos kx \cdot \frac{1}{k^2} \cdot \left[\frac{I_n(kr)K_n(kR)}{I_n'^2} + \left(1 - r^2 - R^2 - \frac{4}{k^2}\right)\delta_{0n}\right] \tag{4.25}$$

and outside the cell:

$$V_0 = 0 \tag{4.26}$$

$$V_1 = 0 \tag{4.27}$$

$$V_{3/2} = -\frac{\alpha\sqrt{2}}{8\pi} \tag{4.28}$$

$$V_2 = \frac{\alpha\sqrt{2}}{4\pi}\left(\gamma + \frac{1}{2}\log\frac{r^2}{2}\right) \tag{4.29}$$

$$V_3 = -\frac{\alpha x}{2\pi}\left(1 - 2\gamma + \log\frac{2x}{r}\right) - \frac{\alpha}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\,\frac{\cos kx}{k^2} \cdot \left[\frac{K_n(kr)I_n(kR)}{I_n'K_n'} - 2\left(\gamma + \log\frac{kr}{2}\right)\delta_{n0}\right] \tag{4.30}$$

Direct substitution of the potentials (4.21) - (4.30) in the sequence of problems (4.16) - (4.20) demonstrates that the potentials do indeed satisfy the respective problems.

D. Physical Interpretation of the Near Field

It may be seen from (4.21) - (4.30) that the first two orders of the solution, $O(\epsilon^{1/2})$ in (4.21) and (4.26) and $O(1)$ in (4.22) and (4.27), are independent of α, the ratio of conductivities. The results are therefore identical to the $\alpha = 0$ results of Reference 2, up to $O(1)$. To this order, the cylindrical cell behaves precisely as if the outside medium were a perfect conductor; that is to $O(1)$, the entire outside surface of the membrane is at zero potential. The physical interpretation of the first two problems, (4.16) for V_0 and (4.17) for V_1, is therefore identical to that in Reference 2 for the $\alpha = 0$ case.

The $O(\varepsilon^{-1/2})$ problem (4.16) for V_0 contains no source and has no current crossing the membrane. The solution, (4.21) and (4.26), is a constant inside and zero potential outside the cell. There is no current flow in this order.

The O(1) problem for V_1, (4.17), contains the source, but again, the boundary condition at $r = 1$ requires no current crossing the membrane. Since there is a source, but no current leaves the cell, all the current must flow toward $x = \pm \infty$. Consequently, there must be a term in the potential (4.22) inside the cell which decreases linearly with increasing x. Since no current flows outside the cell, the potential (4.27) outside must be zero.

The $O(\varepsilon^{1/2} \log \varepsilon)$ problem (4.18) for $V_{3/2}$, is similar to the problem for V_0. There is no source, no current crossing the membrane, and the potential approaches a constant as $x \to \pm \infty$ for both $r > 1$ and $r < 1$. The solution, (4.23) and (4.28), is a constant potential inside, and a different constant potential (twice as large) outside.

In all subsequent problems there is no source, but there is a current crossing the membrane equal to the transmembrane potential in an earlier problem. In the $O(\varepsilon^{1/2})$ problem (4.19) for V_2, the current crossing the membrane is the transmembrane potential in the $O(\varepsilon^{-1/2})$ problem. Substituting (4.21) and (4.26) in the $r = 1$ boundary condition in (4.19), the boundary condition becomes

$$\frac{\partial V_2^-}{\partial r} = \frac{1}{\alpha} \frac{\partial V_2^+}{\partial r} = - \frac{\sqrt{2}}{4\pi} ,$$

so that there is a constant current crossing the membrane in the $O(\varepsilon^{1/2})$ problem.

Similarly, substituting (4.22) and (4.27) in the $r = 1$ boundary condition in the $O(\varepsilon)$ problem (4.20) for V_3, we obtain a current crossing the membrane, which is a function of x.

It is interesting to compare the results in the near field given by (4.21) - (4.30) with the results of a similar analysis of the spherical cell in Reference 1. In the sphere, the leading term is also a finite constant inside and zero outside, but the inside potential is $O(\varepsilon^{-1})$ rather than $O(\varepsilon^{-1/2})$. In the sphere, the second, $O(1)$ term in the inside potential contains an additive constant equal to α (in nondimensional units). The second term immediately outside the membrane, at $r = 1+$, is α. Consequently, in the sphere, as in the cylinder, the second term in the <u>transmembrane</u> potential expansion is independent of α. In both the spherical and cylindrical cell, then, the transmembrane potential is independent of α until the third term. The significant difference, however, is that the consecutive terms in the spherical case increase in order by ε, so that the third term is $O(\varepsilon^2)$ times the leading term, and is almost always negligible. In the cylindrical case the third term is $O(\varepsilon \log \varepsilon)$ times the leading term and so may be significant.

Similarly, substituting (4.22) and (4.27) in the $r = 1$ boundary condition in the $O(\varepsilon)$ problem (4.20) for V_3, we obtain a current crossing the membrane, which is a function of x.

It is interesting to compare the results in the near field given by (4.21) - (4.30) with the results of a similar analysis of the spherical cell in Reference 1. In the sphere, the leading term is also a finite constant inside and zero outside, but the inside potential is $O(\varepsilon^{-1})$ rather than $O(\varepsilon^{-1/2})$. In the sphere, the second, $O(1)$ term in the inside potential contains an additive constant equal to α (in nondimensional units). The second term immediately outside the membrane, at $r = 1+$, is α. Consequently, in the sphere, as in the cylinder, the second term in the <u>transmembrane</u> potential expansion is independent of α. In both the spherical and cylindrical cell, then, the transmembrane potential is independent of α until the third term. The significant difference, however, is that the consecutive terms in the spherical case increase in order by ε, so that the third term is $O(\varepsilon^2)$ times the leading term, and is almost always negligible. In the cylindrical case the third term is $O(\varepsilon \log \varepsilon)$ times the leading term and so may be significant.

V. SINUSOIDAL STEADY STATE

The results of Section IV for the steady state potential can be easily extended to the sinusoidal steady state by generalizing the small parameter ε to a complex parameter ε^*. In the sinusoidal steady state, using complex notation, the potential may be written in the form

$$V(x,r,\theta,t) = v(x,r,\theta)\, e^{i\omega t} \tag{5.1}$$

Replacing $u(t)$ in (2.1) by $e^{i\omega t}$, and V by (5.1), the equation and boundary conditions for v are then:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial x^2} = -\frac{1}{r}\,\delta(x)\;\delta(r-R)\;\delta(\theta) \tag{5.2}$$

$$\frac{\partial v^-}{\partial r} = \frac{1}{\alpha}\frac{\partial v^+}{\partial r} = \varepsilon^*\,(v^+ - v^-) \tag{5.3}$$

$$v(x,r,\theta) = 0 \text{ at } x = \pm\infty \text{ or } r = \infty \tag{5.4}$$

where, in (5.3)

$$\varepsilon^* = \varepsilon(1 + i\omega) \tag{5.5}$$

The problem represented by (5.2) - (5.4) is identical to the $t \to \infty$ limit of (2.1) - (2.3), except for the replacement of ε in (2.2) by ε^*. (Note that ε is the real part of ε^*.) The real part of ε^* is independent of frequency and is simply the parameter ε which appeared in the preceding sections; the imaginary part increases linearly with frequency. The asymptotic expansions already obtained are valid if $|\varepsilon^*|$ is small and the phase of ε^* is less than π in magnitude. In Equation (3.10d), for example, if the phase of ε^* is π there are poles on the integration path at $k = \sqrt{2|\varepsilon^*|}$, resulting in a Stokes' phenomenon.[5] The dependence of ε^* on the frequency ω, will limit the results to frequencies for which

$$\varepsilon\omega << 1 \tag{5.6}$$

We assume, as above that $\varepsilon << 1$.

In order to see what this frequency limitation is numerically, we must transform back to dimensional units. Recalling that the real time t' is related to the dimensionless time t by $t = (\sigma_m/C_m\delta)t'$, the dimensionless frequency ω therefore is related to the real frequency ω' (rad/sec) by

$$\omega = \frac{C_m\delta}{\sigma_m} \cdot \omega' \tag{5.7}$$

Substituting (5.7) and the expression for ε in terms of cell parameters, $\varepsilon = \sigma_m a/\sigma_i\delta$, in the inequality (5.6), we obtain the requirement on the frequency

$$\omega' << \frac{\sigma_i}{aC_m} . \tag{5.8}$$

Using typical values of the parameters:

$$\sigma_m = 3 \times 10^{-10} \text{ mho-cm}^{-1}$$
$$\sigma_i = 10^{-2} \text{ mho-cm}^{-1}$$
$$a = 5 \times 10^{-3} \text{ cm}$$
$$C_m = 10^{-6} \text{ farad-cm}^{-2}$$
$$\delta = 10^{-6} \text{ cm}$$

(5.7) becomes $\omega = 3 \times 10^{-3}\ \omega'$. Above $\omega' = 300$ rad/sec ($\omega = 1$), the frequency dependent part of ε^* becomes dominant in (5.5), that is, ε^* becomes pure imaginary. The restriction (5.8) on the angular frequency becomes $\omega << 2 \times 10^6$ rad/sec or, dividing by 2π, the frequency f' satisfies $f' << 3 \times 10^5$ Hz. With the same choice of parameters, we have $|\varepsilon^*| = 3 \times 10^{-6} f'$, so that, for example, at 3 kHz, $|\varepsilon^*| = 10^{-2}$.

Now let us examine what effect a complex value of ε has on the solutions already obtained in Sections III and IV. First, it should be noticed that for complex ε^*, some reordering of the terms in the expansions (4.1) and (4.15) is necessary.

If we write

$$\varepsilon^* = |\varepsilon^*| \; e^{i\phi}$$

then the second term in the far field expansion (4.1), for example, can be separated into two parts,

$$\varepsilon^{*1/2} \log \varepsilon^* \, W_{1/2} \,(X,r) = |\varepsilon^*|^{1/2} \, e^{i\phi/2} \log |\varepsilon^*| \, W_{1/2} \,(X,r)$$
$$+ \, i\phi \, |\varepsilon^*|^{1/2} \, e^{i\phi/2} \, W_{1/2} \,(X,r)$$

where the second part is of higher order than the first part and actually belongs with the $\varepsilon^{1/2} \, W_1(X,r)$ term in (4.1). This occurs for all the terms in (4.1) and (4.15) containing a log ε.

The transformation (3.17) from near-field to far-field longitudinal coordinate is now complex

$$X = \sqrt{2|\varepsilon^*|} \; x \; e^{i\phi/2}\left[1 - |\varepsilon^*| \; e^{i\phi}\left(\frac{1}{8} - \frac{\alpha\gamma}{2} - \frac{\alpha}{4}\log\frac{|\varepsilon^*|}{2} - \frac{i\alpha\phi}{4}\right) + \ldots\right].$$

If this is substituted in the first term in the far field potential expansion, (4.9), the real part of X produces an exponential decay in the magnitude of W_0, and the imaginary part of X produces a linearly increasing phase delay with increasing x. The same applies to $W_{1/2}$ in (4.10) and (4.13), and the behavior of W_1 in (4.11) and (4.14) is similar, but more complicated because of the presence of the exponential integrals.

Let us now consider an experimental measurement of the near-field potential in a cell with the parameter values given following Equation (5.8), at a longitudinal distance $x = 0.2$ from a sinusoidal source of frequency $\omega' = 2\pi \times 10^4$ rad/sec. To get some idea of the relative importance of the terms in the near field expansion, we calculate the ratio $|\varepsilon^*| V_2/V_0$. Substituting the value of γ and log 2, we have

$$|\varepsilon^*| \; V_2/V_0 = \frac{1}{2} \, |\varepsilon^*| \left[\frac{5}{4} - r^2 - R^2 + 2x^2 + .77\alpha\right].$$

Taking $r^2 = R^2 = 0$ to obtain an upper bound, $\alpha = 1/3$ and $|\varepsilon^*| = 3 \times 10^{-2}$, yields

$$|\varepsilon^*| \; V_2/V_0 = .02$$

or, the correction in the magnitude of the potential due to V_2 is equal to 2% of the magnitude of V_0, and since ε^* is pure imaginary at $\omega' = 2\pi \times 10^4$, there is a 2° change in the phase of the potential (i.e., .02 × 90° = 2°).

Since the asymptotic expansions break down as $\omega \to \infty$, it is not possible to obtain the solution for an arbitrary time dependent source directly by Fourier transforming the expansions valid in the sinusoidal steady state for finite ω. In the next section we will consider the case of a source which is a step function in time using a different method.

VI. TIME DEPENDENT POTENTIAL FOR INFINITE EXTERNAL CONDUCTIVITY

A. Long time expansion

In this section we will develop an asymptotic representation of the time dependence of the potential for the special case in which the cell is surrounded by a perfectly conducting medium, that is, $\sigma_o = \infty$ or $\alpha = \sigma_i/\sigma_o = 0$. We will obtain an expansion which is asymptotic to the potential for any fixed value of t, in the limit of $\varepsilon \to 0$. The expansion will thus be useful for obtaining the potential in cases where the ratio t/ε is large. In addition, the validity of the expansion will be restricted to positions where the axial variable satisfies the inequality $\varepsilon x^2 << 16t^2$.

Setting $\alpha = 0$ in (2.28), we obtain for the inside potential, for $0 \leq r < 1$, $t > 0$,

$$V(x,r,\theta,t) = \frac{1}{4\pi}\left(x^2+ r^2+ R^2- 2rR\cos\theta\right)^{-1/2}$$
$$-\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^\infty dk \cos kx \frac{I_n(kr)I_n(kR)}{kI_n'(k)+\varepsilon I_n(k)}$$
$$\cdot\left[\varepsilon K_n + kK_n' + \frac{1}{I_n} e^{-\left\{(kI_n'/\varepsilon I_n) + 1\right\} t}\right]. \tag{6.1}$$

Setting $\alpha = 0$ in the formula (2.29) for the outside potential, because of the multiplicative factor α, we obtain $V = 0$ for $1 < r < \infty$. This, of course, must be true because there can be no potential drop in a perfect conductor with finite current density. The entire external medium is therefore an equipotential; zero potential, to conform to the boundary condition at $r = \infty$ and at $x = \infty$.

We see that (6.1), except for the last term in the square brackets, is just the steady state potential for $\alpha = 0$, which was studied earlier.[2] To extend our earlier work on the steady state potential to the transient case we have the additional task of studying the asymptotic behavior of the additive time dependent part of the potential (6.1),

$$v(x,r,\theta,t) \equiv -\frac{e^{-t}}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\ \cos kx\ \frac{I_n(kr)I_n(kR)}{I_n[kI_n'+\varepsilon I_n]}\ e^{-kI_n' t/\varepsilon I_n} \tag{6.2}$$

The quantity $kI_n'(k)/I_n(k)$ in the exponent in (6.2) is nonzero for all real k when $n \neq 0$. In the case of $n = 0$, it is nonzero for $k \neq 0$, but we have for small k,

$$\frac{kI_0'(k)}{I_0(k)} = \frac{kI_1(k)}{I_0(k)} \sim \frac{k^2}{2},$$

so that in the $k \to 0$ limit the exponent vanishes. If we consider now the "long time" behavior of (6.2), that is, the limit as $\varepsilon \to 0$ for fixed t, all terms with $n \geq 1$ are exponentially small, (the exponents go to minus infinity) and hence are absent from the asymptotic expansion. On the other hand, in the vicinity of $k = 0$, more specifically, $0 < k^2 \leq \varepsilon$, the $n = 0$ term is not exponentially small. The long time behavior of the potential, (6.1) or (6.2), is therefore determined entirely by a consideration of the $n = 0$ term in the vicinity of $k = 0$. In the limit $t/\varepsilon \to \infty$, the time dependent part of the potential (6.2) is given by the asymptotic representation.

$$v(x,r,t) = \frac{-1}{2\pi^2\varepsilon}\int_0^{\infty} dk\ \cos kx \cdot \frac{I_0(kr)I_0(kR)}{I_0^2(k)\left[(kI_1/\varepsilon I_0)+1\right]} \cdot e^{-\left[(kI_1/\varepsilon I_0)+1\right]t} + \ldots \tag{6.3}$$

and furthermore we have noted that the major contribution to the integral comes from a small range of k; $0 \leq k \lesssim \varepsilon^{1/2}$. Because of the absence of $n \neq 0$ terms in (6.3), the long time asymptotic expansion has no θ dependence.

From the expansion of the modified Bessel functions around $k = 0$, it follows that

$$\frac{kI_1(k)}{I_0(k)} = \frac{k^2}{2} - \frac{k^4}{16} + \frac{k^6}{96} - \cdots \tag{6.4}$$

Defining a new variable ζ by

$$\varepsilon\zeta^2 = \frac{kI_1(k)}{I_0(k)} \tag{6.5}$$

reversion of the series (6.4) yields

$$k(\zeta) = \sqrt{2\varepsilon}\ \zeta \left(1 + \frac{\varepsilon\zeta^2}{8} + \frac{5\varepsilon^2\zeta^4}{384} + \cdots\right) \tag{6.6}$$

and

$$dk = \sqrt{2\varepsilon}\ d\zeta \left(1 + \frac{3\varepsilon\zeta^2}{8} + \frac{25\varepsilon^2\zeta^4}{384} + \cdots\right). \tag{6.7}$$

Substituting (6.6) in the power series expansion of I_0 results in

$$I_0(k) = 1 + \frac{k^2}{4} + \frac{k^4}{64} + \cdots$$

$$= 1 + \frac{\varepsilon\zeta^2}{2} + \frac{3\varepsilon^2\zeta^4}{16} + \cdots \tag{6.8a}$$

and

$$I_0(kr) = 1 + \frac{\varepsilon\zeta^2 r^2}{2} + \frac{3\varepsilon^2\zeta^4 r^4}{16} + \cdots \tag{6.8b}$$

$$I_0(kR) = 1 + \frac{\varepsilon\zeta^2 R^2}{2} + \frac{3\varepsilon^2\zeta^4 R^4}{16} + \cdots \tag{6.8c}$$

Substituting (6.4) - (6.8) in (6.3), yields an asymptotic form for v written in terms of an integral over the new variable ζ,

$$v(x,r,t) = \frac{-1}{\pi^2\sqrt{2\varepsilon}} \int_0^\infty d\zeta \left(1 + \frac{3\varepsilon\zeta^2}{8} + \frac{25\varepsilon^2\zeta^4}{384} + \cdots\right) \cos\left[\sqrt{2\varepsilon}\ \zeta\left(1 + \frac{\varepsilon\zeta^2}{8} + \frac{5\varepsilon^2\zeta^4}{384} + \cdots\right) x\right]$$

$$\cdot \left\{1 + \frac{\varepsilon\zeta^2}{2}\left(r^2 + R^2 - 2\right) + \frac{\varepsilon^2\zeta^4}{16}\left[3\left(r^4 + R^4\right) + 4r^2R^2 - 8\left(r^2 + R^2\right) + 6\right] + \cdots\right\}$$

$$\cdot \frac{e^{-(\zeta^2+1)t}}{\zeta^2 + 1} \tag{6.9}$$

Multiplying the two series in (6.9) and replacing the cosine by complex exponentials leads to

$$v(x,r,t) = \frac{-1}{2\pi^2\sqrt{2\varepsilon}}\int_{-\infty}^{\infty} d\zeta \cdot \left\{ 1 + \frac{\varepsilon\zeta^2}{2}\left(r^2 + R^2 - \frac{5}{4}\right) + \frac{\varepsilon^2\zeta^4}{16}\left[3\left(r^4 + R^4\right) + 4r^2R^2 - 5\left(r^2+R^2\right)\right] + \dots \right\}$$

$$\cdot \frac{1}{\zeta^2+1} \cdot e^{i\zeta\sqrt{2\varepsilon}\, x\left(1 + \frac{\varepsilon\zeta^2}{8} + \frac{5\varepsilon^2\zeta^4}{384} + \dots\right) - (\zeta^2+1)t} \tag{6.10}$$

We now truncate the first series in (6.10) after the $O(\varepsilon)$ term, and the series in the exponent after the $O(\varepsilon^{3/2})$ term. Equivalently, in the "far field" ($\sqrt{2\varepsilon}$ x fixed as $\varepsilon \to 0$) we truncate both series after the $O(\varepsilon)$ term. This results in the asymptotic relation for $t/\varepsilon \to \infty$, $\varepsilon \to 0$,

$$v(x,r,t) = \frac{-1}{2\pi^2\sqrt{2\varepsilon}}\int_{-\infty}^{\infty} d\zeta\, \frac{1 + \frac{\varepsilon\zeta^2}{2}\left(r^2+R^2-\frac{5}{4}\right)}{\zeta^2+1}\, e^{i\zeta\sqrt{2\varepsilon}\, x\left(1 + \frac{\varepsilon\zeta^2}{8}\right) - \left(\zeta^2+1\right)t} + \dots \tag{6.11}$$

We would now like to evaluate the integral in (6.11) in terms of tabulated functions. First it should be observed that[8]

$$\int_{-\infty}^{\infty} d\zeta\, \left(\zeta^2+1\right)^{-1} e^{i\zeta X - (\zeta^2+1)T} = \frac{\pi}{2}\left[e^{-X}\operatorname{erfc}\left(\sqrt{T} - \frac{X}{2\sqrt{T}}\right) + e^{X}\operatorname{erfc}\left(\sqrt{T} + \frac{X}{2\sqrt{T}}\right)\right] \tag{6.12}$$

is a known integral. The most straightforward procedure would be to remove the $\varepsilon\zeta^2/8$ term from the exponent by using the expansion

$$e^{i\zeta\sqrt{2\varepsilon}\, x \cdot \frac{\varepsilon\zeta^2}{8}} = 1 + i\zeta\sqrt{2\varepsilon}\, x \cdot \frac{\varepsilon\zeta^2}{8} + \dots ,$$

multiplying this expansion by the other expansion in (6.11) and then evaluating the resulting integral by relating it to the appropriate derivatives of (6.12). This is a lengthy procedure and leads to an asymptotic expression which is not in the most convenient form, but can be shown to equal (6.19).

It is preferable to begin by making a change of variables in (6.11) to new space and time coordinates. Doing this it is possible to transform (6.11) into precisely the form (6.12), so that no differentiation of the complementary error functions in (6.12) is necessary. Let

$$\sqrt{2\varepsilon}\, x = X\ (1 + f\varepsilon + \ldots) \tag{6.13a}$$

$$t = T\ (1 + g\varepsilon + \ldots) \tag{6.13b}$$

where f and g will be determined to cause the greatest simplification of (6.11). Substituting (6.13a) and (6.13b) in (6.11), expanding the $O(\varepsilon)$ term in the exponential, and rearranging terms, we obtain

$$v(x,r,t) = \frac{-1}{2\pi^2\sqrt{2\varepsilon}} \int_{-\infty}^{\infty} d\zeta e^{i\zeta X - (\zeta^2+1)T}\, e^{\varepsilon\left(\frac{i\zeta^3 X}{8} + i\zeta X f - T(\zeta^2+1)g\right)} \cdot \frac{1 + \frac{\varepsilon\zeta^2}{2}\left(r^2 + R^2 - \frac{5}{4}\right)}{\zeta^2 + 1}$$

$$+ \ldots$$

$$= \frac{-1}{2\pi^2\sqrt{2\varepsilon}} \int_{-\infty}^{\infty} d\zeta e^{i\zeta X - (\zeta^2+1)T}\, \frac{1 + \varepsilon\left[i\zeta X\left(\frac{\zeta^2}{8} + f\right) - Tg(\zeta^2+1) + \frac{\zeta^2}{2}\left(r^2+R^2-\frac{5}{4}\right)\right] + \ldots}{\zeta^2 + 1}$$

$$= \frac{-1}{2\pi^2\sqrt{2\varepsilon}} \left\{\left[1 - \frac{\varepsilon}{2}\left(r^2 + R^2 - \frac{5}{4}\right)\right] \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta^2+1}\, e^{i\zeta X - (\zeta^2+1)T}\right.$$

$$\left. + \varepsilon \int_{-\infty}^{\infty} d\zeta e^{i\zeta X - (\zeta^2+1)T} \cdot \left[\frac{i\zeta X}{8} \cdot \frac{\zeta^2+8f}{\zeta^2+1} - Tg + \frac{1}{2}\left(r^2+R^2 - \frac{5}{4}\right)\right] + \ldots\right\}$$

(6.14)

A judicious choice of f and g will now make the last integral in (6.14) vanish. If we let

$$f = \frac{1}{8} \tag{6.15a}$$

$$g = \frac{1}{2T}\left(r^2 + r^2 - \frac{5}{4}\right) - \frac{X^2}{16T^2} \tag{6.15b}$$

the expression in square brackets in the last integral in (6.14) becomes

$$\frac{i\zeta X}{8} + \frac{X^2}{16T} = -\frac{iX}{16T}(iX - 2\zeta T)$$

$$= -\frac{iX}{16T}\frac{d}{d\zeta}[iX\zeta - (\zeta^2+1)T]. \tag{6.16}$$

Since this is proportional to the derivative of the exponent in (6.14), the last integral vanishes. Therefore,

$$v(x,r,t) = \frac{-1}{2\pi^2\sqrt{2\varepsilon}} \cdot \left[1 - \frac{\varepsilon}{2}\left(r^2 + R^2 - \frac{5}{4}\right)\right] \int_{-\infty}^{\infty} d\zeta e^{i\zeta X - (\zeta^2+1)T} \cdot (\zeta^2+1)^{-1} + \ldots$$

$$= \frac{-1}{4\pi\sqrt{2\varepsilon}} \cdot \left[1 - \frac{\varepsilon}{2}\left(r^2 + r^2 - \frac{5}{4}\right)\right]\left[e^{-X} \operatorname{erfc}\left(\sqrt{T} - \frac{X}{2\sqrt{T}}\right) + e^{X} \operatorname{erfc}\left(\sqrt{T} + \frac{X}{2\sqrt{T}}\right)\right] + \ldots \tag{6.17}$$

where from (6.13) and (6.15) it follows that the new variables are

$$X = \sqrt{2\varepsilon}\, x\left(1 - \frac{\varepsilon}{8} + \ldots\right) \tag{6.18a}$$

$$T = t\left[1 - \frac{\varepsilon}{2t}\left(r^2 + R^2 - \frac{5}{4}\right) + \frac{\varepsilon x^2}{16t^2} + \ldots\right] \tag{6.18b}$$

X is the same far field longitudinal coordinate that was found earlier for the steady state problem.

Substituting (6.17) for the time dependent part and Equation (5.3) of Reference 2 for the steady state part in (6.1), the asymptotic expansion of the time dependent potential is

$$V(x,r,\theta,t) = \frac{1}{4\pi\sqrt{2\varepsilon}} \cdot \left[1 - \frac{\varepsilon}{2}\left(r^2 + R^2 - \frac{5}{4}\right) + \ldots\right] \cdot \left[e^{-X} \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} - \sqrt{T}\right) - e^{X} \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} + \sqrt{T}\right)\right]$$

$$+ \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2}$$

$$- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\, \cos kx \left[\frac{K_n'(k)}{I_n'(k)} I_n(kR) I_n(kr) + \frac{2\delta_{0n}}{k^2}\right.$$

$$\left. + \frac{\varepsilon}{k^2}\left\{\frac{I_n(kR)I_n(kr)}{I_n'^2(k)} - \left(r^2 + R^2 - 1 + \frac{4}{k^2}\right)\delta_{0n}\right\} + \ldots\right] \tag{6.19}$$

with X and T given in (6.18a) and (6.18b).

The first term in (6.19) results from combining the far field part of Equation (5.3), Reference 2, with the first erfc in (6.17):

$$\frac{1}{2\pi\sqrt{2\varepsilon}} e^{-X}\left(1 + \frac{\varepsilon}{2}\left\{\frac{5}{4} - r^2 - R^2\right\}\right) - \frac{1}{4\pi\sqrt{2\varepsilon}}\left(1 + \frac{\varepsilon}{2}\left\{\frac{5}{4} - r^2 - R^2\right\}\right) \cdot \operatorname{erfc}\left(\sqrt{T} - \frac{X}{2\sqrt{T}}\right)$$

$$= \frac{e^{-X}}{4\pi\sqrt{2\varepsilon}}\left[1 + \frac{\varepsilon}{2}\left(\frac{5}{4} - r^2 - R^2\right)\right]\left[2 - \operatorname{erfc}\left(\sqrt{T} - \frac{X}{2\sqrt{T}}\right)\right],$$

and changing the sign in the erfc argument leads to the result in (6.19).

The combination of exponentials and error functions in (6.19) is identical to the classical result of one-dimensional cable theory.[9] The space time variables given by (6.18a) and (6.18b), however, contain higher order corrections to the classical variables. With (6.19), we can now state precisely the range of validity of the classical one-dimensional cable theory. This was not previously possible because of the nature of the one-dimensional approximation, which did not give any clue to the errors in the approximation, or how the higher order terms might be obtained.

As was concluded in Reference 2, by converting the last three lines of (6.19) to the equivalent double sums (4.31) and (C.9) of Reference 2, we find that this part of (6.19) decays exponentially with increasing x. Consequently, for

$$x \text{ or } \frac{X}{\sqrt{\varepsilon}} \to \infty, \text{ as } \varepsilon \to 0, \tag{6.20}$$

which defines the "far field," the last three lines of (6.19) are exponentially small, and only the error function terms survive in the far field. The far field potential is therefore given by

$$\begin{aligned} W(X,r,T) = \frac{1}{4\pi\sqrt{2\varepsilon}} &\left[1 - \frac{\varepsilon}{2}\left(r^2 + R^2 - \frac{5}{4}\right) + \ldots\right] \\ &\times\left[e^{-X} \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} - \sqrt{T}\right) - e^{-X} \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} + \sqrt{T}\right)\right] \end{aligned} \tag{6.21}$$

with X and T related to x and t by (6.18a) and (6.18b). If we let $T \to \infty$ in (6.21) we recover the steady state result given in Reference 2, Equation (5.1).

If we only retain the leading terms in (6.18a), (6.18b) and (6.21), we obtain

$$W = \frac{1}{4\pi\sqrt{2\varepsilon}}\left[e^{-\sqrt{2\varepsilon}\,x} \operatorname{erfc}\left(\frac{\sqrt{2\varepsilon}\,x}{2\sqrt{t}} - \sqrt{t}\right) - e^{\sqrt{2\varepsilon}\,x} \operatorname{erfc}\left(\frac{\sqrt{2\varepsilon}\,x}{2\sqrt{t}} + \sqrt{t}\right)\right] \tag{6.22}$$

or precisely the result of one-dimensional cable theory. Thus we see that in addition to the far field restriction (6.20), it is also required that

$$\frac{\varepsilon x^2}{t^2} \to 0 \text{ as } \varepsilon \to 0 \tag{6.23}$$

for validity of the one-dimensional cable theory formula.

It should also be recalled that to obtain (6.3) from (6.2), it was assumed that

$$\frac{t}{\varepsilon} \to \infty, \tag{6.24}$$

which is what we defined as the "long time" in Reference 1.

In summary, the classical result of one-dimensional cable theory, given in (6.22), has been shown to be valid in a well defined range of the independent variables x and t. It is valid under three constraints:

1. Away from the point source in the far field, defined by (6.20).
2. For long times, defined by (6.24).
3. For longitudinal positions and times, related by (6.23).

With these constraints on x and t, we have been able to extend the classical result (6.22) to include correction terms, as given in (6.21).

Equation (6.19) is not subject to the far field limitation (6.20), but is restricted to the long time condition (6.24), and to the constraint (6.23).

B. Short Time Behavior

We have shown that Equation (6.19) for the inside potential follows from (6.1) if $\varepsilon \to 0$, $t/\varepsilon \to \infty$, and $\varepsilon x^2/t^2 \to 0$. We would now like to investigate the "short time" period, for which it is convenient to define the short time variable

$$\tilde{t} = \frac{t}{\varepsilon} \tag{6.25}$$

Making this change of variable, (6.1) can be written

$$\begin{aligned} V(x,r,\theta,\varepsilon\tilde{t}) &= \frac{1}{4\pi}\left(x^2+r^2+R^2-2rR\cos\theta\right)^{-1/2} \\ &- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk \cos kx \,\frac{I_n(kr)I_n(kR)}{kI_n'(k)+\varepsilon I_n(k)} \\ &\cdot \left[\varepsilon K_n + kK_n' + \frac{1}{I_n}\, e^{-\left\{(kI_n'/I_n)+\varepsilon\right\}\tilde{t}}\right] \end{aligned} \tag{6.26}$$

For small $\tilde{t}$, we expand the exponential in (6.26) so that the bracketed expression becomes

$$\varepsilon K_n + kK_n' + \frac{1}{I_n} - \frac{kI_n' + \varepsilon I_n}{I_n^2} \cdot \tilde{t} + \ldots$$

$$= \frac{\varepsilon I_n K_n + kI_n K_n' + 1}{I_n} - \frac{kI_n' + \varepsilon I_n}{I_n^2} \cdot \tilde{t} + \ldots$$

$$= \left(\frac{K_n}{I_n} - \frac{\tilde{t}}{I_n^2} \right) \left(kI_n' + \varepsilon I_n \right) + \ldots \tag{6.27}$$

and substituting (6.27) in (6.26), we obtain

$$V(x,r,\theta,\varepsilon\tilde{t}) = \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2}$$

$$- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk \cos kx \, I_n(kr)\, I_n(kR) \cdot \left[\frac{K_n(k)}{I_n(k)} - \frac{\tilde{t}}{I_n^2(k)} + \ldots\right] \tag{6.28}$$

From (6.28) it is easy to see that at $\tilde{t} = 0^+$, the potential at the inside surface of the membrane, $r = 1^-$, is zero. Setting $\tilde{t} = 0^+$ and $r = 1^-$ in (6.28), yields

$$V(x,1^-,\theta,0^+) = \frac{1}{4\pi}\left(x^2 + 1 + R^2 - 2R\cos\theta\right)$$

$$- \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk \cos kx \, K_n(k) I_n(kR)$$

$$= 0,$$

where we have used (2.27) to obtain the equality to zero.

Thus, (6.28) with $\tilde{t} = 0$, [or (6.1) with $t = 0$] is the solution of the Dirichlet problem of a point source inside an infinite cylinder surrounded by a perfect conductor held at zero potential. This should be constrasted with the O(1) term in the asymptotic expansion of the near field potential when $t = \infty$, Equation (4.30), (4.31) or (D.14) of Reference 2, which satisfies the Neumann problem (4.3) of Reference 2. The physical basis of this behavior is that for very short times ($\tilde{t} \to 0$) the membrane capacitance acts as a short circuit (to all orders of ε) whereas for very long times ($t \to \infty$) the large membrane resistance appears as an open circuit [to $O(1)$].

According to (6.28), at $t = 0^+$, immediately after switching on the current, the potential everywhere inside the cell instantaneously becomes equal to that of a conducting cylinder surrounded by a grounded perfectly conducting boundary. The current lines initially are normal to the membrane. The initial time derivative is given by the second term in (6.28). It is interesting to note that ε does not appear in (6.28). This means that both the potential and its initial time derivative are independent of the membrane resistance. Higher derivatives, of course, will depend on ε.

Equation (6.28) is useful for times t smaller than $O(\varepsilon)$. For intermediate times (of order between ε and 1) the exact expression (6.1) must be used to obtain the potential; for long times (of order 1 or larger) the asymptotic form (6.19) is applicable, subject to the additional constraint that $\varepsilon x^2/16t^2$ be small.

C. Computable Representation of the Short-Time Potential

We will now derive infinite sum representations of the two Fourier cosine transforms appearing in (6.28),

$$F_1 = \int_0^\infty dk \cos kx \; I_n(kr) I_n(kR) \frac{K_n(k)}{I_n(k)} \tag{6.29}$$

and

$$F_2 = \int_0^\infty dk \cos kx \; \frac{I_n(kr) I_n(kR)}{I_n^2(k)} \tag{6.30}$$

By considering the integrals in (6.29) and (6.30) as portions along the real axis of contour integrals in the complex $z = k + i\lambda$ plane, (6.29) and (6.30) can be related to the infinite sum of residues which occur along the λ axis wherever $I_n(z)$ has a zero. This will be done for (6.30). In order to obtain expansions for (6.29) and (6.30) which converge at $x = 0$, however, it is necessary to consider a modification of the integrands. This is analagous to the treatment necessary for Equation (4.30) of Reference 2 in order to avoid the divergence of (4.31) at $x = 0$. We will do this for (6.29), but not for (6.30) which is of somewhat less interest.

We consider the contour integral

$$\frac{1}{2\pi i} \int_C dz \cos zx \, I_n(zr) I_n(zR) \frac{K_n(z)}{I_n(z)} \frac{z}{(z^2 - \tau^2) \cos\left(\frac{\pi z}{\beta}\right)} \tag{6.31}$$

in which β and τ are real parameters, $\beta > 0$, and $\cos(\pi\tau/\beta) \neq 0$ so that the poles introduced by the modification of the integrand are distinct. The integration path is shown in Figure 2. The branch cut can be taken between the origin and infinity, anywhere in the left half-plane. If $|x| < \pi/\beta$, the integrand along the semicircle vanishes exponentially as the radius tends to infinity. Since there are no singularities enclosed by the contour, the contour integral (6.30) is zero. Therefore, the sum of the principal value of the integral up the imaginary axis, the residues on the real axis, and one-half times the residues on the imaginary axis is zero. The method here is exactly analagous to the method introduced in Appendix D of Reference 2 and the reader is referred to that reference for the intermediate steps and discussion, some of which are omitted from the present derivation.

First, we consider the integral up the imaginary axis, which is separated into two pieces:

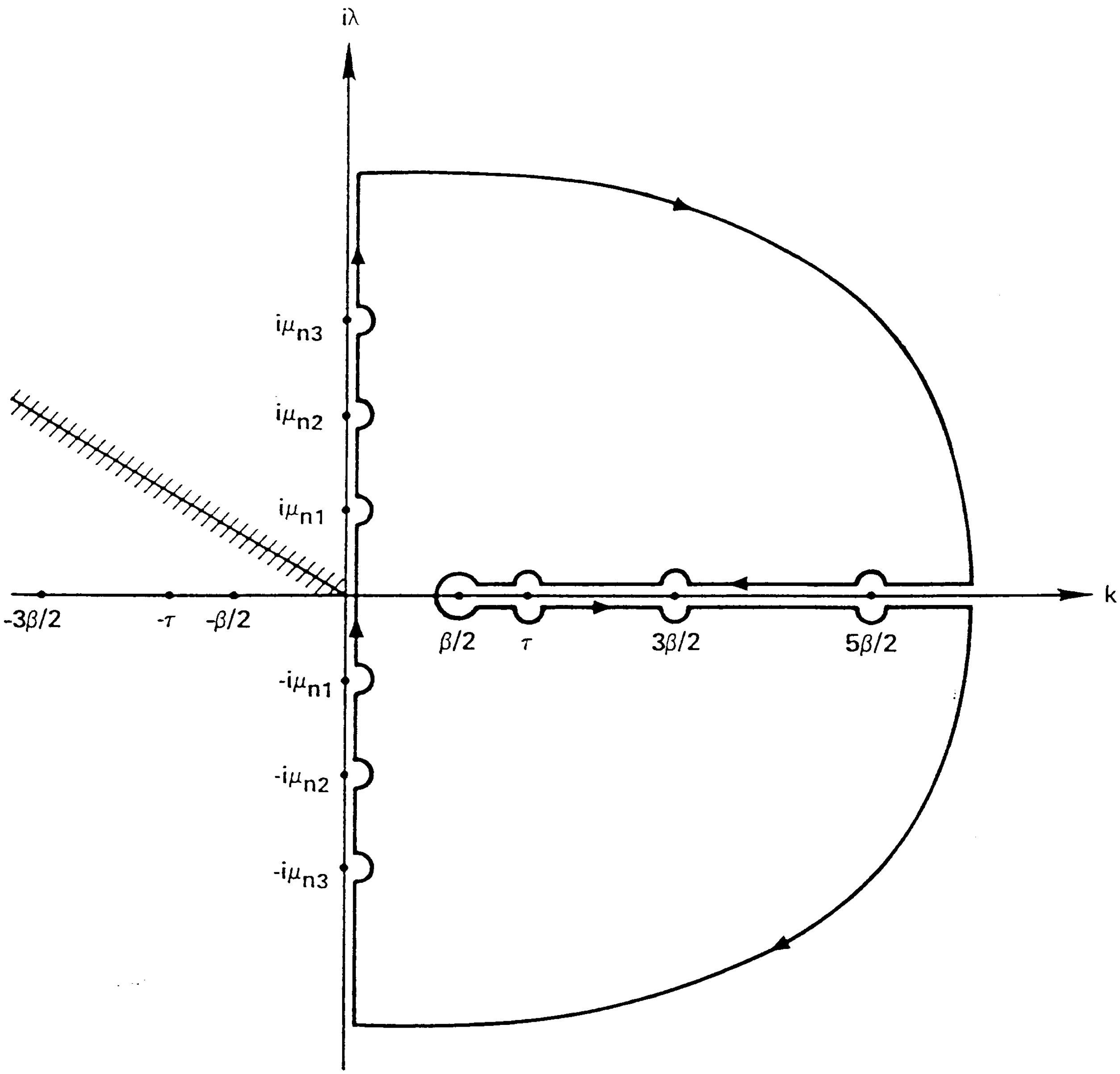

Figure 2. Contour for (6.31).

$$\frac{1}{2\pi i}\int_{-\infty}^{0} i d\lambda \cosh \lambda x \frac{K_n\left(e^{-i\pi/2}|\lambda|\right)}{I_n(i\lambda)} I_n(i\lambda r) I_n(i\lambda R) \frac{i\lambda}{(-\lambda^2-\tau^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$-\frac{1}{2\pi i}\int_{0}^{\infty} i d\lambda \cosh \lambda x \frac{K_n\left(e^{i\pi/2}|g|\right)}{I_n(i\lambda)} I_n(i\lambda r) I_n(i\lambda R) \frac{i\lambda}{(-\lambda^2-\tau^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} d\lambda \cosh \lambda x \left[\frac{K_n\left(e^{-i\pi/2}\lambda\right)}{I_n(-i\lambda)} I_n(-i\lambda r)\, I_n(-i\lambda R) - \frac{K_n\left(e^{i\pi/2}\lambda\right)}{I_n(i\lambda)} I_n(i\lambda r)\, I_n(i\lambda R)\right] \frac{i\lambda}{(\lambda^2+\tau^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)} .$$

Using the properties of the Bessel functions:

$$I_n(z) = i^{-n} J_n(iz)$$

$$K_n(z) = \frac{\pi}{2} i^{n+1}\left[J_n(iz) + iY_n\left(e^{i\pi/2} z\right)\right]$$

$$J_n(-\lambda) = (-1)^n J_n(\lambda)$$

the integral becomes

$$\frac{1}{2\pi}\int_{0}^{\infty} d\lambda \cosh \lambda x \left(\frac{i\pi}{2}\right)\left[\frac{J_n(\lambda) + iY_n(\lambda)}{J_n(\lambda)} - \frac{J_n\left(\lambda e^{i\pi}\right) + iY_n\left(\lambda e^{i\pi}\right)}{J_n\left(\lambda e^{i\pi}\right)}\right] \cdot \frac{i\lambda J_n(\lambda r) J_n(\lambda R)}{(\lambda^2+\tau^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)} .$$

Using the relation

$$Y_n\left(\lambda e^{i\pi}\right) = (-1)^n\left[Y_n(\lambda) + 2iJ_n(\lambda)\right]$$

the integral up the imaginary axis finally becomes

$$-\frac{1}{2}\int_{0}^{\infty} d\lambda \cosh \lambda x \frac{\lambda J_n(\lambda r) J_n(\lambda R)}{(\lambda^2+\tau^2)\cosh\left(\frac{\pi\lambda}{\beta}\right)} . \tag{6.32}$$

The residue at $z = \tau$ is

$$\frac{K_n(\tau)}{I_n(\tau)} I_n(\tau r) I_n(\tau R) \frac{\cos \tau x}{2 \cos\left(\frac{\pi\tau}{\beta}\right)} \tag{6.33}$$

The residue at the zeroes of $\cos(\pi z/\beta)$ which occur at $z = (\nu + \frac{1}{2})\beta$ is

$$-\frac{\beta}{\pi}(-1)^{\nu} \frac{K_n\left[(\nu+\frac{1}{2})\beta\right]}{I_n\left[(\nu+\frac{1}{2})\beta\right]} I_n\left[(\nu+\frac{1}{2})\beta r\right] I_n\left[(\nu+\frac{1}{2})\beta R\right] \frac{\cos\left[(\nu+\frac{1}{2})\beta x\right](\nu+\frac{1}{2})\beta}{(\nu+\frac{1}{2})^2\beta^2-\tau^2} \tag{6.34}$$

The zeroes of $I_n(z)$ occur in pairs, along the imaginary axis at $z = \pm i\mu_{ns}$ where $\lambda = \mu_{ns}$ is the s^{th} root of $J_n(\lambda) = 0$. The residue at $z = i\mu_{ns}$ is obtained by noting that near $z = i\mu_{ns}$ we can expand the Bessel function in a Taylor series,

$$I_n(i\lambda) = i^{-n} J_n(-\lambda) = i^n J_n(\lambda) = i^n(\lambda-\mu_{ns}) J_n'(\mu_{ns}) + O(\lambda-\mu_{ns})^2,$$

where the leading term is missing because $J_n(\mu_{ns}) = 0$. From this we obtain

$$\operatorname{Res}\left\{\frac{1}{I_n(z)}\right\}_{z=i\mu_{ns}} = \left.\frac{z - i\mu_{ns}}{i^n(\lambda-\mu_{ns})J_n'(\mu_{ns}) + \ldots}\right|_{z=i\mu_{ns}}$$

$$= \frac{i^{1-n}}{J_n'(\mu_{ns})}$$

To obtain the residue of the integrand in (6.31) at the simple pole, $z = i\mu_{ns}$, the above residue is to be multiplied by the rest of the integrand evaluated at $z = i\mu_{ns}$. The $K_n(z)$ term is

$$K_n(i\mu_{ns}) = \frac{\pi}{2} i^{n+1}\left[J_n(-\mu_{ns}) + iY_n\left(e^{i\pi}\mu_{ns}\right)\right]$$

$$= -\frac{\pi}{2} i^{-n} Y_n(\mu_{ns})$$

$$= \frac{i^{-n}}{\mu_{ns} J_n'(\mu_{ns})}$$

which follows using the Wronskian of J_n and Y_n and noting again that $J_n(\mu_{ns}) = 0$. Consequently, the residue of the integrand in (6.31) at $z = i\mu_{ns}$ is

$$\frac{\cosh(\mu_{ns}x)\, J_n(\mu_{ns}r)\, J_n(\mu_{ns}R)}{\left(\mu_{ns}^2+\tau^2\right)+\cosh\left(\frac{\pi\mu_{ns}}{\beta}\right) J_n'^2(\mu_{ns})} \tag{6.35}$$

Equating the negative of the integral up the imaginary axis, (6.32), to the sum of the residue at $z = \tau$, (6.33), the residues at $z = (\nu + \frac{1}{2})\beta$, (6.34), and one-half of the residues at $z = \pm i\mu_{ns}$, we obtain

$$\frac{1}{2}\int_0^\infty d\lambda \cosh\lambda x \frac{\lambda J_n(\lambda r)J_n(\lambda R)}{\left(\lambda^2+\tau^2\right)\cosh\left(\frac{\pi\lambda}{\beta}\right)}$$

$$= \frac{K_n(\tau)}{I_n(\tau)} I_n(\tau r) I_n(\tau R) \frac{\cos\tau x}{2\cos\left(\frac{\pi\tau}{\beta}\right)}$$

$$+ \sum_{s=1}^{\infty} \frac{\cosh(\mu_{ns}x)\, J_n(\mu_{ns}r)\, J_n(\mu_{ns}R)}{\left(\mu_{ns}^2+\tau^2\right)+\cosh\left(\frac{\pi\mu_{ns}}{\beta}\right) J_n'^2(\mu_{ns})}$$

$$- \frac{\beta}{\pi}\sum_{\nu=0}^{\infty}(-1)^\nu \frac{K_n\left[(\nu+\frac{1}{2})\beta\right]}{I_n\left[(\nu+\frac{1}{2})\beta\right]} I_n\left[(\nu+\tfrac{1}{2})\beta r\right] I_n\left[(\nu+\tfrac{1}{2})\beta R\right] \frac{\cos\left[(\nu+\frac{1}{2})\beta x\right](\nu+\frac{1}{2})\beta}{(\nu+\frac{1}{2})^2\beta^2-\tau^2} \tag{6.36}$$

The derivation now follows exactly the steps on pp. 61-63 of Reference 2. We repeat the arguments for completeness.

The left hand side of (6.36) is analytic everywhere in the finite right-half complex τ-plane. It equals the right hand side everywhere in this region, except perhaps at the points $\tau = (\mu + \frac{1}{2})\beta$, μ an integer, which were excluded originally, to make the poles introduced in (6.31) distinct. At $\tau = (\mu + \frac{1}{2})\beta$, however, the residue of the first term on the right hand side is seen to cancel the residue of the $\nu = \mu$ term in the sum over ν, and hence the right hand side of (6.36) is analytic everywhere in the finite right-half τ-plane. By analytic continuation, (6.36) must then be valid even when $\tau = (\mu + \frac{1}{2})\beta$.

If we multiply (6.36) by $2\cos(\pi\tau/\beta)$, and integrate over τ from 0 to ∞, the first term on the right is just the integral F_1 in (6.29) which we wish to evaluate.

Using the formulas

$$\int_0^\infty \frac{\cos\ (\pi\tau/\beta)}{\lambda^2+\tau^2}\, d\tau = \frac{\pi}{2\lambda}\, e^{-\pi\lambda/\beta}$$

$$\int_0^\infty \frac{\cos\ (\pi\tau/\beta)}{(\nu+\frac{1}{2})^2\ \beta^2-\tau^2}\, d\tau = \frac{(-1)^\nu\ \pi}{2(\nu+\frac{1}{2})\beta}$$

yields

$$\int_0^\infty d\tau\ \cos(\tau x)\ \frac{K_n(\tau)}{I_n(\tau)}\ I_n(\tau r) I_n(\tau R)$$

$$= \frac{\pi}{2} \int_0^\infty d\lambda\ \cos(\lambda x)\ \frac{e^{-\lambda\pi/\beta}}{\cosh(\pi\lambda/\beta)}\ J_n(\lambda r) J_n(\lambda R)$$

$$- \pi \sum_{s=1}^{\infty} \frac{J_n(\mu_{ns} r) J_n(\mu_{ns} R)\ e^{-\pi\mu_{ns}/\beta}\ \cosh(\mu_{ns} x)}{\mu_{ns} J_n'^{\,2}(\mu_{ns})\ \cosh(\pi\mu_{ns}/\beta)}$$

$$+ \beta \sum_{\nu=0}^{\infty} \cos\left[(\nu+\tfrac{1}{2})\beta x\right] \frac{K_n\left[(\nu+\frac{1}{2})\beta\right]}{I_n\left[(\nu+\frac{1}{2})\beta\right]}\ I_n\left[(\nu+\tfrac{1}{2})\beta r\right] I_n\left[(\nu+\tfrac{1}{2})\beta R\right] \qquad (6.37)$$

We now substitute (6.37) in the formula (6.28) at $\tilde{\tau} = 0$, using the result (Reference 2, page 62)

$$- \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty d\lambda\ \cosh \lambda x\ \frac{e^{-\pi\lambda/\beta}}{\cosh(\pi\lambda/\beta)}\ J_n(\lambda r) J_n(\lambda R)$$

$$= \left(\sum_{m=-\infty}^{-1} + \sum_{m=1}^{\infty} \right) (-1)^m \left[\left(x - \frac{2\pi m}{\beta} \right)^2 + r^2 + R^2 - 2rR\ \cos\theta \right]^{-1/2} ,$$

and obtain

$$V(x,r,\theta,0^+) = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} (-1)^m \left[\left(x - \frac{2\pi m}{\beta}\right)^2 + r^2 + R^2 - 2rR\cos\theta\right]^{-1/2}$$

$$+ \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{s=1}^{\infty} \frac{2\cosh(\mu_{ns}x)}{1+e^{2\pi\mu_{ns}/\beta}} \frac{J_n(\mu_{ns}r)J_n(\mu_{ns}R)}{\mu_{ns}J_n'^2(\mu_{ns})}$$

$$- \frac{\beta}{2\pi^2} \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{\nu=0}^{\infty} \cos\left[(\nu+\tfrac{1}{2})\beta x\right] \frac{K_n\left[(\nu+\frac{1}{2})\beta\right]}{I_n\left[(\nu+\frac{1}{2})\beta\right]}$$

$$\cdot I_n\left[(\nu+\tfrac{1}{2})\beta r\right] I_n\left[(\nu+\tfrac{1}{2})\beta R\right] \tag{6.38}$$

For large ν, the Fourier coefficients in the ν sum approach

$$\frac{1}{(2\nu+1)\beta\sqrt{rR}} \cdot e^{-(\nu+\frac{1}{2})\beta(2-r-R)}$$

so that the terms in the ν sum decrease exponentially with increasing ν, and the sum converges rapidly, unless $r = R = 1$. For fixed ν, the terms in the sum over n approach $r^n R^n/2n$ for large n. Consequently the sum over n is absolutely convergent unless $r = R = 1$.

The sum over m converges everywhere except the points $(2\pi m/\beta, R, 0)$, when one term in the sum is infinite. When $m = 0$, this divergence describes the infinite potential at the location of the point source.

The double sum over n and s converges for $|x| < 2\pi/\beta$, in which case the terms decrease exponentially with increasing n and s. This is twice the range of $|x|$ implied by our derivation; however, by analytic continuation (6.38) must be correct over this larger range of $|x|$.

When $\beta \to 0$, all but the $m = 0$ term in the first sum vanish, the second sum vanishes, and the third sum approaches the integral in (6.28) with

$\tilde{t} = 0$. Hence for $\beta = 0$, (6.38) reduces to the integral form (6.28) for $V(x,r,\theta,0^+)$.

When $\beta = \pi/|x|$, the sum over m vanishes because of term by term cancellation of the m = 1, 2, 3,... terms with the m = 0, -1, -2,... terms; the sum over ν vanishes because of the factor $\cos[(\nu + \frac{1}{2})\pi]$, and noting that in this case,

$$\frac{2\cosh(\mu_{ns}x)}{1 + e^{2\pi\mu_{ns}/\beta}} = e^{-\mu_{ns}|x|} ,$$

(6.28) reduces to

$$V(x,r,\theta,0^+) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{s=1}^{\infty} \frac{e^{-\mu_{ns}|x|}\, J_n(\mu_{ns}r)J_n(\mu_{ns}R)}{\mu_{ns}\, J_n'^{\,2}(\mu_{ns})} , \tag{6.39}$$

which is the well known eigenfunction expansion of the Dirichlet problem of a point source inside a conducting cylinder surrounded by a grounded perfect conductor. Equation (6.39) is preferable to (6.38) for calculating the potential for large values of x. However, as $x \to 0$, its convergence becomes increasingly poor, and for small x, with an appropriate choice of β, (6.38) can be made rapidly convergent and is far superior.

Returning to the integral F_2 in (6.30), which determines the time derivative of the potential immediately after the initial jump, we derive an equivalent eigenfunction expansion for F_2. Since there is no branch point in the integrand of (6.30) [as there was at k = 0 in the integrand of (6.29)], it is convenient to write an equivalent integral with k ranging from $-\infty$ to $+\infty$, and consider the contour which follows the real axis from $-\infty$ to $+\infty$ and then traverses counterclockwise along the large semicircle in the upper half plane, back to $-\infty$. Thus, since the integral along the semicircular portion of the contour approaches zero as the radius approaches infinity, we have

$$F_2 = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty} dk\; e^{ikx}\;\frac{I_n(kr)I_n(kR)}{I_n^2(k)}$$

$$= \mathrm{Re}\left\{\pi i\sum_{s=1}^{\infty}\mathrm{Res}\left[e^{izx}\;\frac{I_n(zr)I_n(zR)}{I_n^2(z)}\right]_{z=i\mu_{ns}}\right\}$$

$$= -\pi\sum_{s=1}^{\infty}\mathrm{Res}\left[e^{-\lambda x}\;\frac{J_n(\lambda r)J_n(\lambda R)}{J_n^2(\lambda)}\right]_{\lambda=\mu_{ns}} \tag{6.40}$$

In the vicinity of $\lambda = \mu_{ns}$,

$$J_n(\lambda) = (\lambda-\mu_{ns})\;J_n'(\mu_{ns}) + \frac{1}{2}\;(\lambda-\mu_{ns})^2\;J_n''(\mu_{ns}) + \ldots$$

Using Bessel's equation and $J_n(\mu_{ns}) = 0$, $J_n''(\mu_{ns})$ can be eliminated, yielding

$$J_n(\lambda) = (\lambda-\mu_{ns})\;J_n'(\mu_{ns})\left(1 - \frac{\lambda-\mu_{ns}}{2\mu_{ns}} + \ldots\right)$$

Taking the reciprocal and squaring, we have

$$\frac{1}{J_n^2(\lambda)} = \frac{1}{J_n'^2(\mu_{ns})}\cdot\left[\frac{1}{(\lambda-\mu_{ns})^2} + \frac{1}{\mu_{ns}(\lambda-\mu_{ns})} + \cdots\right] \tag{6.41}$$

In addition, near $\lambda = \mu_{ns}$,

$$J_n(\lambda r)J_n(\lambda R) = J_n(\mu_{ns}r)\;J_n(\mu_{ns}R) + (\lambda-\mu_{ns})\;\frac{d}{d\mu_{ns}}\left[J_n(\mu_{ns}r)J_n(\mu_{ns}R)\right] \tag{6.42}$$

and

$$e^{-\lambda x} = e^{-\mu_{ns}x}\left[1 - (\lambda-\mu_{ns})\;x + \ldots\right]. \tag{6.43}$$

Multiplying (6.41), (6.42) and (6.43) together, and taking the coefficient of the $(\lambda - \mu_{ns})^{-1}$ term for the residue in (6.40), we have

$$F_2 = -\pi \sum_{s=1}^{\infty} \frac{e^{-\mu_{ns}x}}{J_n^2(\mu_{ns})} \cdot \left[\frac{1}{\mu_{ns}} - x + \frac{d}{d\mu_{ns}} \right] \cdot J_n(\mu_{ns}r)J_n(\mu_{ns}R) . \tag{6.44}$$

Equation (6.44) is a convenient formula for computing the Fourier transform (6.30) which determines the initial time rate of change of potential in (6.28), except near $x = 0$, where its convergence is slow. A formula analagous to (6.38) could be obtained to compute F_2 near $x = 0$.

VII. SPECIAL CASE OF EQUAL INTERIOR AND EXTERIOR CONDUCTIVITIES

It is possible experimentally to adjust the conductivity of the bathing solution outside the cell in order to achieve equal conductivities inside and outside. In this case, the ratio α is unity, and considerable mathematical simplification of the exact solution (2.28) and (2.29) is attained.

Letting $\alpha = 1$, and using the Wronskian of $I_n(k)$ and $K_n(k)$, the potential (2.28) inside the cell $(0 < r < 1)$ can be reduced to:

$$V(x,r,\theta,t) = \frac{1}{4\pi}\left(x^2 + r^2 + R^2 - 2rR\cos\theta\right)^{-1/2}$$

$$+ \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\, \cos kx \,\frac{k^2 I_n(kr) I_n(kR) K_n'^2(k)}{\varepsilon - k^2 I_n'(k)K_n'(k)}$$

$$\cdot\left[1 - e^{\left(k^2 I_n' K_n' - \varepsilon\right)t/\varepsilon}\right] \tag{7.1}$$

and the potential (2.29) outside the cell $(1 < r < \infty)$ can be reduced to:

$$V(x,r,\theta,t) = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} e^{in\theta}\int_0^{\infty} dk\, \cos kx \,\frac{K_n(kr) I_n(kR)}{\varepsilon - k^2 I_n'(k)K_n'(k)}$$

$$\cdot\left[\varepsilon - k^2 I_n'(k)K_n'(k)\, e^{(k^2 I_n' K_n' - \varepsilon)t/\varepsilon}\right] \tag{7.2}$$

Letting $\alpha = 1$ in the asymptotic expansions in the steady state, (3.21) and (3.28) results in no significant simplification. However, comparing the complexity of (7.1) and (7.2) with the complexity of the exact solution (6.1) for the case $\alpha = 0$, it appears that the same type of analysis as used in Section VI on (6.1) could be applied to (7.1) and (7.2) to obtain asymptotic expansions for the time-dependent potential when $\alpha = 1$. We have not attempted this.

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