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POISSON–NERNST–PLANCK SYSTEMS FOR ION CHANNELS WITH PERMANENT CHARGES*

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Abstract. Ionic channels and semiconductor devices use atomic scale structures to control macroscopic flows from one reservoir to another. The one-dimensional steady-state Poisson-Nernst-Planck (PNP) system is a useful representation of these devices, but experience shows that describing the reservoirs as boundary conditions is difficult. We study the PNP system for two types of ions with three regions of piecewise constant permanent charge, assuming the Debye number is large, because the electric field is so strong compared to diffusion. Reservoirs are represented by the outer regions with permanent charge zero. If the reciprocal of the Debye number is viewed as a singular parameter, the PNP system can be treated as a singularly perturbed system that has two limiting systems: inner and outer systems (termed fast and slow systems in geometric singular perturbation theory). A complete set of integrals for the inner system is presented that provides information for boundary and internal layers. Application of the exchange lemma from geometric singular perturbation theory gives rise to the existence and (local) uniqueness of the solution of the singular boundary value problem near each singular orbit. A set of simultaneous equations appears in the construction of singular orbits. Multiple solutions of such equations in this or similar problems might explain a variety of multiple valued phenomena seen in biological channels, for example, some forms of gating, and might be involved in other more complex behaviors, for example, some kinds of active transport.

Key words. singular perturbation, boundary layers, internal layers

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1. Introduction.

Electrodiffusion, the diffusion and migration of electric charge, plays a central role in a wide range of our technology and science [53, 11, 54, 14, 15, 67, 41]: semiconductor technology controls the migration and diffusion of quasi-particles of charge in transistors and integrated circuits [75, 62, 71], chemical sciences deal with charged molecules in water [11, 19, 8, 26, 9, 10], all of biology occurs in plasmas of ions and charged organic molecules in water [3, 16, 33, 72]. It is no coincidence that the physics of electrodiffusion is of such general importance: systems of moving charge have a richness of behavior that can be sometimes easily controlled by boundary conditions [67, 71], and the goal of technology (and much of physical science) is to control systems to allow useful behavior.

Control is important to the medical and biological sciences as well. Medicine seeks to control disease and help life. Evolution controls life by selecting those organisms that successfully reproduce. Organisms control their internal environment and external behavior to make reproduction possible, often using electrodiffusion for the mechanism of control [72, 33]. Whatever the reason, it is a fact that nearly all biology occurs in ultrafiltrates of blood called plasmas, in which ions move much as they move in gaseous plasmas, or as quasi-particles move in semiconductors [21, 22, 23, 24].

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The pun between the medical and physical meanings of "plasma" is useful and surprisingly precise.

In semiconductor and biological devices, macroscopic flows of charges are driven through tiny (atomic scale) channels that link one macroscopic reservoir to another. The reservoirs are macroscopic regions in which the concentration of charges is nearly constant (because the dimensions of the reservoirs are macroscopic and so the total number of charges is hardly changed by the flows) and electrical potentials are nearly constant too. The electrical resistance of the macroscopic region is so small that only a tiny electrical potential gradient is needed to drive significant flow of charge in the reservoir. The electric field is strong throughout these systems and only a few charges (ions) are needed to create significant electrical potentials, compared to the enormous number of ions (10^{23}, Avogadro's number) needed to create chemical potentials (and diffusion). That is why the Debye number is so large (see systems (1) and (5)). Semiconductors and evolution take advantage of the strength of the electric field. Engineers and biophysicists control flow by setting the electric potential at the boundaries called terminals, contacts, or baths.

The flow through the atomic scale channel is affected by other variables besides the applied boundary potentials, namely, by the shape of the pore in the channel (through which permanently charged ions flow) and the distribution of permanent and induced (i.e., polarization) charge on the wall of the channel as well as the mobility of ions [49, 25, 17, 40, 32]. A precise description and understanding of flow on an atomic scale is daunting. Enormous numbers of variables are needed to describe atomic scale trajectories that have a fundamental time scale 10^{-16} sec and length scale 10^{-10} m compared to biological function that is typically much slower than 10^{-5} sec. It is not clear what to do with this number of variables and trajectories even if they could be computed accurately or with known inaccuracies.

We are fortunate that description on the atomic time scale is unnecessary. What is needed in fact is a reduced description that focuses attention on the properties that control function in technology and biology. This reduced description needs to describe channel structure on the atomic scale of distance, in all likelihood, but it needs to describe flows and reservoirs only on the macroscopic scale.

Reduced descriptions of this type are familiar in engineering where they are called device equations. Semiconductor manufacturers produce the device behaviors they need by choosing particular structures of permanent charge, using as little atomic structure as possible, so cost is minimized. Device behaviors are described by device equations. It is device equations that we seek as we try to understand and control ion channels (and molecular machines of biology in general).

Device equations are most useful when they predict complex behaviors realistically while using only a few parameters with fixed values (that do not need to be changed to describe the complex behaviors). Fortunately, electrodiffusion allows rich behavior with simple device equations and a fixed set of parameters. Remarkably, the diverse (technologically important) behavior of transistors can be described by simple conservation laws and constitutive relations, the Poisson–Nernst–Planck (PNP) equations using fixed values of parameters. A single transistor can behave as many different devices, each with its own device equation, and this rich behavior can be described quite well by the PNP equations with a fixed set of parameters. Different values of the boundary potentials (i.e., power supply voltages) move the solution of the equations into different domains, each with its own device equation.

The PNP system of equations has been analyzed mathematically to some extent, but the equations have been simulated and computed to a much larger extent [18, 6,

16, 38, 49, 4, 17, 66, 36, 37, 40, 2, 20, 31, 55, 56, 1, 13, 70, 65, 29, 30, 43, 7]. Computational and experimental experience with a variety of PNP-like systems shows that the existing mathematical analysis is unsatisfactory. It is clear from these simulations that macroscopic reservoirs must be included in the mathematical formulation to describe the actual behavior of channels (or useful transistors) [60, 32, 31, 12, 59, 57, 58, 34, 29]. Macroscopic boundary conditions that describe such reservoirs introduce boundary layers of concentration and charge. If those boundary layers reach into the part of the device performing atomic control, they dramatically affect its behavior. Boundary layers of charge are particularly likely to create artifacts over long distances because the electric field spreads a long way. Indeed, transistors, channels, transporters, and receptors are actually built so that the contacts, electrodes, and control systems that maintain the reservoirs are quite distant and distinct from the channel.

In this paper, we construct and analyze the minimal model that includes reservoirs and channels and start the study of its mathematical properties. We begin with simple setups and conditions using geometric singular perturbation theory to extract powerful results. In particular, we consider three regions, two of which are reservoirs, and one of which is the narrow channel (with permanent charge, i.e., doping). And we consider only two species of current carriers. Nonetheless, we find quite complex behavior showing clearly that the reservoirs are inextricably linked to the channel and cannot be replaced by simple boundary conditions. We find general properties of the system and hints that somewhat more complicated systems (with several regions of permanent charge of different density and/or sign) carrying multiple ionic species (with different valence, i.e., with different permanent charge on each type of ion) may have quite rich behavior. Such rich behavior is apparent in biology where channels switch ("gate") between different values of current (one value nearly zero) and where transporters couple the flow of different types of ions in an extremely important, quite robust, but nearly unknown way.

The rest of the paper is organized as follows. In section 2, we begin with a description of a three-dimensional PNP system as the model for ion flow through an ion channel and discuss a one-dimensional reduction as the maximal radius of cross-sections of the channel approaching zero. We then identify the problem to be studied in this paper: steady-states of boundary value problems of the one-dimensional PNP system. In section 3, we cast our problem in the language of geometric singular perturbation theory. By introducing new dependent variables, we write the PNP system as a singularly perturbed system of first order equations. Making use of the inner and outer limiting systems, we then construct singular orbits for the PNP boundary value problem. In section 4, we apply geometric singular perturbation theory to show that, for small $\epsilon > 0$, there is a true solution shadowing each singular orbit. We conclude the paper by a general remark in section 5.

2. Three-dimensional model PNP system and a one-dimensional reduction. We now briefly describe the model PNP system of equations. As discussed above, the key features of an ion channel are the shape of its pore and the distribution of the permanent charge along its interior wall. As a first approximation, we consider a special ion channel modeled by

$$\Omega_\mu = \{(x, y, z) : 0 < x < 1,\ y^2 + z^2 < g^2(x, \mu)\},$$

where g is a smooth function satisfying

$$g(x, 0) = 0 \ \text{ and } \ g_0(x) = \frac{\partial g}{\partial \mu}(x, 0) > 0 \ \text{ for } \ x \in [0, 1].$$

The small parameter μ measures the maximal radius of cross-sections of the channel. The boundary $\partial\Omega_\mu$ of Ω_μ consists of three portions:

$$\begin{aligned}\mathcal{L}_\mu &= \{(x,y,z) \in \partial\Omega_\mu : x = 0\},\\ \mathcal{R}_\mu &= \{(x,y,z) \in \partial\Omega_\mu : x = 1\},\\ \mathcal{M}_\mu &= \{(x,y,z) \in \partial\Omega_\mu : y^2 + z^2 = g^2(x,\mu)\}.\end{aligned}$$

Here, $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$ are viewed as the two ends of the reservoirs and $\mathcal{M}_\mu$ the wall of the channel and the reservoirs.

Then the model employed for flow through the channel is the PNP system (see [5] for a derivation from Boltzmann transport equation; see [66] for a derivation including correlations from coupled Langevin–Poisson equations; see [11, p. 773, eq. 26.64] for the classical description of the system at thermodynamic equilibrium, when all fluxes are zero):

$$\begin{aligned}\Delta\phi &= -\lambda\left(\sum_{i=1}^{n}\alpha_i c_i + Q\right),\\ \frac{\partial c_i}{\partial t} &= D_i\nabla\cdot(\nabla c_i + \alpha_i c_i\nabla\phi),\end{aligned} \tag{1}$$

where ϕ is the electric potential; c_i's are the concentrations of the n species, and α_i's are the valences, i.e., charge on one ion; D_i's are the diffusion constants; λ is the Debye number; and Q is the distribution of the permanent charge along the interior wall of the channel.

As mentioned in the introduction, the concentrations of the ions and the electrical potential in the reservoirs are nearly constants, and the wall of the channel is assumed to be perfectly insulated. We thus assume the following boundary conditions:

$$\begin{gathered}\phi|_{\mathcal{L}_\mu} = \nu_0, \quad \phi|_{\mathcal{R}_\mu} = 0, \quad c_i|_{\mathcal{L}_\mu} = L_i, \quad c_i|_{\mathcal{R}_\mu} = R_i,\\ \frac{\partial\phi}{\partial\mathbf{n}}\Big|_{\mathcal{M}_\mu} = \frac{\partial c_k}{\partial\mathbf{n}}\Big|_{\mathcal{M}_\mu} = 0,\end{gathered} \tag{2}$$

where ν_0, L_i, R_i are constants and $\mathbf{n}$ is the outward unit normal vector to $\mathcal{M}_\mu$.

We remark that, typically in the reservoirs, one imposes electroneutrality conditions: $\alpha L_1 - \beta L_2 = 0$ and $\alpha R_1 - \beta R_2 = 0$. In this case, there will be no boundary layers at the two ends although there will be internal layers *where* the permanent charge Q jumps. For mathematical interest, we use the slightly more general boundary conditions.

In [52], for $n = 2$ with $Q = 0$, we obtained the following limiting one-dimensional PNP system as $\mu \to 0$:

$$\begin{aligned}\frac{1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}\phi\right) &= -\lambda(\alpha_1 c_1 + \alpha_2 c_2),\\ \frac{\partial c_1}{\partial t} &= \frac{D_1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}c_1 + \alpha_1 c_1 g_0^2\frac{\partial}{\partial x}\phi\right),\\ \frac{\partial c_2}{\partial t} &= \frac{D_2}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}c_2 + \alpha_2 c_2 g_0^2\frac{\partial}{\partial x}\phi\right)\end{aligned} \tag{3}$$

on $x \in (0,1)$ with the boundary conditions

$$\phi(t,0) = \nu_0, \quad \phi(t,1) = 0, \quad c_i(t,0) = L_i, \quad c_i(t,1) = R_i. \tag{4}$$

In particular, we showed that the attractors $\mathcal{A}_\mu$ of (1) and (2) are upper semicontinuous at $\mu = 0$ to the attractor $\mathcal{A}_0$ of (3) and (4). One-dimensional PNP systems of the form (3) also arise in treatments based on the density functional theory of statistical mechanics [31]. The motivation for such a mathematical treatment is that, first of all, the one-dimensional system is much simpler; second, if the one-dimensional limiting system is structurally stable (i.e., if the global dynamics is robust), then the dynamics for the system on the three-dimensional domain with small μ is essentially the same as that of the limiting one-dimensional system. There is a well-established framework for verification of structural stability although it is by no means trivial. A key step is to understand the behavior of the steady-state of the limiting one-dimensional system.

In light of the above result and discussion, we will then study steady-states of the one-dimensional PNP system for two species of current carriers with valences $\alpha > 0$ and $-\beta < 0$, including now a permanent charge:

$$\begin{aligned} \epsilon^2 h^{-1}(x)\frac{d}{dx}\left(h(x)\frac{d}{dx}\phi\right) &= -(\alpha c_1 - \beta c_2 + Q(x)), \quad \frac{dJ_i}{dx} = 0, \\ h(x)\frac{dc_1}{dx} + \alpha c_1 h(x)\frac{d\phi}{dx} &= -J_1, \\ h(x)\frac{dc_2}{dx} - \beta c_2 h(x)\frac{d\phi}{dx} &= -J_2, \end{aligned} \tag{5}$$

with the boundary conditions

$$\phi(0) = \nu_0, \quad c_i(0) = L_i; \quad \phi(1) = 0, \quad c_i(1) = R_i. \tag{6}$$

Here J_i is the total flux of the ith ion, $Q(x)$ is the permanent charge along the channel, $h(x) = g_0^2(x)$, and ϵ is related to λ via $\lambda = \epsilon^{-2}$.

Many mathematical papers have been written about the existence and uniqueness of solutions of the boundary value problems, and numerical algorithms have been developed to approximate solutions even for high-dimensional systems (see, e.g., [39, 42, 61, 44]). Under the assumption that $\epsilon \ll 1$, the problem can be viewed as a singular perturbation one. In particular, for $\alpha = \beta = 1$, $h(x) = 1$, and $Q(x) = 0$, the boundary value problem for the one-dimensional PNP system (5) was studied in [7] using the method of matched asymptotic expansions as well as numerical simulations, which provide a good quantitative understanding of the problem with one region without permanent charge. In [51], assuming $\epsilon \ll 1$ but for general α, β, $h(x) = 1$ and $Q(x) = 0$, the boundary value problem was treated using geometric theory for singularly perturbed problems (see, e.g., [27, 45, 47, 50]).

We use the geometric framework in paper [51] to investigate PNP systems with multiple regions of permanent charge and with multiple ions. A major difference of the model studied in this paper from those previously studied is the inclusion of multiple regions of permanent charge. The focus will be on the simple case of two ions and two reservoirs (i.e., two regions without permanent charge). The idea is to construct singular orbits for the boundary value problem and apply geometric singular perturbation theory to obtain, for $\epsilon > 0$ small, solutions near singular orbits. Issues of the existence and multiplicity of singular orbits are reduced to the properties of a set of nonlinear algebraic equations (43). To our surprise, for the simple case we study, multiple solutions for the boundary value problem are shown to exist. This contrasts to what was suspected in some early works (see, for example, [63, 64]) which expressed the (entirely reasonable) opinion that multiple solutions cannot occur for the simple

structure of permanent charge considered here. The set of equations (43) governs the multiplicity of solutions to the boundary value problem. We will thoroughly examine the set of algebraic equations in the future.

3. A dynamical system framework and a construction of singular orbits. We will rewrite the PNP system into a standard form for singularly perturbed systems and convert the boundary value problem to a connecting problem.

Denote the derivative with respect to x by overdot and introduce $u = \epsilon\dot{\phi}$ and $\tau = x$. System (5) becomes

$$\begin{aligned} &\epsilon\dot{\phi} = u, \quad \epsilon\dot{u} = \beta c_2 - \alpha c_1 - Q(\tau) - \epsilon\frac{h'(\tau)}{h(\tau)}u, \\ &\epsilon\dot{c}_1 = -\alpha c_1 u - \epsilon h^{-1}(\tau)J_1, \\ &\epsilon\dot{c}_2 = \beta c_2 u - \epsilon h^{-1}(\tau)J_2, \\ &\dot{J}_1 = \dot{J}_2 = 0, \quad \dot{\tau} = 1. \end{aligned} \tag{7}$$

We will treat system (7) as a dynamical system of phase space $\boldsymbol{R}^7$ with state variables $(\phi, u, c_1, c_2, J_1, J_2, \tau)$. The introduction of the extra state variable $\tau = x$ and the τ-equation seems to add complications to the problem, but this has a great advantage that we will explain shortly.

For $\epsilon > 0$, the rescaling $x = \epsilon\xi$ of the independent variable x gives rise to an equivalent system

$$\begin{aligned} &\phi' = u, \quad u' = \beta c_2 - \alpha c_1 - Q(\tau) - \epsilon\frac{h'(\tau)}{h(\tau)}u, \\ &c_1' = -\alpha c_1 u - \epsilon h^{-1}(\tau)J_1, \\ &c_2' = \beta c_2 u - \epsilon h^{-1}(\tau)J_2, \\ &J_1' = J_2' = 0, \quad \tau' = \epsilon, \end{aligned} \tag{8}$$

where prime denotes the derivative with respect to the variable ξ.

For $\epsilon > 0$, systems (7) and (8) have exactly the same phase portrait. But their limits at $\epsilon = 0$ are different and, very often, the two limiting systems provide complementary information on state variables. Therefore, the main task of singularly perturbed problems is to patch the limiting information together to form a solution for the entire $\epsilon > 0$ system. In terms of asymptotic expansions, system (7) and its limit at $\epsilon = 0$ will be used to study *outer or regular layer* solutions. We will call this system the *outer system* and its limit at $\epsilon = 0$ the *outer limit system*. System (8) and its limit at $\epsilon = 0$ will be used to study *inner or singular layer* solutions, and we call the system the *inner system* and its limit system at $\epsilon = 0$ the *inner limit system*. By *a singular orbit*, we mean a continuous and piecewise smooth curve in $\boldsymbol{R}^7$ that is a union of finitely many orbits of the outer limit system or inner limit system. In the theory of geometric singular perturbations, viewing the independent variables x and ξ as slow and fast time variables, the outer system is called the *slow system*, the inner system is called the *fast system*, and a singular orbit is a union of *slow* and *fast* orbits.

Let B_L and B_R be the subsets of the phase space $\boldsymbol{R}^7$ defined by

$$B_L = \{(\nu_0, u, L_1, L_2, J_1, J_2, 0) \in \boldsymbol{R}^7 : \text{ arbitrary } u, J_1, J_2\}, \tag{9}$$

$$B_R = \{(0, u, R_1, R_2, J_1, J_2, 1) \in \boldsymbol{R}^7 : \text{ arbitrary } u, J_1, J_2\}. \tag{10}$$

Then the boundary value problem is equivalent to a *connecting problem*, namely, finding a solution of (7) or (8) from B_L to B_R. To see this, suppose that $(\phi, u, c_1, c_2, J_1, J_2, \tau)$ is an orbit starting at a point on B_L and ending at a point on B_R. Due to the definitions of B_L and B_R, the starting point automatically has $x = \tau = 0$ with the assigned values for ϕ, c_1, and c_2 at $x = 0$, and the ending point has $x = \tau = 1$ with the assigned values for ϕ, c_1, and c_2 at $x = 1$. This solution $(\phi, u, c_1, c_2, J_1, J_2, \tau)$ satisfies the boundary condition automatically. Most importantly, when we *arbitrarily* rescale the independent variable x, the phase portrait will remain the same. Therefore, in searching for a solution from B_L to B_R, we can apply any rescaling of the independent variable x, even a rescaling that depends on each individual solution. (We will use a rescaling that is different for each solution when we derive the system (34) from system (33).) This is the significant advantage of introducing $\tau = x$ and $\dot{\tau} = 1$ promised earlier. The idea of converting a boundary value problem to a connecting one is now rather standard in applied dynamical systems.

In this paper, we will consider the case where the outer regions are reservoirs and the permanent charge is constant along the channel; that is,

$$Q(x) = \begin{cases} 0 & \text{for } 0 < x < a, \\ Q & \text{for } a < x < b, \\ 0 & \text{for } b < x < 1, \end{cases}$$

where Q is a constant. The intervals $[0, a]$ and $[b, 1]$ are the reservoirs, and the interval $[a, b]$ is the channel.

We will be interested in solutions of the connecting problem for system (7) or (8) from B_L to B_R defined in (9) and (10). In view of the jump of Q at $x = a$ and $x = b$, the best one can hope is that the solution is continuous and piecewise differentiable. We therefore require our solutions to be *continuous and piecewise differentiable*. The continuity of u implies that ϕ, c_1, and c_2 are differentiable. Our requirement is motivated by two considerations: (i) the dissipation present in the full PNP system (that includes time evolution) improves the regularity of solutions; in particular, the attractor contains regular solutions. Steady-state solutions, being in the attractor, should have the regularity imposed; (ii) if the requirement is relaxed, say, only requiring ϕ, c_1, c_2 to be piecewise differentiable, then one can preassign any value for (ϕ, c_1, c_2) at any partition points $0 < x_1 < x_2 < \cdots < x_k < 1$ and construct solutions over each subinterval and piece them together to create a solution over $[0, 1]$ with the preassigned values for (ϕ, c_1, c_2) at the partition points. (This assertion follows from the work in [7, 51].) It is clear that the only relevant solutions are those in which ϕ, c_1, c_2 are differentiable.

Our construction of a solution involves two main steps: the first step is to construct a singular orbit to the connecting problem, and the second step is to apply geometric singular perturbation theory to show that there is a unique solution near the singular orbit for $\epsilon > 0$ and small. Here we will give a detailed explanation for the first step and leave the explanation of the second step to section 4.

To construct a singular orbit, we first construct a singular orbit on each of the subinterval $[0, a]$, $[a, b]$, and $[b, 1]$. The reason to split the interval $[0, 1]$ into three subintervals is simply because the permanent charge $Q(x)$ has jumps at $x = a$ and $x = b$. To be able to construct a singular orbit on each subinterval, we need to preassign the values of ϕ, c_1, and c_2 at $x = a$ and $x = b$. Suppose, for the moment, that $\phi = \phi^a$, $c_1 = c_1^a$, and $c_2 = c_2^a$ at $x = a$, and that $\phi = \phi^b$, $c_1 = c_1^b$, and $c_2 = c_2^b$

at $x = b$. Those *six* unknown values

$$\phi^a,\ c_1^a,\ c_2^a;\ \phi^b,\ c_1^b,\ c_2^b \tag{11}$$

will be determined along our construction of a singular orbit on the whole interval $[0, 1]$.

1. On the left subinterval $[0, a]$ where $Q = 0$ or there is no permanent charge, we construct a singular orbit for the boundary value problem with (ϕ, c_1, c_2, τ) being

$$(\nu_0, L_1, L_2, 0) \text{ at } x = 0 \text{ and } (\phi^a, c_1^a, c_2^a, a) \text{ at } x = a.$$

   The orbit consists of two boundary layers Γ_l^0 and Γ_l^a and one regular layer Λ_l. In particular, given (ϕ^a, c_1^a, c_2^a), the flux densities J_1^l, J_2^l and the value $u_l(a)$ are uniquely determined (see section 3.1).
2. On the middle subinterval $[a, b]$, we construct a singular orbit for the boundary value problem with (ϕ, c_1, c_2, τ) being

$$(\phi^a, c_1^a, c_2^a, a) \text{ at } x = a \text{ and } (\phi^b, c_1^b, c_2^b, b) \text{ at } x = b.$$

   The orbit consists of two boundary layers Γ_m^a and Γ_m^b and one regular layer Λ_m. In particular, given (ϕ^a, c_1^a, c_2^a) and (ϕ^b, c_1^b, c_2^b), the flux densities J_1^m, J_2^m and the values $u_m(a)$ and $u_m(b)$ are uniquely determined (see section 3.2).
3. On the right subinterval $[b, 1]$, we construct a singular orbit for the boundary value problem with (ϕ, c_1, c_2, τ) being

$$(\phi^b, c_1^b, c_2^b, b) \text{ at } x = b \text{ and } (0, R_1, R_2, 1) \text{ at } x = 1.$$

   The orbit again consists of two boundary layers Γ_r^b and Γ_r^1 and one regular layer Λ_r. In particular, given (ϕ^b, c_1^b, c_2^b), the flux densities J_1^r, J_2^r and the value $u_r(b)$ are uniquely determined (see section 3.3).
4. Finally, for a singular orbit on the whole interval $[0, 1]$, we require that

$$J_1^l = J_1^m = J_1^r, \quad J_2^l = J_2^m = J_2^r, \quad u_l(a) = u_m(a), \quad u_m(b) = u_r(b).$$

   This consists of *six* conditions. *The number of conditions is exactly the same as the number of unknown values in* (11) (see section 3.4).
   The qualitative properties of these six equations and conditions are of great importance. It turns out that they can have multiple solutions. Different solutions yield different amounts of current for otherwise identical conditions, suggesting that each level might correspond to a different functional state of a transporter, or a different gating state of a channel. Indeed, it seems likely that more complex systems than those considered here would be described by similar systems of equations with multiple solutions. Interesting and very important properties of channels and transporters—each corresponding to a quite distinct device with a quite distinct input-output relation and device equation—might arise this way in systems including Ca^{2+} or in systems with multiple regions of nonzero permanent charge, or in systems with branched, Y-shaped, or adjacent interacting channels.

Remark 3.1. We call Γ_l^a, Γ_m^a, Γ_m^b, and Γ_r^b boundary layers because, relative to each subinterval, they are boundary layers. But, relative to the whole interval $[0, 1]$, they should be termed *internal layers*.

3.1. Singular orbit on $[0, a]$ where $Q(x) = 0$. We consider the case with zero permanent charge on the subinterval $[0, a]$ because $[0, a]$ is viewed as one of the reservoirs. The nonzero Q over the subinterval $[a, b]$ will affect the solution on $[0, a]$ and on $[b, 1]$. This effect will show up when we impose *matching* conditions on ϕ^a, c_1^a, and c_2^a to construct the singular orbit over the *whole* interval $[0, 1]$.

Following the discussion above, we set $\phi(a) = \phi^a$, $c_1(a) = c_1^a$, and $c_2(a) = c_2^a$, where ϕ^a, c_i^a are unknown values to be determined later on. Now let

$$B_a = \{(\phi^a, u, c_1^a, c_2^a, J_1, J_2, a) \in \boldsymbol{R}^7 : u, J_i \text{ arbitrary}\}.$$

In this part, we will construct a singular orbit that connects B_L to B_a. Two boundary layers will be constructed in section 2.1.1 followed by the construction of the regular layer in section 2.1.2. The permanent charge Q is zero in both constructions.

If we set $\epsilon = 0$ in system (7) with $Q(x) = 0$, we get the outer limit system and, in particular, $u = 0$ and $\alpha c_1 = \beta c_2$. The set

$$\mathcal{Z}_l = \{u = 0, \alpha c_1 = \beta c_2\}$$

will be called *the outer manifold*. In the theory of geometric singular perturbations, $\mathcal{Z}_l$ is called *the slow manifold* because if x and ξ are viewed as time variables, the evolution on $\mathcal{Z}_l$ is characterized by the time variable ξ, which is slow.

Remark 3.2. In systems (7) and (8), there appear to be four fast equations and three slow equations. Typically, one would expect a three-dimensional slow manifold. But, in this specific problem, the slow manifold is five-dimensional. This fact indicates some degeneracy of the slow flow, which is reflected in sections 3.1.2 and 3.2.2. The exchange lemma applied in the proof of Theorem 4.1 in section 4 is still valid. In fact, it applies to singular perturbation problems of more general forms than standard ones (see, e.g., [46, p. 562, Remark 1]).

The geometric method for a construction of singular orbits on each of the subintervals $[0, a]$, $[a, b]$, and $[b, 1]$ is the same. Let us explain the approach for constructing the singular orbit that connects B_L to B_a on $[0, a]$ (see Figure 1). Generally, the outer manifold $\mathcal{Z}_l$ will not intersect B_L and B_a. Since every outer or regular layer orbit lies entirely on the outer manifold $\mathcal{Z}_l$, it will not intersect B_L and B_a; that is, it cannot satisfy the boundary conditions. Two boundary or inner layers need to be introduced to connect boundaries B_L and B_a with the outer layer solution on $\mathcal{Z}_l$. These boundary layers should satisfy the inner limit system. The boundary layer orbit Γ_l^0 at $x = 0$ will connect B_L to $\mathcal{Z}_l$. It must lie on the stable manifold $W^s(\mathcal{Z}_l)$; that is, it belongs to the intersection $M_L \cap W^s(\mathcal{Z}_l)$, where M_L is the collection of orbits starting from points on B_L. Similarly, the boundary layer Γ_l^a at $x = a$ will connect $\mathcal{Z}_l$ to B_a and it must lie on the unstable manifold $W^u(\mathcal{Z}_l)$; that is, it belongs to the intersection $M_l^a \cap W^u(\mathcal{Z}_l)$, where M_l^a is the collection of orbits starting from points on B_l^a.

The first step in the construction examines the stability of the outer manifold $\mathcal{Z}_l$ by linearizing along $\mathcal{Z}_l$. ($\mathcal{Z}_l$ is the set of equilibria of the inner limit system.) It turns out that the outer manifold $\mathcal{Z}_l$ has a stable manifold $W^s(\mathcal{Z}_l)$ and an unstable manifold $W^u(\mathcal{Z}_l)$. The next step is to check whether $W^s(\mathcal{Z}_l)$ intersects B_L and whether $W^u(\mathcal{Z}_l)$ intersects B_a. This requires concrete knowledge of the *global* behavior of $W^s(\mathcal{Z}_l)$ and $W^u(\mathcal{Z}_l)$, and the information from the linearization is not enough. Neither is abstract dynamical systems theory (since the inner limit system is *nonlinear*). Luckily, we discovered a complete set of integrals for the inner limit system (see Proposition 3.2). The set of integrals reflects the intrinsic *mathematical*

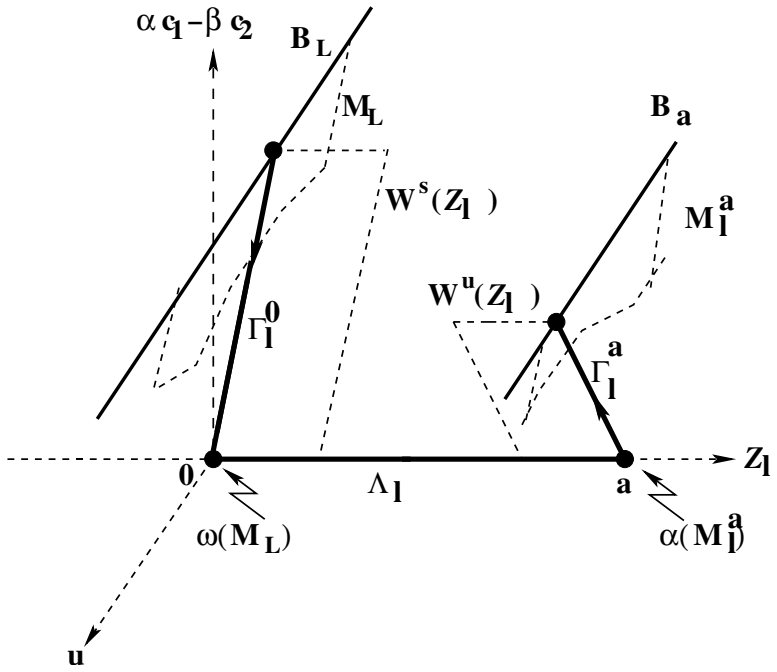

FIG. 1. *Schematic picture of the singular orbit (solid curves) on* $[0,a]$*: one left boundary layer* Γ_l^0*, one regular layer* Λ_l*, and one right boundary layer* Γ_l^a*.*

structure of this particular electrodiffusion system, the channel problem. This mathematical special structure implies particular specific physical and chemical properties of the ion channel. It is irresistible, albeit speculative, to suspect that the special mathematical structure produces biologically important properties of the channel. In that sense, the mathematical structure of the problem provides one possible "device equation" for the channel system.

It is this set of integrals in Proposition 3.2 that allows us to give a complete, global description of the inner limit dynamics; in particular, we are able to establish the required intersections $M_L \cap W^s(\mathcal{Z}_l)$ and $M_l^a \cap W^u(\mathcal{Z}_l)$ and are also able to identify the so-called ω-limit set $\omega(M_L \cap W^s(\mathcal{Z}_l))$ and the α-limit set $\alpha(M_l^a \cap W^u(\mathcal{Z}_l))$ of the intersections. The intersections give the set of candidates for the boundary layers (consisting of two parameter families of inner orbits parameterized by J_1 and J_2). The foot points $\omega(M_L \cap W^s(\mathcal{Z}_l))$ and $\alpha(M_l^a \cap W^u(\mathcal{Z}_l))$ (each parameterized by J_1 and J_2 also) on $\mathcal{Z}_l$ provide the (reduced) boundary conditions for the outer solutions. It turns out there is only one outer orbit Λ_l that connects $\omega(M_L \cap W^s(\mathcal{Z}_l))$ to $\alpha(M_l^a \cap W^u(\mathcal{Z}_l))$ and also determines the pair (J_1, J_2) uniquely. The desired singular orbit connecting B_L to B_a on $[0,a]$ is formed by this outer orbit Λ_l together with the two boundary layers Γ_l^0 and Γ_l^a that are uniquely determined by the pair (J_1, J_2).

We remind the reader that the singular orbit to be constructed on this subinterval with zero permanent charge will *not* be complete *until* the unknowns in (11) are determined through matching conditions implicitly posed by the permanent charge Q on the *whole* interval $[0,1]$, including the channel region where the permanent charge is not zero. The entire system is coupled and must be solved together, suggesting the source of difficulties with earlier treatments, which tried to replace the reservoirs with

boundary conditions. The importance of the coupling of different intervals suggests that the shapes of antechambers commonly found in biological channels may be important to their function. It is interesting that synthetic nanochannels acquire some properties of biological channels when they are built with antechambers of specific shape [68, 69].

3.1.1. Inner dynamics on $[0, a]$: Boundary layers or inner solutions. We start with the examination of boundary layers on the interval $[0, a]$ where $Q = 0$. These will be studied using the inner limit system obtained by setting $\epsilon = 0$ in (8):

$$\begin{aligned} &\phi' = u, \quad u' = \beta c_2 - \alpha c_1, \\ &c_1' = -\alpha c_1 u, \\ &c_2' = \beta c_2 u, \\ &J_1' = J_2' = 0, \quad \tau' = 0. \end{aligned} \tag{12}$$

This inner limit system describes what a chemist would call (thermodynamic) equilibrium. The reader should be warned that the word "equilibrium" is used widely, albeit informally, in computational electronics to describe a system *not* at thermodynamic equilibrium, namely, a system in which the distribution of velocities is a displaced Maxwellian, with displacement given by the flux (in appropriate units). Only when the flux of every species is zero is the "equilibrium" of computational electronics a thermodynamic equilibrium.

The set of equilibria of (12), that is, the set of points at which the vector field of (12) vanishes, is precisely $\mathcal{Z}_l = \{u = 0, \alpha c_1 = \beta c_2\}$. The linearization at points $(\phi, 0, c_1, c_2, J_1, J_2, \tau) \in \mathcal{Z}_l$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & \beta & 0 & 0 & 0 \\ 0 & -\alpha c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This linearization is similar to the Green–Kubo expansion used by physical chemists to describe a nonequilibrium system close to equilibrium [11, 48, 74, 76]. Of course, such a linearization is useful only around some specific (operating) point; here the thermodynamic operating point with zero fluxes. To study nonlinear behavior far from the thermodynamic operating point, one must do a linearization around other points, at which fluxes are not zero. Such analyses have not been attempted, as far as we know for the PNP system, or in physical chemistry in general, perhaps because the locations and properties of operating points other than the thermodynamic one are hard to specify simply. Linearization around general *nonequilibrium* operating points is a crucial method in electrical engineering and has been used to design nonlinear circuits since the invention of electron valves—i.e., vacuum tubes—in the 1930s.

The linearized system has five zero eigenvalues whose generalized eigenspace is the tangent space of the five-dimensional outer manifold $\mathcal{Z}_l$ of equilibria. The two other eigenvalues are $\pm\sqrt{(\alpha + \beta)\alpha c_1} \neq 0$ whose eigenvectors are not tangent to $\mathcal{Z}_l$.

In this sense, $\mathcal{Z}_l$ is called *normally hyperbolic*. The theory of normally hyperbolic invariant manifolds (e.g., [27]) states that

(i) there is a six-dimensional stable manifold $W^s(\mathcal{Z}_l)$ of $\mathcal{Z}_l$ that consists of points approaching $\mathcal{Z}_l$ in forward time;

(ii) there is a six-dimensional unstable manifold $W^u(\mathcal{Z}_l)$ of $\mathcal{Z}_l$ that consists of points approaching $\mathcal{Z}_l$ in backward time;

(iii) $\mathcal{Z}_l$ as well as $W^s(\mathcal{Z}_l)$ and $W^u(\mathcal{Z}_l)$ persists for $\epsilon > 0$ small; that is, for $\epsilon > 0$ small, there exist invariant manifolds $\mathcal{Z}_l^\epsilon$, $W^s(\mathcal{Z}_l^\epsilon)$, and $W^u(\mathcal{Z}_l^\epsilon)$, close to their counterparts.

What this result suggests is that, for a singular orbit connecting B_L to B_a, the boundary layer at $x = 0$ must lie in $M_L \cap W^s(\mathcal{Z}_l)$ and the boundary layer at $x = a$ must lie in $M_l^a \cap W^u(\mathcal{Z}_l)$, where M_L is the collection of orbits from B_L in forward time under the flow of system (12) and M_l^a is the collection of orbits from B_a in backward time under the flow of system (12). This is precisely what we will show.

DEFINITION 3.1. *A smooth function $H : \boldsymbol{R}^n \to \boldsymbol{R}$ is called* an integral *of system $\frac{d}{dt}z = f(z)$, $z \in \boldsymbol{R}^n$, if $\frac{d}{dt}[H(z(t))] = 0$ whenever $z(t)$ is a solution.*

For a system on $\boldsymbol{R}^n$, if there are $(n-1)$ (independent) integrals, then any orbits can be theoretically determined by the intersections of $(n-1)$ level sets of the integrals.

PROPOSITION 3.2. *System* (12) *has the following six integrals:*

$$H_1 = e^{\alpha\phi}c_1, \quad H_2 = e^{-\beta\phi}c_2, \quad H_3 = c_1 + c_2 - \frac{1}{2}u^2,$$
$$H_4 = J_1, \quad H_5 = J_2, \quad H_6 = \tau.$$

Proof. The proof can be verified directly. ∎

The reader seeking physical insight is reminded that α is the valence (i.e., charge) of the ions with number density c_1; $(-\beta)$ is the charge of the ions with number density c_2, $u = \epsilon\dot{\phi}$, $\tau = x$; and ϵ is the Debye length.

These integrals allow one to completely understand the boundary layers (at $x = 0, a$) and characterize landing points of boundary layers on the outer manifold $\mathcal{Z}_l$. The information on landing points is crucial because it provides the boundary conditions that allow the regular layer to connect boundary layers.

COROLLARY 3.3. (i) *Let $\phi = \phi^L$ be the unique solution of*

$$\alpha L_1 e^{\alpha(\nu_0-\phi)} - \beta L_2 e^{-\beta(\nu_0-\phi)} = 0, \quad \textit{that is, } \phi^L = \nu_0 - \frac{1}{\alpha+\beta}\ln\frac{\beta L_2}{\alpha L_1},$$

and let

$$c_1^L = \frac{1}{\alpha}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^L = \frac{1}{\beta}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}}.$$

The stable manifold $W^s(\mathcal{Z}_l)$ intersects B_L transversally at points with

$$\begin{aligned} u_0 &= [sgn(\phi^L - \nu_0)]\sqrt{2(L_1+L_2) - 2(L_1 e^{\alpha(\nu_0-\phi^L)} + L_2 e^{-\beta(\nu_0-\phi^L)})} \\ &= [sgn(\alpha L_1 - \beta L_2)]\sqrt{2\left(L_1 + L_2 - \frac{\alpha+\beta}{\alpha\beta}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}}\right)} \end{aligned} \tag{13}$$

and arbitrary J_i's, where sgn is the sign function (see Figure 1*).*

Let $\phi = \phi^{a,l}$ be the unique solution of

$$\alpha c_1^a e^{\alpha(\phi^a-\phi)} - \beta c_2^a e^{-\beta(\phi^a-\phi)} = 0, \quad \textit{that is, } \phi^{a,l} = \phi^a - \frac{1}{\alpha+\beta}\ln\frac{\beta c_2^a}{\alpha c_1^a},$$

and let

$$c_1^{a,l} = \frac{1}{\alpha}(\alpha c_1^a)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^a)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^{a,l} = \frac{1}{\beta}(\alpha c_1^a)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^a)^{\frac{\alpha}{\alpha+\beta}}.$$

The unstable manifold $W^u(\mathcal{Z}_l)$ intersects B_a transversally at points with

$$\begin{aligned} u_l(a) &= [sgn(\phi^a - \phi^{a,l})]\sqrt{2(c_1^a + c_2^a) - 2(c_1^a e^{\alpha(\phi^a-\phi^{a,l})} + c_2^a e^{-\beta(\phi^a-\phi^{a,l})})} \\ &= [sgn(\beta c_2^a - \alpha c_1^a)]\sqrt{2\left(c_1^a + c_2^a - \frac{\alpha+\beta}{\alpha\beta}(\alpha c_1^a)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^a)^{\frac{\alpha}{\alpha+\beta}}\right)} \end{aligned} \tag{14}$$

and arbitrary J_i's (see Figure 1*).*

(ii) *Potential boundary layers Γ_l^0 at $x = 0$ are determined up to (J_1, J_2) as follows: the ϕ-component satisfies the Hamiltonian system*

$$\phi'' + \alpha L_1 e^{\alpha(\nu_0-\phi)} - \beta L_2 e^{-\beta(\nu_0-\phi)} = 0,$$

together with $\phi(0) = \nu_0$ and $\phi(\xi) \to \phi^L$ as $\xi \to \infty$, $u(\xi) = \phi'(\xi)$, and

$$c_1(\xi) = L_1 e^{\alpha(\nu_0-\phi(\xi))}, \quad c_2(\xi) = L_2 e^{-\beta(\nu_0-\phi(\xi))}.$$

Similarly, potential boundary layers Γ_l^a at $x = a$ are determined in the following way: the ϕ-component satisfies the Hamiltonian system

$$\phi'' + \alpha c_1^a e^{\alpha(\phi^a-\phi)} - \beta c_2^a e^{-\beta(\phi^a-\phi)} = 0,$$

together with $\phi(0) = \phi^a$ and $\phi(\xi) \to \phi^{a,l}$ as $\xi \to -\infty$, $u(\xi) = \phi'(\xi)$, and

$$c_1(\xi) = c_1^a e^{\alpha(\phi^a-\phi(\xi))}, \quad c_2(\xi) = c_2^a e^{-\beta(\phi^a-\phi(\xi))}.$$

(iii) *Let $N_L = M_L \cap W^s(\mathcal{Z}_l)$ and $N_l^a = M_l^a \cap W^u(\mathcal{Z}_l)$. Then,*

$$\omega(N_L) = \left\{\left(\phi^L, 0, c_1^L, c_2^L, J_1, J_2, 0\right) : \text{ all } J_1, J_2\right\},$$

$$\alpha(N_l^a) = \left\{\left(\phi^{a,l}, 0, c_1^{a,l}, c_2^{a,l}, J_1, J_2, a\right) : \text{ all } J_1, J_2\right\},$$

where ϕ^L, c_1^L, c_2^L, $\phi^{a,l}$, $c_1^{a,l}$, and $c_2^{a,l}$ are given explicitly as in part (i)*.*

Proof. We provide a proof for the first part that is related to the boundary layer on the left in each statement.

Let $z(\xi) = (\phi(\xi), u(\xi), c_1(\xi), c_2(\xi), J_1(\xi), J_2(\xi), \tau(\xi))$ be a solution of system (12) with $z(0) \in B_L$ and $z(\xi) \in W^s(\mathcal{Z}_l)$. Then, $J_i(\xi) = J_i$, $\tau(\xi) = 0$ for all ξ, $z(\xi) \to z(\infty) = (\phi^L, 0, c_1^L, c_2^L, J_1, J_2, 0) \in \mathcal{Z}_l$ for some ϕ^L and c_i^L with $\alpha c_1^L = \beta c_2^L$, and

$$\phi(0) = \nu_0, \quad c_1(0) = L_1, \quad c_2(0) = L_2.$$

Using the integrals H_1 and H_2, we have

$$e^{\alpha\phi} c_1 = e^{\alpha\nu_0} L_1, \quad e^{-\beta\phi} c_2 = e^{-\beta\nu_0} L_2.$$

Therefore,

$$c_1 = L_1 e^{\alpha(\nu_0-\phi)}, \quad c_2 = L_2 e^{-\beta(\nu_0-\phi)}. \tag{15}$$

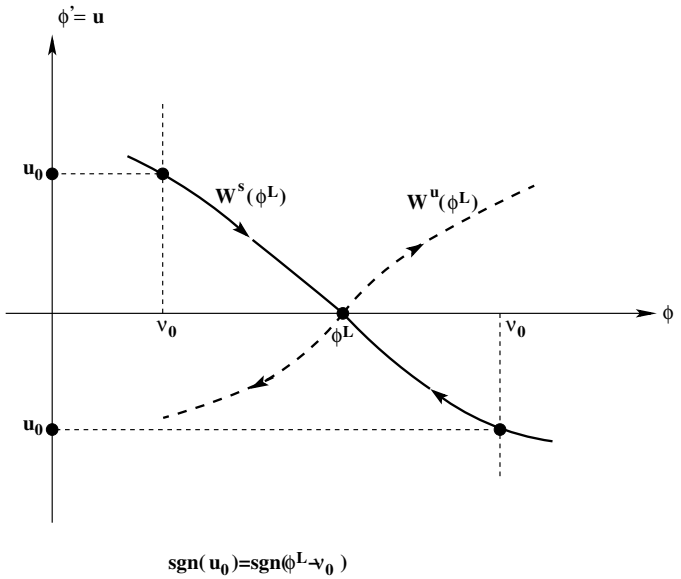

FIG. 2. *The stable manifold $W^s(\phi^L)$ of the equilibrium $(\phi^L, 0)$ is the solid curve, and the unstable manifold $W^u(\phi^L)$ is the dashed curve. The left branch of $W^s(\phi^L)$ has positive u-coordinates, and the right branch has negative u-coordinates; e.g., if $(\phi, u) \in W^s(\phi^L)$, then sign$[u]$ = sign$[\phi^L - \phi]$.*

Taking the limit as $\xi \to \infty$, we have

$$c_1^L = L_1 e^{\alpha(\nu_0 - \phi^L)}, \quad c_2^L = L_2 e^{-\beta(\nu_0 - \phi^L)}.$$

In view of the relation $\alpha c_1^L = \beta c_2^L$, one has

$$\alpha L_1 e^{\alpha(\nu_0 - \phi^L)} = \beta L_2 e^{-\beta(\nu_0 - \phi^L)} \text{ or } \phi^L = \nu_0 - \frac{1}{\alpha + \beta} \ln \frac{\beta L_2}{\alpha L_1}.$$

Hence,

$$c_1^L = \frac{1}{\alpha} (\alpha L_1)^{\frac{\beta}{\alpha+\beta}} (\beta L_2)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^L = \frac{1}{\beta} (\alpha L_1)^{\frac{\beta}{\alpha+\beta}} (\beta L_2)^{\frac{\alpha}{\alpha+\beta}}.$$

Since $\phi'' = \beta c_2 - \alpha c_1$, (15) implies that ϕ satisfies the Hamiltonian equation

$$\phi'' + \alpha L_1 e^{\alpha(\nu_0 - \phi)} - \beta L_2 e^{-\beta(\nu_0 - \phi)} = 0$$

with $\phi(0) = \nu_0$ and $\phi(\xi) \to \phi^L$ as $\xi \to \infty$. The Hamiltonian is

$$H(\phi, u) = \frac{u^2}{2} - L_1 e^{\alpha(\nu_0 - \phi)} + L_2 e^{-\beta(\nu_0 - \phi)}.$$

In terms of ϕ and $u = \phi'$, the equation becomes

$$\phi' = u, \quad u' = \beta L_2 e^{-\beta(\nu_0 - \phi)} - \alpha L_1 e^{\alpha(\nu_0 - \phi)}. \tag{16}$$

The Hamiltonian system has a unique equilibrium $(\phi^L, 0)$ with ϕ^L given above. If $W^s(\phi^L)$ is the stable manifold of $(\phi^L, 0)$, then it is the restriction of $W^s(\mathcal{Z}_l)$ to the (ϕ, u)-plane. In order to have $(\nu_0, u_0) \in W^s(\phi^L)$ (see Figure 2), $H(\phi^L, 0) = H(\nu_0, u_0)$

and one has the expression for u_0 in (13). To determine the sign of u_0, note that the left branch of the stable manifold $W^s(\phi^L)$ lies above the ϕ-axis and hence that $\nu_0 < \phi^L$ implies $u_0 > 0$; similarly, if $\nu_0 > \phi^L$, then $u_0 < 0$. □

Remark 3.3. We claim that the quantities under the square root in the displays (13) and (14) are nonnegative. In fact, quite interestingly, the nonnegativeness is equivalent to Young's inequality

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab \text{ for } a, b \geq 0, \ \frac{1}{p} + \frac{1}{q} = 1; \text{ “ = ” holds if and only if } a^p = b^q.$$

Take (13) for example. If we set

$$a = (\alpha L_1)^{\frac{\beta}{\alpha+\beta}}, \quad b = (\beta L_2)^{\frac{\alpha}{\alpha+\beta}}, \quad p = \frac{\alpha+\beta}{\beta}, \quad q = \frac{\alpha+\beta}{\alpha},$$

then

$$L_1 + L_2 - \frac{\alpha+\beta}{\alpha\beta}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}} = \frac{\alpha+\beta}{\alpha\beta}\left(\frac{a^p}{p} + \frac{b^q}{q} - ab\right).$$

Thus, the quantity is always nonnegative and it is zero if and only if $\alpha L_1 = \beta L_2$.

3.1.2. Outer dynamics on $[0, a]$: Regular layers or outer solutions. We now construct regular layers or outer solutions on $\mathcal{Z}_l$ that connect $\omega(N_L)$ to $\alpha(N_l^a)$. We find that the outer flow on $\mathcal{Z}_l$ is itself a singular perturbation problem. To see this, we zoom in on an $O(\epsilon)$-neighborhood of $\mathcal{Z}_l$ by blowing up the u and $\alpha c_1 - \beta c_2$ coordinates; that is, we make a scaling $u = \epsilon p$ and $\beta c_2 - \alpha c_1 = \epsilon q$. System (7) becomes

$$\begin{aligned}
&\dot{\phi} = p, \quad \epsilon\dot{p} = q - \epsilon\frac{h'(\tau)}{h(\tau)}p,\\
&\epsilon\dot{q} = (\alpha(\alpha+\beta)c_1 + \epsilon\beta q)p - h^{-1}(\tau)(\beta J_2 - \alpha J_1),\\
&\dot{c}_1 = -\alpha c_1 p - h^{-1}(\tau)J_1,\\
&\dot{J}_i = 0, \quad \dot{\tau} = 1,
\end{aligned} \tag{17}$$

which is indeed a singular perturbation problem due to the factor ϵ in front of $\dot{p}$ and $\dot{q}$. Its limit, as $\epsilon \to 0$, is

$$\begin{aligned}
&\dot{\phi} = p, \quad 0 = q,\\
&0 = \alpha(\alpha+\beta)c_1 p - h^{-1}(\tau)(\beta J_2 - \alpha J_1),\\
&\dot{c}_1 = -\alpha c_1 p - h^{-1}(\tau)J_1,\\
&\dot{J}_i = 0, \quad \dot{\tau} = 1.
\end{aligned} \tag{18}$$

For this system, the outer manifold is

$$\mathcal{S}_l = \left\{ p = \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)h(\tau)c_1}, \ q = 0 \right\}.$$

The outer limit dynamics on $\mathcal{S}_l$ is

$$\begin{aligned}
&\dot{\phi} = \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)h(\tau)c_1},\\
&\dot{c}_1 = -\frac{\beta(J_1 + J_2)}{(\alpha+\beta)h(\tau)},\\
&\dot{J}_i = 0, \quad \dot{\tau} = 1.
\end{aligned} \tag{19}$$

Remark 3.4. Following the suggestion of one of the referees, we give a sketch of an alternative and more standard way of deriving the outer limit dynamics (19).

Introduce $\hat{q} = \beta c_2 - \alpha c_1$. In terms of the variables $(\phi, u, \hat{q}, c_1, J_i, \tau)$, system (8) (with $Q = 0$) becomes

$$\begin{aligned} &\phi' = u, \quad u' = \hat{q} - \epsilon \frac{h'(\tau)}{h(\tau)} u, \\ &\hat{q}' = (\alpha(\alpha+\beta)c_1 + \beta\hat{q})u - \epsilon h^{-1}(\tau)(\beta J_2 - \alpha J_1), \\ &c_1' = -\alpha c_1 u - \epsilon h^{-1}(\tau) J_1, \\ &J_i' = 0, \quad \tau' = \epsilon. \end{aligned} \tag{20}$$

For $\epsilon = 0$, the set $\{u = \hat{q} = 0\}$ is a normally hyperbolic invariant manifold consisting of equilibria. By Fenichel's theory, the manifold persists for $\epsilon > 0$ small and is given by

$$u = \epsilon A(\phi, c_1, J_i, \tau) + O(\epsilon^2), \quad \hat{q} = \epsilon B(\phi, c_1, J_i, \tau) + O(\epsilon^2).$$

Using the invariance of the manifold and substituting the above expressions for u and $\hat{q}$ into system (20), one obtains

$$B = O(\epsilon), \quad A = \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)h(\tau)c_1} + O(\epsilon).$$

System (20) on the perturbed invariant manifold can be obtained by substituting the expression of u and $\hat{q}$ with the approximations of A and B above. It reads as follows:

$$\begin{aligned} &\phi' = \epsilon \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)h(\tau)c_1} + O(\epsilon^2), \\ &c_1' = -\epsilon \frac{\beta(J_1 + J_2)}{(\alpha+\beta)h(\tau)} + O(\epsilon^2), \\ &J_i' = 0, \quad \tau' = \epsilon. \end{aligned} \tag{21}$$

The corresponding outer dynamics is

$$\begin{aligned} &\dot{\phi} = \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)h(\tau)c_1} + O(\epsilon), \\ &\dot{c}_1 = -\frac{\beta(J_1 + J_2)}{(\alpha+\beta)h(\tau)} + O(\epsilon), \\ &\dot{J}_i = 0, \quad \dot{\tau} = 1. \end{aligned} \tag{22}$$

Its limiting dynamics at $\epsilon = 0$ is exactly system (19).

The outer limit dynamics (33) in section 3.2.2 can also be derived this way.

The solution of (19) with the initial condition $(\phi^L, c_1^L, J_1, J_2, 0)$ that corresponds to the point $(\phi^L, 0, c_1^L, c_2^L, J_1, J_2, 0) \in \omega(N_L)$ is

$$\begin{aligned} \tau(x) &= x, \quad c_1(x) = c_1^L - \frac{\beta(J_1+J_2)}{\alpha+\beta} \int_0^x h^{-1}(s) ds, \\ \phi(x) &= \phi^L + \frac{\beta J_2 - \alpha J_1}{\alpha(\alpha+\beta)} \int_0^x h^{-1}(s) c_1^{-1}(s) ds \\ &= \phi^L - \frac{\beta J_2 - \alpha J_1}{\alpha\beta(J_1+J_2)} \int_0^x \frac{\dot{c}_1(s)}{c_1(s)} ds \quad ((19) \text{ is used here}) \\ &= \phi^L - \frac{\beta J_2 - \alpha J_1}{\alpha\beta(J_1+J_2)} \ln \frac{c_1(x)}{c_1^L}. \end{aligned}$$

Recall that we are looking for solutions that belong to $\alpha(N_l^a)$ when $\tau = a$. Evaluating the solution at $\tau = x = a$, we have

$$c_1^{a,l} = c_1^L - \frac{\beta(J_1+J_2)}{\alpha+\beta}\int_0^a h^{-1}(s)ds,$$
$$\phi^{a,l} = \phi^L - \frac{\beta J_2 - \alpha J_1}{\alpha\beta(J_1+J_2)}\ln\frac{c_1^{a,l}}{c_1^L};$$

in particular,

$$J_1 = \frac{(c_1^L - c_1^{a,l})}{\int_0^a h^{-1}(s)ds}\left(1 + \frac{\alpha(\phi^L - \phi^{a,l})}{\ln c_1^L - \ln c_1^{a,l}}\right),$$
$$J_2 = \frac{(c_2^L - c_2^{a,l})}{\int_0^a h^{-1}(s)ds}\left(1 - \frac{\beta(\phi^L - \phi^{a,l})}{\ln c_2^L - \ln c_2^{a,l}}\right). \tag{23}$$

We have used the relations $\alpha c_1^L = \beta c_2^L$ and $\alpha c_1^{a,l} = \beta c_2^{a,l}$ to get this more symmetric form for J_2.

The regular layer Λ_l is given by

$$\begin{aligned}
&\phi(x) = \phi^L - \frac{\beta J_2 - \alpha J_1}{\alpha\beta(J_1+J_2)}\ln\frac{c_1(x)}{c_1^L},\\
&u(x) = 0, \quad \alpha c_1(x) = \beta c_2(x),\\
&c_1(x) = c_1^L - \frac{\beta(J_1+J_2)}{\alpha+\beta}\int_0^x h^{-1}(s)ds,\\
&\tau(x) = x
\end{aligned} \tag{24}$$

with J_1 and J_2 determined by (23).

To summarize, for given values (ϕ^a, c_1^a, c_2^a), we have constructed a unique singular orbit on the left subinterval $[0,a]$ that connects B_L to B_a. It consists of two boundary layer orbits Γ_l^0 from the point $(\nu_0, u_0, L_1, L_2, J_1, J_2, 0) \in B_L$ to the point $(\phi^L, 0, c_1^L, c_2^L, J_1, J_2, 0) \in \omega(N_L) \subset \mathcal{Z}_l$ and Γ_l^a from the point $(\phi^{a,l}, 0, c_1^{a,l}, c_2^{a,l}, J_1, J_2, a) \in \alpha(N_l^a) \subset \mathcal{Z}_l$ to the point $(\phi^a, u_l(a), c_1^a, c_2^a, J_1, J_2, a) \in B_a$, and a regular layer Λ_l on $\mathcal{Z}_l$ that connects the two foot points $(\phi^L, 0, c_1^L, c_2^L, J_1, J_2, 0) \in \omega(N_L)$ and $(\phi^{a,l}, 0, c_1^{a,l}, c_2^{a,l}, J_1, J_2, a) \in \alpha(N_l^a)$ of the two boundary layers.

3.2. Singular orbits on $[a,b]$ with $Q(x) = Q$. We now construct a singular orbit on the subinterval $[a,b]$ viewed as the channel where the permanent charge $Q(x) = Q$ is a nonzero constant. The construction is nearly the same as that for singular orbits on $[0,a]$.

We set $\phi(b) = \phi^b$, $c_1(b) = c_1^b$, and $c_2(b) = c_2^b$, where ϕ^b, c_i^b are unknowns to be determined later. Let

$$B_b = \{(\phi^b, u, c_1^b, c_2^b, J_1, J_2, b) \in \boldsymbol{R}^7 : \text{ arbitrary } u, J_1, J_2\}.$$

The singular orbit to be constructed will be a connecting orbit from B_a to B_b over $[a,b]$.

3.2.1. Inner dynamics on $[a,b]$: Boundary layers or inner solutions. By setting $\epsilon = 0$ in system (7) with $Q(x) = Q$, we get $u = 0$ and $\alpha c_1 + Q = \beta c_2$. The outer manifold is

$$\mathcal{Z}_m = \{u = 0, \alpha c_1 + Q = \beta c_2\}.$$

In terms of ξ, we obtain the inner system of (7):

$$\begin{aligned}
&\phi' = u, \quad u' = \beta c_2 - \alpha c_1 - Q - \epsilon \frac{h'(\tau)}{h(\tau)} u,\\
&c_1' = -\alpha c_1 u - \epsilon h^{-1}(\tau) J_1,\\
&c_2' = \beta c_2 u - \epsilon h^{-1}(\tau) J_2,\\
&J_1' = J_2' = 0, \quad \tau' = \epsilon.
\end{aligned} \tag{25}$$

The limiting system at $\epsilon = 0$ is

$$\begin{aligned}
&\phi' = u, \quad u' = \beta c_2 - \alpha c_1 - Q,\\
&c_1' = -\alpha c_1 u,\\
&c_2' = \beta c_2 u,\\
&J_1' = J_2' = 0, \quad \tau' = 0.
\end{aligned} \tag{26}$$

The set of equilibria of (26) is precisely $\mathcal{Z}_m$, and $\mathcal{Z}_m$ is normally hyperbolic with a six-dimensional stable manifold $W^s(\mathcal{Z}_m)$ and a six-dimensional unstable manifold $W^u(\mathcal{Z}_m)$. The manifolds $\mathcal{Z}_m$, $W^s(\mathcal{Z}_m)$, and $W^s(\mathcal{Z}_m)$ persist for $\epsilon > 0$ small.

PROPOSITION 3.4. (i) *System* (26) *has the following six integrals:*

$$\begin{aligned}
&H_1 = e^{\alpha\phi} c_1, \quad H_2 = e^{-\beta\phi} c_2, \quad H_3 = c_1 + c_2 - \frac{1}{2}u^2 - Q\phi,\\
&H_4 = J_1, \quad H_5 = J_2, \quad H_6 = \tau.
\end{aligned}$$

(ii) *Let $\phi = \phi^{a,m}$ be the unique solution of*

$$\alpha c_1^a e^{\alpha(\phi^a - \phi)} - \beta c_2^a e^{-\beta(\phi^a - \phi)} + Q = 0, \tag{27}$$

and let

$$c_1^{a,m} = e^{\alpha(\phi^a - \phi^{a,m})} c_1^a, \quad c_2^{a,m} = e^{-\beta(\phi^a - \phi^{a,m})} c_2^a.$$

The stable manifold $W^s(\mathcal{Z}_m)$ intersects B_a transversally at points with

(28)

$$\begin{aligned}
&u_m(a)\\
&= [sgn(\phi^{a,m} - \phi^a)] \sqrt{2c_1^a(1 - e^{\alpha(\phi^a - \phi^{a,m})}) + 2c_2^a(1 - e^{-\beta(\phi^a - \phi^{a,m})}) - 2Q(\phi^a - \phi^{a,m})}.
\end{aligned}$$

and arbitrary J_i's.

Let $\phi = \phi^{b,m}$ be the unique solution of

$$\alpha c_1^b e^{\alpha(\phi^b - \phi)} - \beta c_2^b e^{-\beta(\phi^b - \phi)} + Q = 0, \tag{29}$$

and let

$$c_1^{b,m} = e^{\alpha(\phi^b - \phi^{b,m})} c_1^b, \quad c_2^{b,m} = e^{-\beta(\phi^b - \phi^{b,m})} c_2^b.$$

The unstable manifold $W^u(\mathcal{Z}_m)$ intersects B_b transversally at points with

(30)

$$u_m(b) = [sgn(\phi^b - \phi^{b,m})]\sqrt{2c_1^b(1 - e^{\alpha(\phi^b-\phi^{b,m})}) + 2c_2^b(1 - e^{-\beta(\phi^b-\phi^{b,m})}) - 2Q(\phi^b - \phi^{b,m})}.$$

and arbitrary J_i's.

(iii) *Potential boundary layers Γ_m^a at $x = a$ can be determined in the following way: the ϕ-component satisfies the Hamiltonian system*

$$\phi'' + \alpha c_1^a e^{\alpha(\phi^a-\phi)} - \beta c_2^a e^{-\beta(\phi^a-\phi)} + Q = 0,$$

together with $\phi(0) = \phi^a$ and $\phi(\xi) \to \phi^{a,m}$ as $\xi \to \infty$, $u(\xi) = \phi'(\xi)$, and

$$c_1(\xi) = c_1^a e^{\alpha(\phi^a-\phi(\xi))}, \quad c_2(\xi) = c_2^a e^{-\beta(\phi^a-\phi(\xi))}.$$

Similarly, potential boundary layers Γ_m^b at $x = b$ can be determined in the following way: the ϕ-component satisfies the Hamiltonian system

$$\phi'' + \alpha c_1^b e^{\alpha(\phi^b-\phi)} - \beta c_2^b e^{-\beta(\phi^b-\phi)} + Q = 0,$$

together with $\phi(0) = \phi^b$ and $\phi(\xi) \to \phi^{b,m}$ as $\xi \to -\infty$, $u(\xi) = \phi'(\xi)$, and

$$c_1(\xi) = c_1^b e^{\alpha(\phi^b-\phi(\xi))}, \quad c_2(\xi) = c_2^b e^{-\beta(\phi^b-\phi(\xi))}.$$

(iv) *Let $N_m^a = M_m^a \cap W^s(\mathcal{Z}_m)$ and $N_m^b = M_m^b \cap W^u(\mathcal{Z}_m)$, where M_m^a is the collection of orbits from B_a in forward time under the flow* (26) *and M_m^b is the collection of orbits from B_b in backward time under the flow* (26)*. Then,*

$$\omega(N_m^a) = \{(\phi^{a,m}, 0, c_1^{a,m}, c_2^{a,m}, J_1, J_2, a) : \text{ all } J_i\},$$
$$\alpha(N_m^b) = \{(\phi^{b,m}, 0, c_1^{b,m}, c_2^{b,m}, J_1, J_2, b) : \text{ all } J_i\}.$$

Remark 3.5. To show that the quantity under the square root in the display (28) is nonnegative, we assume $c_1^a > 0$ and $c_2^a > 0$ for the moment and let

$$f(x) = c_1^a + c_2^a - c_1^a e^{\alpha(\phi^a-x)} - c_2^a e^{-\beta(\phi^a-x)} - Q(\phi^a - x).$$

Then,

$$f'(x) = \alpha c_1^a e^{\alpha(\phi^a-x)} - \beta c_2^a e^{-\beta(\phi^a-x)} + Q$$

and

$$f''(x) = -\alpha^2 c_1^a e^{\alpha(\phi^a-x)} - \beta^2 c_2^a e^{-\beta(\phi^a-x)} < 0.$$

Therefore $f(x)$ is concave downward. Note that $f'(x) \to +\infty$ as $x \to -\infty$ and $f'(x) \to -\infty$ as $x \to +\infty$. Hence, $f(x)$ has a unique critical point and it must have a global maximum at this critical point. Since $x = \phi_m^a$ is the critical point, we have

$$f(\phi_m^a) \geq f(\phi^a) = 0.$$

By continuity, we have $f(\phi_m^a) \geq 0$ even if $c_1^a = 0$ and/or $c_2^a = 0$. Similarly, the quantity under the square root in the display (30) is nonnegative.

3.2.2. Outer dynamics on $[a, b]$: Regular layers or outer solutions. We now study the flow in the vicinity of the outer manifold $\mathcal{Z}_m$. Following the treatment of the outer flow on $\mathcal{Z}_l$ in section 3.1.2 (see also Remark 3.4), we make a scaling $u = \epsilon p$ and $\beta c_2 - \alpha c_1 - Q = \epsilon q$. System (7) becomes

$$
\begin{aligned}
&\dot{\phi} = p, \quad \epsilon \dot{p} = q - \epsilon \frac{h'(\tau)}{h(\tau)} p,\\
&\epsilon \dot{q} = ((\alpha+\beta)\alpha c_1 + \beta Q + \epsilon \beta q)p - h^{-1}(\tau)(\beta J_2 - \alpha J_1),\\
&\dot{c}_1 = -\alpha c_1 p - h^{-1}(\tau) J_1,\\
&\dot{J}_1 = \dot{J}_2 = 0, \quad \dot{\tau} = 1.
\end{aligned}
\tag{31}
$$

Its limit, as $\epsilon \to 0$, is

$$
\begin{aligned}
&\dot{\phi} = p, \quad 0 = q,\\
&0 = ((\alpha+\beta)\alpha c_1 + \beta Q)p - h^{-1}(\tau)(\beta J_2 - \alpha J_1),\\
&\dot{c}_1 = -\alpha c_1 p - h^{-1}(\tau) J_1,\\
&\dot{J}_i = 0, \quad \dot{\tau} = 1.
\end{aligned}
\tag{32}
$$

For this system, the outer manifold is

$$
\mathcal{S}_m = \left\{ p = \frac{\beta J_2 - \alpha J_1}{h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q)},\ q = 0 \right\}.
$$

The outer limit dynamics on $\mathcal{S}_m$ is governed by system (32), which reads as follows:

$$
\begin{aligned}
\dot{\phi} &= \frac{\beta J_2 - \alpha J_1}{h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q)},\\
\dot{c}_1 &= -\frac{(\beta J_2 - \alpha J_1)\alpha c_1}{h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q)} - h^{-1}(\tau) J_1\\
&= -\frac{\alpha\beta(J_1 + J_2)c_1 + \beta Q J_1}{h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q)},\\
\dot{J}_i &= 0, \quad \dot{\tau} = 1.
\end{aligned}
\tag{33}
$$

Since $h(\tau) > 0$ and $\beta c_2 = \alpha c_1 + Q > 0$, system (33) has the same phase portrait as that of the following system obtained by multiplying $h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q)$ on the right-hand side of system (33) (here we see the reason why $\tau = x$ and $\dot{\tau} = 1$ were introduced into the analysis; see (7)):

$$
\begin{aligned}
&\frac{d}{dy}\phi = \beta J_2 - \alpha J_1,\\
&\frac{d}{dy} c_1 = -\alpha\beta(J_1 + J_2)c_1 - \beta Q J_1,\\
&\frac{d}{dy} J_i = 0, \quad \frac{d}{dy}\tau = h(\tau)((\alpha+\beta)\alpha c_1 + \beta Q).
\end{aligned}
\tag{34}
$$

The solution with the initial condition $(\phi^{a,m}, c_1^{a,m}, J_1, J_2, a)$ that corresponds to the point $(\phi^{a,m}, 0, c_1^{a,m}, c_2^{a,m}, J_1, J_2, a) \in \omega(N_m^a)$ is

$$
\begin{aligned}
&\phi(y) = \phi^{a,m} + (\beta J_2 - \alpha J_1)y,\\
&c_1(y) = e^{-\alpha\beta(J_1+J_2)y}c_1^{a,m} - \frac{QJ_1}{\alpha(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y}\right),\\
&\int_a^\tau h^{-1}(s)ds = (\alpha+\beta)\alpha\int_0^y c_1 ds + \beta Q y \qquad (35)\\
&\qquad = \frac{(\alpha+\beta)c_1^{a,m}}{\beta(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y}\right)\\
&\qquad\quad - \frac{(\alpha+\beta)QJ_1}{J_1+J_2}\left(y - \frac{1}{\alpha\beta(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y}\right)\right) + \beta Q y.
\end{aligned}
$$

We are looking for solutions to reach $\alpha(N_m^b)$; that is, whenever $\tau(y) = b$, we require $\phi(y) = \phi^{b,m}$ and $c_1(y) = c_1^{b,m}$. Assume $\tau(y_0) = b$ for some $y_0 > 0$. Then, $\phi(y_0) = \phi^{b,m}$ and $c_1(y_0) = c_1^{b,m}$, and hence,

$$
\begin{aligned}
&\phi^{b,m} = \phi^{a,m} + (\beta J_2 - \alpha J_1)y_0,\\
&c_1^{b,m} = e^{-\alpha\beta(J_1+J_2)y_0}c_1^{a,m} - \frac{QJ_1}{\alpha(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y_0}\right),\\
&\int_a^b h^{-1}(s)ds = \frac{(\alpha+\beta)c_1^{a,m}}{\beta(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y_0}\right) \qquad (36)\\
&\qquad - \frac{(\alpha+\beta)QJ_1}{J_1+J_2}\left(y_0 - \frac{1}{\alpha\beta(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y_0}\right)\right) + \beta Q y_0.
\end{aligned}
$$

System (36) is equivalent to

$$
\begin{aligned}
&\phi^{b,m} = \phi^{a,m} + (\beta J_2 - \alpha J_1)y_0,\\
&c_1^{b,m} = e^{-\alpha\beta(J_1+J_2)y_0}c_1^{a,m} - \frac{QJ_1}{\alpha(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y_0}\right), \qquad (37)\\
&J_1 + J_2 = \frac{\alpha(\alpha+\beta)(c_1^{a,m} - c_1^{b,m}) - \alpha\beta Q(\phi^{a,m} - \phi^{b,m})}{\alpha\beta\int_a^b h^{-1}(s)ds}.
\end{aligned}
$$

Therefore, the outer or regular layer solution Λ_m on $[a, b]$ is given by (35) with J_1 and J_2 determined by (37). Together with the boundary layers Γ_m^a and Γ_m^b in statement (iii) of Proposition 3.4, this gives the singular orbit on the interval $[a, b]$.

3.3. Singular orbits on $[b, 1]$ with $Q(x) = 0$. The construction of singular orbits on $[b, 1]$ is virtually identical to the construction of singular orbits on $[0, a]$ in section 3.1. We will state only the results for later use.

3.3.1. Inner dynamics on $[b, 1]$: Boundary layers or inner solutions. The inner limit system is

$$
\begin{aligned}
&\phi' = u, \quad u' = \beta c_2 - \alpha c_1,\\
&c_1' = -\alpha c_1 u,\\
&c_2' = \beta c_2 u, \qquad (38)\\
&J_1' = J_2' = 0, \quad \tau' = 0.
\end{aligned}
$$

The outer manifold is

$$\mathcal{Z}_r = \{u = 0, \alpha c_1 = \beta c_2\}.$$

It consists of equilibria of system (38) and is normally hyperbolic with a six-dimensional stable manifold $W^s(\mathcal{Z}_r)$ and a six-dimensional unstable manifold $W^u(\mathcal{Z}_r)$. Concerning the boundary layers, we have the following proposition.

PROPOSITION 3.5. (i) *System* (38) *has the following six integrals:*

$$H_1 = e^{\alpha\phi}c_1, \quad H_2 = e^{-\beta\phi}c_2, \quad H_3 = c_1 + c_2 - \frac{1}{2}u^2,$$
$$H_4 = J_1, \quad H_5 = J_2, \quad H_6 = \tau.$$

(ii) *Let* $\phi = \phi^{b,r}$ *be the unique solution of*

$$\alpha c_1^b e^{\alpha(\phi^b - \phi)} - \beta c_2^b e^{-\beta(\phi^b - \phi)} = 0, \quad \textit{that is, } \phi^{b,r} = \phi^b - \frac{1}{\alpha+\beta}\ln\frac{\beta c_2^b}{\alpha c_1^b},$$

and let

$$c_1^{b,r} = \frac{1}{\alpha}(\alpha c_1^b)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^b)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^{b,r} = \frac{1}{\beta}(\alpha c_1^b)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^b)^{\frac{\alpha}{\alpha+\beta}}.$$

The stable manifold $W^s(\mathcal{Z}_r)$ *intersects* B_b *transversally at points with*

$$u_r(b) = [sgn(\alpha c_1^b - \beta c_2^b)]\sqrt{2\left(c_1^b + c_2^b - \frac{\alpha+\beta}{\alpha\beta}(\alpha c_1^b)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^b)^{\frac{\alpha}{\alpha+\beta}}\right)} \tag{39}$$

and arbitrary J_i*'s.*

Let $\phi = \phi^R$ *be the unique solution of*

$$\alpha R_1 e^{-\alpha\phi} - \beta R_2 e^{\beta\phi} = 0, \quad \textit{that is, } \phi^R = -\frac{1}{\alpha+\beta}\ln\frac{\beta R_2}{\alpha R_1},$$

and let

$$c_1^R = \frac{1}{\alpha}(\alpha R_1)^{\frac{\beta}{\alpha+\beta}}(\beta R_2)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^R = \frac{1}{\beta}(\alpha R_1)^{\frac{\beta}{\alpha+\beta}}(\beta R_2)^{\frac{\alpha}{\alpha+\beta}}.$$

The unstable manifold $W^u(\mathcal{Z}_r)$ *intersects* B_R *transversally at points with*

$$u_1 = [sgn(\beta R_2 - \alpha R_1)]\sqrt{2\left(R_1 + R_2 - \frac{\alpha+\beta}{\alpha\beta}(\alpha R_1)^{\frac{\beta}{\alpha+\beta}}(\beta R_2)^{\frac{\alpha}{\alpha+\beta}}\right)} \tag{40}$$

and arbitrary J_i*'s.*

(iii) *Potential boundary layers* Γ_r^b *at* $x = b$ *can be determined in the following way: the* ϕ*-component satisfies the Hamiltonian system*

$$\phi'' + \alpha c_1^b e^{\alpha(\phi^b - \phi)} - \beta c_2^b e^{-\beta(\phi^b - \phi)} = 0,$$

together with $\phi(0) = \phi^b$ *and* $\phi(\xi) \to \phi^{b,r}$ *as* $\xi \to \infty$, $u(\xi) = \phi'(\xi)$, *and*

$$c_1(\xi) = c_1^b e^{\alpha(\phi^b - \phi(\xi))}, \quad c_2(\xi) = c_2^b e^{-\beta(\phi^b - \phi(\xi))}.$$

Similarly, potential boundary layers Γ_r^1 at $x = 1$ can be determined in the following way: the ϕ-component satisfies the Hamiltonian system

$$\phi'' + \alpha R_1 e^{-\alpha\phi} - \beta R_2 e^{\beta\phi} = 0,$$

together with $\phi(0) = 0$ and $\phi(\xi) \to \phi^R$ as $\xi \to -\infty$, $u(\xi) = \phi'(\xi)$, and

$$c_1(\xi) = R_1 e^{-\alpha\phi(\xi)}, \quad c_2(\xi) = R_2 e^{\beta\phi(\xi)}.$$

(iv) *Let $N_r^b = M_r^b \cap W^s(\mathcal{Z}_r)$ and $N_R = M_R \cap W^u(\mathcal{Z}_r)$, where M_r^b is the collection of orbits from B_b in forward time under the flow* (38) *and M_R is the collection of orbits from B_R in backward time under the flow* (38). *Then,*

$$\omega(N_r^b) = \{(\phi^{b,r}, 0, c_1^{b,r}, c_2^{b,r}, J_1, J_2, b) : \text{ all } J_i\},$$
$$\alpha(N_R) = \left\{\left(\phi^R, 0, c_1^R, c_2^R, J_1, J_2, 1\right) : \text{ all } J_i\right\}.$$

3.3.2. Outer dynamics on $[b, 1]$: Regular layers or outer solutions. We now examine the existence of regular layers or outer solutions that connect $\omega(N_r^b)$ to $\alpha(N_R)$. Following exactly the same analysis as in section 3.1.2, we find that the outer limit dynamics is

$$\begin{aligned} \dot{\phi} &= \frac{\beta J_2 - \alpha J_1}{(\alpha+\beta)\alpha h(\tau) c_1}, \\ \dot{c}_1 &= -\frac{\beta(J_1 + J_2)}{(\alpha+\beta)h(\tau)}, \\ \dot{J}_i &= 0, \quad \dot{\tau} = 1, \end{aligned} \tag{41}$$

and the outer solution Λ_r on $[b, 1]$ with the initial condition $(\phi^{b,r}, c_1^{b,r}, J_1, J_2, b)$ that corresponds to the point $(\phi^{b,r}, 0c_1^{b,r}, c_2^{b,r}, J_1, J_2, b) \in \omega(N_r^b)$ is given by

$$\begin{aligned} \phi(\xi) &= \phi^{b,r} - \frac{\beta J_2 - \alpha J_1}{\alpha\beta(J_1 + J_2)} \ln \frac{c_1(\xi)}{c_1^{b,r}}, \\ u(\xi) &= 0, \quad \alpha c_1(\xi) = \beta c_2(\xi), \\ c_1(\xi) &= c_1^{b,r} - \frac{\beta(J_1 + J_2)}{\alpha + \beta} \int_b^\xi h^{-1}(s) ds, \\ \tau(\xi) &= \xi. \end{aligned}$$

The outer solution Λ_r hits the point $(\phi^R, 0, c_1^R, c_2^R, J_1, J_2, 1) \in \alpha(N_R)$ if and only if

$$\begin{aligned} J_1 &= \frac{c_1^{b,r} - c_1^R}{\int_b^1 h^{-1}(s) ds} \left(1 + \frac{\alpha(\phi^{b,r} - \phi^R)}{\ln c_1^{b,r} - \ln c_1^R}\right), \\ J_2 &= \frac{c_2^{b,r} - c_2^R}{\int_b^1 h^{-1}(s) ds} \left(1 - \frac{\beta(\phi^{b,r} - \phi^R)}{\ln c_2^{b,r} - \ln c_2^R}\right). \end{aligned} \tag{42}$$

The outer solution Λ_r together with the inner solutions Γ_r^b and Γ_r^1 in statement (iii) of Proposition 3.5 gives the singular orbit on $[b, 1]$.

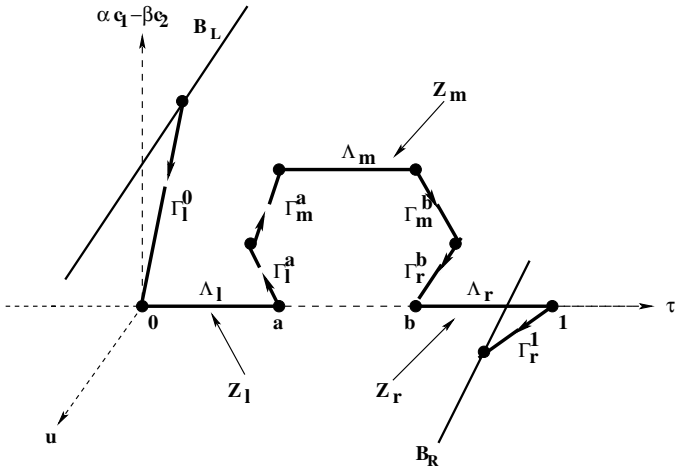

FIG. 3. *Schematic picture of the singular orbit (solid curves): left boundary layer Γ_l^0; right boundary layer Γ_r^1; four internal layers Γ_l^a, Γ_m^a, Γ_m^b, and Γ_r^b, and three regular layers Λ_l, Λ_m, and Λ_r.*

3.4. Matching and singular orbits on [0, 1]. A singular orbit on the whole interval $[0,1]$ will be the union of the singular orbits constructed on each of the subintervals (see Figure 3). The matching conditions are $u_l(a) = u_m(a)$, $u_m(b) = u_r(b)$, and J_1 and J_2 have to be the same on all subintervals; that is, from formulas (14), (23), (27), (28), (29), (30), (37), (39), and (42),

$$
\begin{aligned}
\alpha c_1^a e^{\alpha(\phi^a-\phi^{a,m})} - \beta c_2^a e^{-\beta(\phi^a-\phi^{a,m})} + Q &= 0,\\
\alpha c_1^b e^{\alpha(\phi^b-\phi^{b,m})} - \beta c_2^b e^{-\beta(\phi^b-\phi^{b,m})} + Q &= 0,\\
\frac{\alpha+\beta}{\beta} c_1^{a,l} &= c_1^a e^{\alpha(\phi^a-\phi^{a,m})} + c_2^a e^{-\beta(\phi^a-\phi^{a,m})} + Q(\phi^a-\phi^{a,m}),\\
\frac{\alpha+\beta}{\beta} c_1^{b,r} &= c_1^b e^{\alpha(\phi^b-\phi^{b,m})} + c_2^b e^{-\beta(\phi^b-\phi^{b,m})} + Q(\phi^b-\phi^{b,m}),\\
J_1 &= \frac{(c_1^L - c_1^{a,l})}{\int_0^a h^{-1}(s)ds}\left(1 + \frac{\alpha(\phi^L-\phi^{a,l})}{\ln c_1^L - \ln c_1^{a,l}}\right)\\
&= \frac{c_1^{b,r} - c_1^R}{\int_b^1 h^{-1}(s)ds}\left(1 + \frac{\alpha(\phi^{b,r}-\phi^R)}{\ln c_1^{b,r} - \ln c_1^R}\right),\\
J_2 &= \frac{(c_2^L - c_2^{a,l})}{\int_0^a h^{-1}(s)ds}\left(1 - \frac{\beta(\phi^L-\phi^{a,l})}{\ln c_2^L - \ln c_2^{a,l}}\right)\\
&= \frac{c_2^{b,r} - c_2^R}{\int_b^1 h^{-1}(s)ds}\left(1 - \frac{\beta(\phi^{b,r}-\phi^R)}{\ln c_2^{b,r} - \ln c_2^R}\right),\\
\phi^{b,m} &= \phi^{a,m} + (\beta J_2 - \alpha J_1)y_0,\\
c_1^{b,m} &= e^{-\alpha\beta(J_1+J_2)y_0} c_1^{a,m} - \frac{QJ_1}{\alpha(J_1+J_2)}\left(1 - e^{-\alpha\beta(J_1+J_2)y_0}\right),\\
J_1 + J_2 &= \frac{\alpha(\alpha+\beta)(c_1^{a,m} - c_1^{b,m}) - \alpha\beta Q(\phi^{a,m}-\phi^{b,m})}{\alpha\beta\int_a^b h^{-1}(s)ds},
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
c_1^L &= \frac{1}{\alpha}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^L = \frac{1}{\beta}(\alpha L_1)^{\frac{\beta}{\alpha+\beta}}(\beta L_2)^{\frac{\alpha}{\alpha+\beta}},\\
c_1^{a,l} &= \frac{1}{\alpha}(\alpha c_1^a)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^a)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^{a,l} = \frac{1}{\beta}(\alpha c_1^a)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^a)^{\frac{\alpha}{\alpha+\beta}},\\
c_1^{b,r} &= \frac{1}{\alpha}(\alpha c_1^b)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^b)^{\frac{\alpha}{\alpha+\beta}}, \quad c_2^{b,r} = \frac{1}{\beta}(\alpha c_1^b)^{\frac{\beta}{\alpha+\beta}}(\beta c_2^b)^{\frac{\alpha}{\alpha+\beta}},\\
c_1^{a,m} &= e^{\alpha(\phi^a-\phi^{a,m})}c_1^a, \quad c_1^{b,m} = e^{\alpha(\phi^b-\phi^{b,m})}c_1^b.
\end{aligned}
$$

Recall that $h(x) = g_0^2(x)$, where $g_0(x)$ is the radius of the cross-section of the channel at x, Q is the concentration of the permanent charge over the interval $[a, b]$, (ϕ^a, c_1^a, c_2^a) and (ϕ^b, c_1^b, c_2^b) are the unknown values preassigned at $x = a$ and $x = b$, and J_1 and J_2 are the unknown values for the flux densities of the two types of ions.

There are also three auxiliary unknowns $\phi^{a,m}$, $\phi^{b,m}$, and y_0 in the set of equations (43). The total number of unknowns in (43) is eleven, which matches the total number of equations.

A qualitative important question is whether the set of nonlinear equations (43) has a unique solution. Next, we will consider a special case and demonstrate that (43) can have multiple solutions.

3.4.1. $\alpha = \beta = 1$, and $a = 1/3$, $b = 2/3$, and $h = 1$. We now consider a special case where $\alpha = \beta = 1$. It turns out that the nonlinear system of algebraic equations (43) in eleven unknowns can be reduced to a single algebraic equation with only one unknown. Further restrictions that $a = 1/3$, $b = 2/3$, and $h = 1$ will be posted later merely for simplicity.

Set $c_1^a c_2^a = A^2$, $c_1^b c_2^b = B^2$, $L_1 L_2 = L^2$, $R_1 R_2 = R^2$, and $Q = 2Q_0$. From the first two equations in (43), one has

$$
\begin{aligned}
\phi^a - \phi^{a,m} &= \ln \frac{\sqrt{Q_0^2 + A^2} - Q_0}{c_1^a},\\
\phi^b - \phi^{b,m} &= \ln \frac{\sqrt{Q_0^2 + B^2} - Q_0}{c_1^b}.
\end{aligned}
$$

System (43) becomes

$$
\begin{aligned}
A &= \sqrt{Q_0^2 + A^2} + Q_0 \ln \frac{\sqrt{Q_0^2 + A^2} - Q_0}{c_1^a},\\
B &= \sqrt{Q_0^2 + B^2} + Q_0 \ln \frac{\sqrt{Q_0^2 + B^2} - Q_0}{c_1^b},\\
J_1 &= \frac{L - A}{\int_0^a h^{-1}(s)ds} \cdot \frac{\nu_0 - \phi^a + \ln L_1 - \ln c_1^a}{\ln L - \ln A}\\
&= \frac{B - R}{\int_b^1 h^{-1}(s)ds} \cdot \frac{\phi^b + \ln c_1^b - \ln R_1}{\ln B - \ln R},
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \frac{L-A}{\int_0^a h^{-1}(s)ds}\left(2-\frac{\nu_0-\phi^a+\ln L_1-\ln c_1^a}{\ln L-\ln A}\right)\\
&= \frac{B-R}{\int_b^1 h^{-1}(s)ds}\left(2-\frac{\phi^b+\ln c_1^b-\ln R_1}{\ln B-\ln R}\right),\\
(J_2-J_1)y_0 &= \phi^b-\phi^a+\ln\frac{c_1^b(\sqrt{Q_0^2+A^2}-Q_0)}{c_1^a(\sqrt{Q_0^2+B^2}-Q_0)},\\
J_1+J_2 &= \frac{2(\sqrt{Q_0^2+A^2}-\sqrt{Q_0^2+B^2})-2Q_0(J_1-J_2)y_0}{\int_a^b h^{-1}(s)ds},\\
\sqrt{Q_0^2+B^2}-Q_0 &= e^{-(J_1+J_2)y_0}\left(\sqrt{Q_0^2+A^2}-Q_0\right)\\
&\quad -\frac{2Q_0J_1}{J_1+J_2}\left(1-e^{-(J_1+J_2)y_0}\right).
\end{aligned}
\tag{44}
$$

Add the J_1 and J_2 equations in (44) to get

$$
J_1+J_2 = 2\frac{L-A}{\int_0^a h^{-1}} = 2\frac{B-R}{\int_b^1 h^{-1}};\ \text{hence, } B=\frac{\int_b^1 h^{-1}}{\int_0^a h^{-1}}(L-A)+R.
$$

The first two equations in (44) give

$$
\begin{aligned}
c_1^a &= \left(\sqrt{Q_0^2+A^2}-Q_0\right)\exp\left\{\frac{\sqrt{Q_0^2+A^2}-A}{Q_0}\right\},\\
c_1^b &= \left(\sqrt{Q_0^2+B^2}-Q_0\right)\exp\left\{\frac{\sqrt{Q_0^2+B^2}-B}{Q_0}\right\}.
\end{aligned}
\tag{45}
$$

The first two equations together with $(J_2-J_1)y_0$ and the J_1+J_2 equations give

$$
J_1+J_2 = 2\frac{L-A}{\int_0^a h^{-1}} = 2\frac{B-R}{\int_b^1 h^{-1}} = 2\frac{A-B-Q_0(\phi^a-\phi^b)}{\int_a^b h^{-1}}.
$$

Hence,

$$
J_1+J_2 = 2\frac{L-R-Q_0(\phi^a-\phi^b)}{\int_0^1 h^{-1}},
$$

$$
\phi^b-\phi^a = \frac{(L-A)\int_0^1 h^{-1}-(L-R)\int_0^a h^{-1}}{Q_0\int_0^a h^{-1}},
\tag{46}
$$

and

$$
\begin{aligned}
(J_2-J_1)y_0 &= \phi^b-\phi^a-\ln\frac{\sqrt{Q_0^2+B^2}-Q_0}{c_1^b}+\ln\frac{\sqrt{Q_0^2+A^2}-Q_0}{c_1^a}\\
&= \frac{(L-A)\int_0^1 h^{-1}-(L-R)\int_0^a h^{-1}}{Q_0\int_0^a h^{-1}}+\ln\frac{(\sqrt{Q_0^2+A^2}-Q_0)c_1^b}{(\sqrt{Q_0^2+B^2}-Q_0)c_1^a}\\
&= \frac{(L-A)\int_a^b h^{-1}}{Q_0\int_0^a h^{-1}}+\frac{\sqrt{Q_0^2+B^2}-\sqrt{Q_0^2+A^2}}{Q_0}.
\end{aligned}
$$

Using

$$\frac{L-A}{\int_0^a h^{-1}} = \frac{B-R}{\int_b^1 h^{-1}}$$

and the second equality in the J_1 equation in (44), one has

$$\frac{\nu_0 - \phi^a + \ln L_1 - \ln c_1^a}{\ln L - \ln A} = \frac{\phi^b + \ln c_1^b - \ln R_1}{\ln B - \ln R}.$$

Hence,

$$\frac{\phi^b + \ln c_1^b - \ln R_1}{\ln B - \ln R} = \frac{\nu_0 + \phi^b - \phi^a + \ln(L_1 c_1^b) - \ln(R_1 c_1^a)}{\ln(BL) - \ln(AR)}.$$

The latter together with (46) and (45) gives

$$\begin{aligned}\phi^b &= \frac{\ln \frac{B}{R}}{\ln \frac{BL}{AR}} \left(\nu_0 + \ln \frac{L_1 c_1^b}{R_1 c_1^a} + \frac{(L-A)\int_0^1 h^{-1} - (L-R)\int_0^a h^{-1}}{Q_0 \int_0^a h^{-1}} \right) + \ln \frac{R_1}{c_1^b} \\ &= \frac{\ln \frac{B}{R}}{\ln \frac{BL}{AR}} \left(\nu_0 + \ln \frac{L_1(\sqrt{Q_0^2+B^2} - Q_0)}{R_1(\sqrt{Q_0^2+A^2} - Q_0)} + \frac{\sqrt{Q_0^2+B^2} - \sqrt{Q_0^2+A^2}}{Q_0} \right. \\ &\quad \left. + \frac{(L-A)\int_a^b h^{-1}}{Q_0 \int_0^a h^{-1}} \right) + \ln R_1 - \ln\left(\sqrt{Q_0^2+B^2} - Q_0\right) - \frac{\sqrt{Q_0^2+B^2} - B}{Q_0}.\end{aligned}$$

Note that all the variables in (44) can be expressed in terms of A. Substituting into the last equation in (44) we will get an equation $F(A) = 0$ in the variable A only. The expression of $F(A)$ is complicated but can be explicitly given.

We now suppose further that $a = 1/3$, $b = 2/3$, and $h = 1$. Then,

$$B = L + R - A, \quad J_1 + J_2 = 6(L - A), \tag{47}$$

$$c_1^a = \left(\sqrt{Q_0^2+A^2} - Q_0\right) \exp\left\{ \frac{\sqrt{Q_0^2+A^2} - A}{Q_0} \right\},$$

$$c_1^b = \left(\sqrt{Q_0^2+B^2} - Q_0\right) \exp\left\{ \frac{\sqrt{Q_0^2+B^2} - B}{Q_0} \right\},$$

$$\phi^b - \phi^a = \frac{2L + R - 3A}{Q_0},$$

$$\begin{aligned}\phi^b = {} & \frac{\ln \frac{B}{R}}{\ln \frac{BL}{AR}} \left(\nu_0 + \ln \frac{L_1(\sqrt{Q_0^2+B^2} - Q_0)}{R_1(\sqrt{Q_0^2+A^2} - Q_0)} + \frac{\sqrt{Q_0^2+B^2} - \sqrt{Q_0^2+A^2} + L - A}{Q_0} \right) \\ & + \ln R_1 - \ln\left(\sqrt{Q_0^2+B^2} - Q_0\right) - \frac{\sqrt{Q_0^2+B^2} - B}{Q_0},\end{aligned}$$

$$J_2 - J_1 = 6(L-A) - \frac{6(L-A)}{\ln \frac{BL}{AR}} \left(\nu_0 + \ln \frac{L_1(\sqrt{Q_0^2+B^2} - Q_0)}{R_1(\sqrt{Q_0^2+A^2} - Q_0)} \right) - \frac{6(L-A)(\sqrt{Q_0^2+B^2} - \sqrt{Q_0^2+A^2} + L - A)}{Q_0 \ln \frac{BL}{AR}},$$

$$(J_2 - J_1) y_0 = \frac{\sqrt{Q_0^2+B^2} - \sqrt{Q_0^2+A^2} + L - A}{Q_0}.$$

The final equation involving the only unknown A is $F(A) = 0$, where

$$F(A) = e^{K(A)} \left(\sqrt{Q_0^2+A^2} - \frac{Q_0(J_2-J_1)}{6(L-A)} \right) + \frac{Q_0(J_2-J_1)}{6(L-A)} - \sqrt{Q_0^2+B^2}, \tag{48}$$

where

$$K(A) = -6(L-A) \frac{\sqrt{Q_0^2+B^2} - \sqrt{Q_0^2+A^2} + L - A}{Q_0(J_2-J_1)},$$

$B = L + R - A$, and $J_2 - J_1$ is given above.

To summarize, for the special case where

$$\alpha = \beta = 1, \quad a = 1/3, \quad b = 2/3, \quad h = 1,$$

the set of nonlinear algebraic equations is equivalent to $F(A) = 0$, where $F(A)$ is given in (48). The formula $F(A)$, although terribly complicated, involves only one unknown $A = \sqrt{c_1^a c_2^a}$. Other parameters in $F(A)$ are L_1, $L = \sqrt{L_1 L_2}$, R_1, $R = \sqrt{R_1 R_2}$, ν_0, and Q_0.

For $L = L_1 = 2$, $R = R_1 = 3$, $Q = 2Q_0 = 2$, and $\nu_0 = -20$, we find, numerically, two solutions of $F(A) = 0$: $A_1 = 0.6858357$ and $A_2 = 2$ (the latter is a removable singularity of the functions $F(A)$, J_i's, ϕ^b, and ϕ^a).

Once a feasible value for A is determined, all the unknowns will be determined. We then get a singular orbit that consists of nine pieces $\Gamma_l^0 \cup \Lambda_l \cup \Gamma_l^a \cup \Gamma_m^a \cup \Lambda_m \cup \Gamma_m^b \cup \Gamma_r^b \cup \Lambda_r \cup \Gamma_r^1$ (see Figure 3).

4. Main results and numerical simulations. Any solution of the set of algebraic equations determines a singular orbit for the connecting problem. Once a singular orbit is constructed, we apply geometric singular perturbation theory to show that, for $\epsilon > 0$ small, there is a unique solution that is close to the singular orbit. Before giving the precise statement of our result and its proof, let us explain the ideas behind it.

Let $\Gamma_l^0 \cup \Lambda_l \cup \Gamma_l^a \cup \Gamma_m^a \cup \Lambda_m \cup \Gamma_m^b \cup \Gamma_r^b \cup \Lambda_r \cup \Gamma_r^1$ be a singular orbit to the connecting problem (7) associated to B_L and B_R. For $\epsilon > 0$ small, let $M_L(\epsilon)$ be the forward trace of B_L under the flow of system (7) or, equivalently, system (8). To establish the existence of a unique solution to the boundary value problem near the singular orbit, we will show that $M_L(\epsilon)$ intersects B_R transversally in a neighborhood of the singular orbit.

Roughly speaking, the evolution of $M_L(\epsilon)$ from $x = 0$ to $x = 1$ undergoes the following nine stages with each stage guided by one of the nine pieces of the singular orbit (see Figure 3):

(l1) Along Γ_l^0: Since B_L intersects $W^s(\mathcal{Z}_l)$ transversally, $M_L(\epsilon)$ will first follow the orbit Γ_l^0 towards the vicinity of $\mathcal{Z}_l$ under the inner limit flow (12) near $x = 0$.

(l2) Along Λ_l: Once $M_L(\epsilon)$ gets close to $\mathcal{Z}_l$, the outer limit flow (19) takes over, and $M_L(\epsilon)$ will then follow the outer flow on $\mathcal{Z}_l$ or $\mathcal{S}_l$ along the orbit Λ_l towards the hypersurface $\{x = a\}$.

(l3) Along Γ_l^a: Near but before $\{x = a\}$, $M_L(\epsilon)$ will leave the vicinity of $\mathcal{Z}_l$, follow the orbit Γ_l^a under the inner limit flow (12) near $x = a$, and hit the hypersurface $\{x = a\}$.

(m1) Along Γ_m^a: Upon hitting the hypersurface $\{x = a\}$, the flow switches to the inner limit flow (26) with $Q(x) = Q$. $M_L(\epsilon)$ then follows Γ_m^a towards the vicinity of $\mathcal{Z}_m$.

(m2) Along Λ_m: Once $M_L(\epsilon)$ gets close to $\mathcal{Z}_m$, the outer limit flow (33) takes over, and $M_L(\epsilon)$ will then follow the outer flow on $\mathcal{Z}_m$ or $\mathcal{S}_m$ along the orbit Λ_m towards the hypersurface $\{x = b\}$.

(m3) Along Γ_m^b: Near but before $\{x = b\}$, $M_L(\epsilon)$ will leave the vicinity of $\mathcal{Z}_m$, follow the orbit Γ_m^b under the inner limit flow (26) near $x = b$, and hit the hypersurface $\{x = b\}$.

(r1) Along Γ_r^b: Upon hitting the hypersurface $\{x = b\}$, the flow switches to the inner limit flow (38) with $Q(x) = 0$. $M_L(\epsilon)$ then follows Γ_r^b towards the vicinity of $\mathcal{Z}_r$.

(r2) Along Λ_r: Once $M_L(\epsilon)$ gets close to $\mathcal{Z}_r$, the outer limit flow (41) takes over, and $M_L(\epsilon)$ will then follow the outer flow on $\mathcal{Z}_r$ or $\mathcal{S}_r$ along the orbit Λ_r towards the hypersurface $\{x = 1\}$.

(r3) Along Γ_r^1: Near but before $\{x = 1\}$, $M_L(\epsilon)$ will leave the vicinity of $\mathcal{Z}_r$ and follow the orbit Γ_r^1 under the inner limit flow (38) near $x = 1$. If it hits B_R, then we get our solution.

The main task is to justify the above description of the stages that $M_L(\epsilon)$ undergoes. The exchange lemma—see, for example, [47, 45, 46, 50, 51, 73]—of geometric singular perturbation theory is a result that precisely characterizes the configuration of $M_L(\epsilon)$ during its evolution through the above stages. To apply this abstract theory, one need only verify certain transversality conditions of some limiting objects.

We now state our results and provide a proof using the geometric singular perturbation theory described above.

THEOREM 4.1. *Let* $\Gamma_l^0 \cup \Lambda_l \cup \Gamma_l^a \cup \Gamma_m^a \cup \Lambda_m \cup \Gamma_m^b \cup \Gamma_r^b \cup \Lambda_r \cup \Gamma_r^1$ *be a singular orbit to the connecting problem* (7) *associated to* B_L *and* B_R. *Then, for* $\epsilon > 0$ *small, the boundary value problem* (5) *and* (6) *has a unique continuous and piecewise smooth solution near the singular orbit.*

Proof. For $\epsilon > 0$ small, choose $\delta > 0$ small. Let

$$B_L(\delta) = \{(\nu_0, u, L_1, L_2, J_1, J_2, 0) : |u - u_0| < \delta, |J_i - J_i^0| < \delta\},$$

and let $M_L(\epsilon)$ be the forward trace of $B_L(\delta)$ under the flow of system (7) or, equivalently, system (8). To prove the theorem, we need to show that $M_L(\epsilon)$ intersects B_R transversally in a neighborhood of the singular orbit. Indeed, if we let $M_R(\epsilon)$ be the backward trace of B_R near the singular orbit, then $M_L(\epsilon)$ and $M_R(\epsilon)$ intersect transversally too. The transversality implies that $\dim(M_L(\epsilon) \cap M_R(\epsilon)) =$

$\dim M_L(\epsilon) + \dim M_R(\epsilon) - 7 = 1$. Therefore, the intersection $M_L(\epsilon) \cap M_R(\epsilon)$ consists of precisely one solution to the boundary value problem, and the solution is near the singular orbit.

To establish the transversal intersection of $M_L(\epsilon)$ with B_R near the singular orbit, we apply the exchange lemma successively along the stages described above. The first application of the exchange lemma verifies the descriptions for stages (l1), (l2), and (l3); the second one for stages (m1), (m2), and (m3); and the last application verifies the descriptions for stages (r1), (r2), and (r3).

Note that $\dim B_L(\delta) = 3$. Since the fast flow is not tangent to $B_L(\delta)$, one has $\dim M_L(\epsilon) = 4$. The transversality of the intersection $B_L \cap W^s(\mathcal{Z}_l)$ along Γ_l^0 implies the transversality of the intersection $M_L(0) \cap W^s(\mathcal{Z}_l)$. The exchange lemma implies that $M_L(\epsilon)$ will first follow Γ_l^0 towards $N_L \subset \mathcal{Z}_l$, then follow $N_L \cdot x$ in the vicinity of Λ_l towards $x = a$, and leave the vicinity of $\mathcal{Z}_l$. And upon exit, $M_L(\epsilon)$ is C^1 $O(\epsilon)$-close to $W^u(N_L \times (a - \delta, a))$ in the vicinity of Γ_l^a.

Denote the intersection of $W^u(N_L \times (a - \delta, a))$ with $\{x = a\}$ by $I(a)$. Then $I(a)$ intersects $W^s(\mathcal{Z}_m)$ transversally for the flow (26). Let $K(a)$ be the forward trace of $I(a)$ under (25). The exchange lemma implies that $M_L(\epsilon)$ will first follow $K(a)$ in the vicinity of Γ_m^a towards $N_m^a \subset \mathcal{Z}_m$, then follow $N_m^a \cdot x$ in the vicinity of Λ_m towards $x = b$, and leave the vicinity of $\mathcal{Z}_m$. And upon exit, $M_L(\epsilon)$ is C^1 $O(\epsilon)$-close to $W^u(N_m^a \times (b - \delta, b))$ in the vicinity of Γ_m^b.

Denote the intersection of $W^u(N_m^a \times (b - \delta, b))$ with $\{x = b\}$ by $I(b)$. Then $I(b)$ intersects $W^s(\mathcal{Z}_r)$ transversally for the flow (38). Let $K(b)$ be the forward trace of $I(b)$ under the full system. Then exchange lemma implies that $M_L(\epsilon)$ will first follow $K(b)$ in the vicinity of Γ_r^b towards $N_r^b \subset \mathcal{Z}_r$, then follow $N_r^b \cdot x$ in the vicinity of Λ_r towards $x = 1$, and leave the vicinity of $\mathcal{Z}_r$. And upon exit, $M_L(\epsilon)$ is C^1 $O(\epsilon)$-close to $W^u(N_R \times (1 - \delta, 1))$ in the vicinity of Γ_r^1.

In summary, after three applications of the exchange lemma, we determine that $M_L(\epsilon)$ is C^1 $O(\epsilon)$-close to $W^u(N_R \times (1 - \delta, 1))$ in the vicinity of Γ_r^1. Since $W^u(N_R \times (1 - \delta, 1))$ intersects B_R transversally along Γ_r^1, we have shown that $M_L(\epsilon)$ intersects B_R transversally. The proof is complete. □

Numerical simulations are performed for $A_1 = 0.6858357$ and $A_2 = 2$ (see Figures 4 and 5). The following properties of the two solutions are predicted from the analytical results and can be observed from the numerical simulations:

(i) For both A_1 and A_2, approximately $c_2(x) - c_1(x) = Q(x)$ for $x \in (0, 1)$ except around $x = 1/3$ and $x = 2/3$—the jumping points of Q.

(ii) For $A_2 = L$, $J_1 + J_2 = 0$ from (47). As a consequence of (19) and (41), $c_1(x) = c_2(x) = L = 2$ for $x \in (0, 1/3)$ and $c_1(x) = c_2(x) = R = 3$ for $x \in (2/3, 1)$. The decreasing behavior of $c_1(x) = c_2(x)$ for $x \in (0, 1/3) \cup (2/3, 1)$ can be also predicted from that of the singular orbit corresponding to A_1.

(iii) There is a significant difference between the two solutions for $A_1 \neq L$ and $A_2 = L$: the solution for A_1 has two internal layers with limit orbits Γ_l^a and Γ_m^a at $x = a = 1/3$ that match at a point on B_a (see Figure 3); the solution for A_2 has only one internal layer $\Gamma_l^a = \Gamma_m^a$ at $x = 1/3$. This analytical consequence is not clearly shown in the figures but is indicated by the different behaviors of the ϕ-component: for A_1, with the extra transition through B_a, the layers near $x = 1/3$ are smoother than the one layer for A_2. The same remarks are true for the two solutions near $x = b = 2/3$.

5. Remarks. The defining equation $F(A) = 0$ in (48) that determines multiplicity of steady-states of the PNP system should be investigated thoroughly.

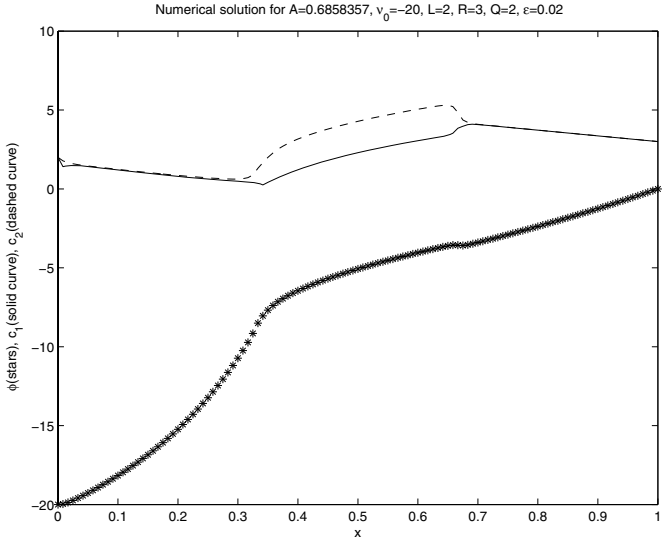

FIG. 4. *ϕ (stars), c_1 (solid curve), and c_2 (dashed curve) for $A_1 = 0.6858357$ with $L_1 = L_2 = 2$, $R_1 = R_2 = 3$, $Q = 2Q_0 = 2$, $\nu_0 = -20$, and $\epsilon = 0.02$.*

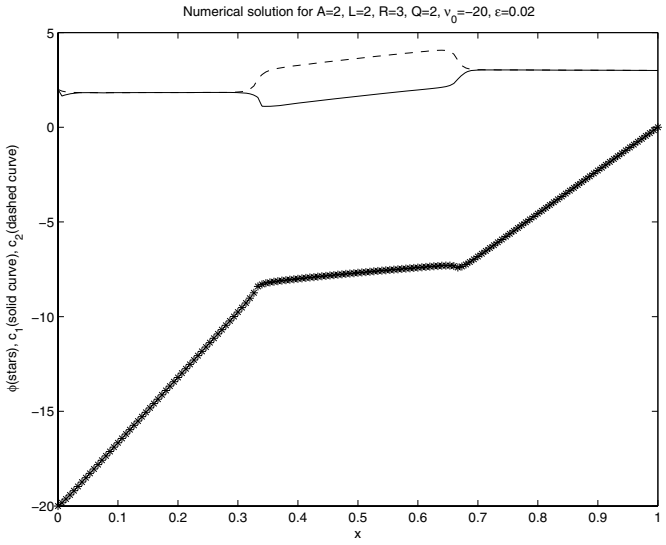

FIG. 5. *ϕ (stars), c_1 (solid curve), and c_2 (dashed curve) for $A_2 = 2$ with $L_1 = L_2 = 2$, $R_1 = R_2 = 3$, $Q = 2Q_0 = 2$, $\nu_0 = -20$, and $\epsilon = 0.02$.*

This could be studied using bifurcation theory of dynamical systems and numerical tools (e.g., AUTO) due to the presence of multiple parameters (L_i, R_i, ν_0, Q, etc. should be viewed as perturbation parameters). Another important problem is the stability of each solution in the full time evolution PNP system. Both multiplicity and stability have important biological consequences for ion channels. Single channels are in fact often defined in the laboratory by their characteristic current signal which switches from one nearly zero level ("the closed channel") to another nonzero level ("the open channel") in a random telegraph signal, with brief incomplete

spiky interruptions. Different types of channels perform their functions by controlling the open probability and/or mean duration of the stochastic signal. These gating phenomena are central to the biological function of channels and are almost always explained by saying the channel changes shape ("conformation") when it switches current level. Another explanation could be that the steady-state solutions of the PNP equations themselves have multiple solutions, and the different current levels correspond to those different solutions. Because the actual current data is stochastic, it is not clear whether the "open channel" state is stationary or not. Indeed, the open probability and/or duration of the open state might be stochastic representations of the instability of the PNP equations. Ion channels also act (in many cases) as if they have two spatially distinct gates, one of which is normally open and the other normally closed. The opening and closing processes of these gates do not overlap in ion channels so there is always a time when both gates are open and current flows through the channel. The stability properties of the equations may determine many of these gating properties. It is hard to see how the stability properties of the equations (and underlying physics) could not be involved to some significant extent, even if that gating is modulated by other processes and involves additional physics or conformational changes. Finally, there is a vitally important class of "channel" proteins in which the two gates open and close in ping pong fashion, so current can never flow right through the channel pore. These channels form mediated transporters of the greatest biological importance. It is hard to imagine that the stability of multiple solutions of the PNP equations (and the underlying physics) is not involved in the correlated gating properties of transporters, even if that gating is modulated by other processes and involves additional physics or even conformational changes.

Clearly our methods will be challenged when we try to extend them to other geometries of channels, multiple regions with nonzero permanent charges, and the even more important problems of three or more ions of different charge (e.g., Na^+, Ca^{2+}, Cl^-). The depletion layers that then occur allow the wide diversity of devices (from amplifier, to limiter, to multiplier, etc.) that can be built from a single PNP transistor, and that can be described by numerical solutions of the PNP equations [67, 35, 41, 28, 71, 34]. An alarming diversity of treatments must arise from any perturbation analysis of PNP systems because such a diversity of real devices actually exist and are built on that (physical and intellectual) substrate! Existing mathematical analysis of the PNP equations will need to be extended to show how those different devices can be built on one substrate. That is to say, analysis is needed to show how different devices arise from different values of the boundary potential but just one set of differential equations (and boundary equations), with one set of parameters (other than boundary potentials). Many useful applications in the design of channels and semiconductors depend on this analysis.

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