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Electrodiffusion and Osmotic Water flow and its Variational Structure

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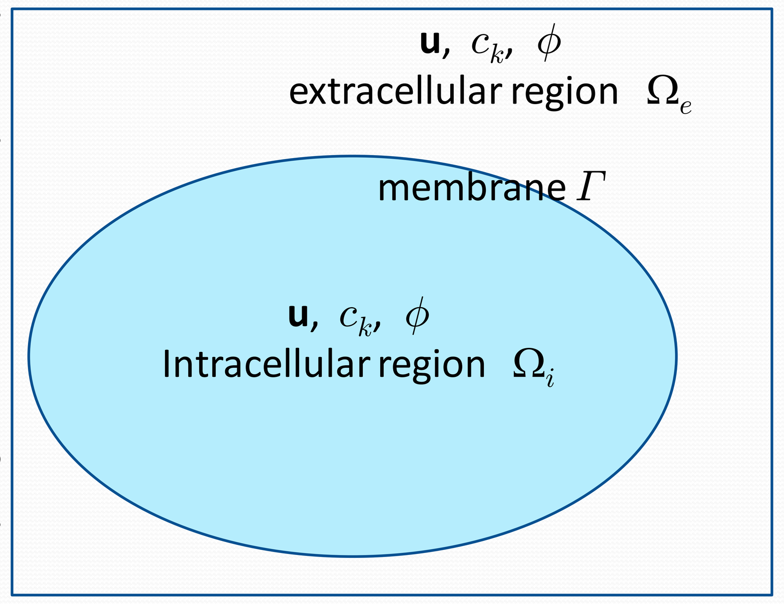
Background and Overview

We propose a system of partial differential equations (PDE) that describe electrodiffusion and osmotic water flow[1]. From a physical standpoint, this is a far-reaching generalization of the standard treatment of osmosis and electrodiffusion in irreversible thermodynamics to spatially extended systems[2,3]. To the best of the authors' knowledge, this is the first model in which osmotic water flow and electrodiffusion with deformable and capacitance-carrying membranes have been treated in a mechanically and thermodynamically consistent fashion.

Systems in which both electrodiffusion and osmotic water flow are important abound in physiology[4]. These include brain ionic homeostasis, fluid secretion by epithelial systems, electrolyte regulation in the kidney, fluid circulation in ocular systems, gastric protection, water uptake by plants etc. We believe the proposed system will have wide-ranging applications in the description of such physiological systems.

Model

Consider biological tissue as a three-dimensional region demarcated by the cell membrane Γ, separating the intra- and extracellular spaces Ω_i and Ω_e. Ionic concentrations c_k satisfy the following drift diffusion equation with the Poisson equation in Ω_i and Ω_e. We solve for **u** (fluid velocity), c_k, ϕ (electrostatic potential) in a self-consistent fashion.



$$\frac{\partial c_k}{\partial t} + \mathbf{u}\cdot\nabla c_k = \nabla\cdot(c_k D_k \nabla\mu_k), \quad \mu_k = k_B T \ln c_k + q z_k \phi$$

$$-\nabla\cdot(\epsilon\nabla\phi) = \sum_{k=1}^{N} q z_k c_k$$

The electrolyte fluid is assumed incompressible and **u** satisfies a fluid force balance equation (e.g. Stokes equation). On Γ, we impose boundary conditions on both sides of the membrane.

- Continuity of the electric flux density across and through the dielectric membrane, thus including membrane capacitance.
- Continuity of the ionic flux, where the ions pass through the membrane via ionic channels/transporters/pumps. Channel currents are driven by voltage and concentration gradients, and transporter/pump currents may be driven by energy supplied to the system (e.g. via ATP).
- Slip boundary conditions for the fluid velocity, taking into account the transmembrane water flow. The membrane thus moves with the flow.
- A stress-jump boundary condition across the membrane. Mechanical properties of the membrane are included.

Variational Structure and Thermodynamic Consistency

An important property of this model is that it satisfies the following free energy equality.

$$\frac{d}{dt}(G_S + E_{\mathrm{elas}} + E_{\mathrm{elec}}) = -I_p - J_p + J_a$$

G_S : entropic contribution to free energy, E_{elas}: elastic energy of membrane, E_{elec}: electrical energy (capacitive and bulk), I_p: dissipation due to electrodiffusion and viscous fluid flow, J_p: dissipation through passive transmembrane currents and transmembrane water flow, J_a: energy generation through active transmembrane currents

Proof: Integrate by parts. Treatment of boundary terms is tricky due to the nonlocal nature of electrostatic interactions and the fact that the membrane is dynamic; a novel calculus identity is proved to overcome this difficulty.■

Significance/Implications:

- We recover van t'Hoff's law of osmotic pressure (Π=cRT where Π is the osmotic pressure and c=$\sum c_k$) as the Legendre transform of the entropic free energy of ions.
- The free energy identity makes it straightforward to identify equality of cross coefficients as dictated by irreversible thermodynamics[2,3].
- Osmotic effects are typically slow, in which case electroneutrality is a very good assumption. The model can be formulated under this assumption without destroying the variational structure. ϕ becomes a Lagrange multiplier.
- Assumption of spatial homogeneity leads to a system of classical ODEs used in the study of cell volume control and electrolyte balance[5]. In this context, the above and its modifications serves as a hitherto unknown Lyapunov function[6]. We have recently established the stability of steady states of a large class of such pump-leak models, a longstanding problem in mathematical physiology[7].

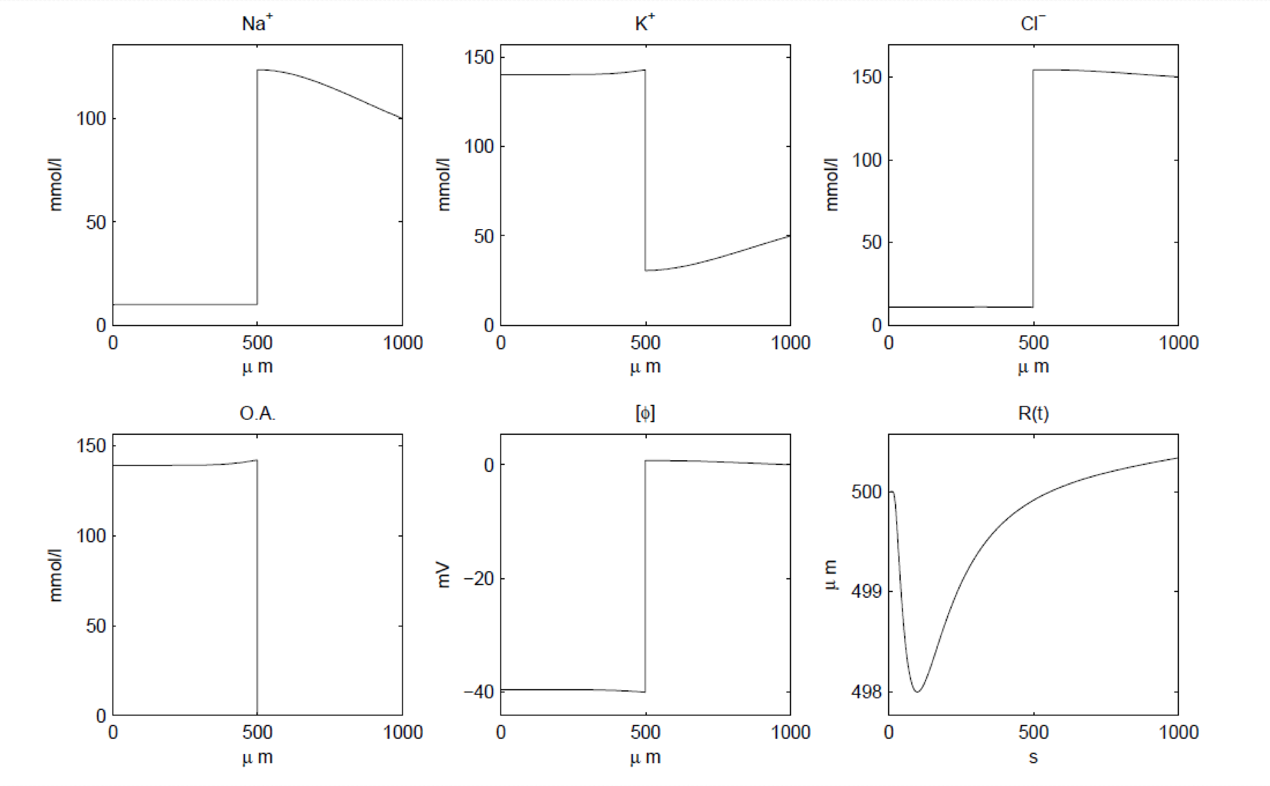
Tissue Level Model

To describe tissue level physiological phenomena, it is important to develop macroscopic models that do not track the biophysical details at the cellular or subcellular spatial scales. Such a model can be obtained by taking a homogenization limit of the cellular model. This procedure parallels the derivation of the bidomain model in cardiac and lens electrophysiology from three dimensional cable theory[8,9]. The tissue level model retains the variational structure of the cellular level model.

We are currently working to apply this theory to the study of ocular fluid circulation[10](cornea and lens) as well as to cortical spreading depression[11] in the brain, a (patho)physiological phenomenon associated with massive redistribution of ionic concentrations across the neuronal and glial membranes.

One Dimensional Simulation

As a computational demonstration of the foregoing theory, we simulate cell volume control for a spherically symmetric cell taking into account the effects of electrodiffusion and water flow through the cell membrane[1]. The cell is subject to a hypotonic stimulus, and the resulting changes in cell volume and electrolyte composition is given in the figure.



Abstract:

We propose a system of partial differential equations (PDE) that describe electrodiffusion and osmotic water flow. From a physical standpoint, this is a far-reaching generalization of the standard treatment of osmosis and electrodiffusion in irreversible thermodynamics to spatially extended systems. As far as we know, this is the first mechanically and thermodynamically consistent model of osmotic water flow and electrodiffusion in systems with deformable cells and membranes with capacitance and conductance. We use an energetic variational approach to enforce consistency and derive a field theory describing the flow, diffusion, and migration of ions, water, and the solution itself. The variational approach is particularly useful because it treats interactions automatically and consistently with a minimal number of arbitrary parameters. Electrodiffusion and osmotic water flow are involved in a wide range of biological functions of organs, tissues, cells, and organelles, including the homeostasis of ions in the brain, fluid secretion by epithelial systems, electrolyte regulation in the kidney, fluid circulation in ocular systems, gastric protection, water uptake by plants, etc. The field equations can be written with boundary conditions and parameters appropriate for the anatomy of each system. The field equations then form a physically and anatomically consistent model of biological function in the variational framework of modern field theory. The variational approach deals naturally with the many ionic solutions (containing a multitude of interacting components in a wide range of concentrations) and the wide range of conditions and forces used in experiments. Solving the PDEs will help suggest and interpret new experiments to understand the interaction of components, conditions, structure, and forces. In the view of classical physiology and biophysics, these interactions are the essence of biological function.

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