Poisson-Fermi Model of Single Ion Activities

Jinn-Liang Liu

Department of Applied Mathematics, National Hsinchu University of Education, Hsinchu 300, Taiwan. E-mail: jinnliu@mail.nhcue.edu.tw

Bob Eisenberg

Department of Molecular Biophysics and Physiology, Rush University, Chicago, IL 60612 USA. E-mail: beisenbe@rush.edu

Abstract

A Poisson-Fermi model is proposed for calculating activity coefficients of single ions in strong electrolyte solutions based on the experimental Born radii and hydration shells of ions in aqueous solutions. The steric effect of water molecules and interstitial voids in the first and second hydration shells play an important role in our model. The screening and polarization effects of water are also included in the model that can thus describe spatial variations of dielectric permittivity, water density, void volume, and ionic concentration. The activity coefficients obtained by the Poisson-Fermi model with only one adjustable parameter are shown to agree with experimental data, which vary nonmonotonically with salt concentrations.

1   Introduction

Comprehensive discussions of theoretical and experimental studies on the activity coefficient of single ions in electrolyte solutions have been recently given by Fraenkel [1], Valikó and Boda [2], and Rowland et al. [3], where more references can also be found. The Poisson-Fermi (PF) model proposed in this paper belongs to the continuum approach that traces back to the simple, elegant, but very coarse theory — the Debye-Hückel (DH) theory. As mentioned by Fraenkel, the continuum theory has evolved in the past century into a series of modified Poisson-Boltzmann (PB) equations that can involve an overwhelmingly large number of parameters in order to fit Monte Carlo (MC), molecular

dynamics (MD), or experimental data. Many expressions of those parameters are rather long and tedious and do not have clear physical meaning [1].

The Debye-Hückel model is derived from a linearized PB equation [4]. Extended from the DH model, the Pitzer model [5] is the most eminent approach to modeling the thermodynamic properties of multicomponent electrolyte solutions due to its unmatched precision over wide ranges of temperature and pressure [3]. However, the combinatorial explosion of adjustable parameters in the extended DH modeling functions (including Pitzer) can cause profound difficulties in fitting experimental data and independent verification because the parameters are very sensitive to numerous related thermodynamic properties in multicomponent systems [3]. The Poisson-Fermi model proposed here involves only one adjustable parameter.

The ineffectiveness of previous Poisson-Boltzmann models is mainly due to inaccurate treatments of the steric and correlation effects of ions and water molecules whose nonuniform charges and sizes can have significant impact on the activities of all particles in an electrolyte system. Unfortunately, the point charge particles of PB theories have electric fields that are most approximate where they are largest, near the point. PB theories are not an appealing choice for the leading terms in a series of approximations, for that reason. The PF theory developed in our papers [6,7,8,9,10] demonstrates how these two effects can be described by a simple steric potential and a correlation length of ions. The parameters of the PF theory describe distinct physical properties of the system in a clear way [9]. The Gibbs-Fermi free energy of the PF model reduces to the classical Gibbs free energy of the PB model when the steric potential and correlation length are omitted [9]. The PF model has been verified with either MC, MD or double layer data at (more or less) equilibrium [6,7,8], and nonequilibrium data from calcium and gramicidin channels [9,10].

Here, we apply the PF theory to study the activity properties of individual ions in strong electrolytes. The steric effect of all particles and the interstitial voids that accompany them are described by a Fermi-like distribution that defines the water densities in the hydration shell of a solvated ion and the particle concentrations in the solvent region outside the hydration shell. The resulting correlations produce a dielectric function that shows variations in permittivity around the solvated ion. The experimental concentration-dependent dielectric constant model proposed in [2] is used to define the concentration-dependent Born radii of the solvated ion in the present work. The experimental data of the activity coefficients of NaCl and $CaCl_2$ electrolytes reported in [11] are used to test the PF model.

2   Theory

The activity coefficient γ_i of an ion of species i in electrolyte solutions describes the deviation of the chemical potential of the ion from ideality ($\gamma_i = 1$). The excess chemical potential is $\mu_i^{\text{ex}} = k_B T \ln \gamma_i$, where k_B is the Boltzmann constant and T is an absolute temperature. In Poisson-Boltzmann theory, the excess chemical potential can be calculated by [12]

$$\mu_i^{\text{ex}} = \Delta G_i^{\text{PB}} - \Delta G_i^0, \;\; \Delta G_i^{\text{PB}} = \frac{1}{2} q_i \phi^{\text{PB}}(\mathbf{0}), \;\; \Delta G_i^0 = \frac{1}{2} q_i \phi^0(\mathbf{0}), \tag{1}$$

where the center of the hydrated ion (also denoted by i) is set to the origin $\mathbf{0}$ for convenience in the following discussion and q_i is the ionic charge. The potential function $\phi^{\text{PB}}(\mathbf{r})$ of spatial variable $\mathbf{r}$ is found by solving the Poisson-Boltzmann equation

$$-\epsilon_s \nabla^2 \phi^{\text{PB}}(\mathbf{r}) = \sum_{j=1}^{K} q_j C_j(\mathbf{r}) = \rho(\mathbf{r}), \tag{2}$$

$$C_j(\mathbf{r}) = C_j^{\text{B}} \exp\left(-\beta_j \phi^{\text{PB}}(\mathbf{r})\right), \tag{3}$$

where the concentration function $C_j(\mathbf{r})$ is described by a Boltzmann distribution (3) with a constant bulk concentration C_j^{B}, $\epsilon_s = \epsilon_{\text{w}} \epsilon_0$, ϵ_{w} is the dielectric constant of bulk water, and ϵ_0 is the vacuum permittivity. The potential $\phi^0(\mathbf{r})$ of the ideal system is obtained by setting $\rho(\mathbf{r}) = 0$ in (2), i.e., all ions of K species in the system do not electrostatically interact with each other since $q_j = 0$ for all j. We consider a large domain Ω of the system in which $\phi^{\text{PB}}(\mathbf{r}) = 0$ on the boundary of the domain $\partial \Omega$. The ideal potential $\phi^0(\mathbf{r})$ is then a constant, i.e., ΔG_i^0 is a constant reference chemical potential independent of C_j^{B}.

For an equivalent binary system, the Debye-Hückel theory simplifies the calculation by analytically solving a linearized equation of (2) so that the potential function $\phi^{\text{PB}}(\mathbf{r})$ becomes a constant [4]

$$\phi^{\text{DH}} = -\frac{q_i \kappa}{4\pi\epsilon_s}, \;\; \frac{1}{\kappa} = \left(\frac{\epsilon_s k_B T}{\sum_{j=1}^{2} q_j^2 C_j^{\text{B}} L}\right)^{1/2} \tag{4}$$

dependent of the bulk concentration C_j^{B}, where L is the Avogadro constant.

The Poisson-Fermi equation proposed in [9] is

$$\epsilon_s \left(l_c^2 \nabla^2 - 1\right) \nabla^2 \phi^{\text{PF}}(\mathbf{r}) = \sum_{j=1}^{K+1} q_j C_j(\mathbf{r}) = \rho(\mathbf{r}), \; \forall \mathbf{r} \in \Omega_s \tag{5}$$

$$C_j(\mathbf{r}) = C_j^{\text{B}} \exp\left(-\beta_j \phi^{\text{PF}}(\mathbf{r}) + S^{\text{trc}}(\mathbf{r})\right), \;\; S^{\text{trc}}(\mathbf{r}) = \ln \frac{\Gamma(\mathbf{r})}{\Gamma^{\text{B}}}, \tag{6}$$

where $S^{\text{trc}}(\mathbf{r})$ is called the steric potential, $\Gamma(\mathbf{r}) = 1 - \sum_{j=1}^{K+1} v_j C_j(\mathbf{r})$ is a void fraction function, $\Gamma^{\text{B}} = 1 - \sum_{j=1}^{K+1} v_j C_j^{\text{B}}$ is a constant void fraction, and v_j is the volume of a species j particle (hard sphere). Note that the PF equation includes water as the last species of particles with the zero charge $q_{K+1} = 0$. The polarization of the water and solution is an output of the theory. The water can be described more realistically, for example, as a quadrupole in later versions of the theory. The distribution (6) is of Fermi type since all concentration functions are bounded above, i.e., $C_j(\mathbf{r}) < 1/v_j$ for all particle species with any arbitrary (or even infinite) potential $\phi(\mathbf{r})$ at any location $\mathbf{r}$ in the domain Ω [9]. The Boltzmann distribution (3) would however diverge if $\phi(\mathbf{r})$ tends to infinity. This is a major deficiency of PB theory for modeling a system with strong local electric fields or interactions. The PF equation (5) and the Fermi distribution reduce to the PB equation (2) and the Boltzmann distribution (3), respectively, when $l_c = S^{\text{trc}} = 0$, i.e., when the correlation and steric effects are not considered.

If the correlation length $l_c = 2a_i \neq 0$, the dielectric operator $\widehat{\epsilon} = \epsilon_s(1 - l_c^2 \nabla^2)$ approximates the permittivity of the bulk solvent and the linear response of correlated ions [6,7,13,14], where a_i is the radius of the ion. The dielectric function $\widetilde{\epsilon}(\mathbf{r}) = \epsilon_s/(1 + \eta/\rho)$ is a further approximation of $\widehat{\epsilon}$. It is found by transforming (5) into two second-order PDEs [6]

$$\epsilon_s \left(l_c^2 \nabla^2 - 1 \right) \Psi(\mathbf{r}) = \rho(\mathbf{r}) \tag{7}$$

$$\nabla^2 \phi^{\text{PF}}(\mathbf{r}) = \Psi(\mathbf{r}) \tag{8}$$

by introducing a density like variable Ψ that yields a polarization charge density $\eta = -\epsilon_s \Psi - \rho$ of water using Maxwell's first equation [7]. Boundary conditions of the new variable Ψ on the boundary $\partial\Omega$ were derived from the global charge neutrality condition [6].

To obtain more accurate potentials at the origin $\mathbf{0}$, i.e., $\phi^{\text{PF}}(\mathbf{0})$, we need to consider the size and hydration shell of the hydrated ion i. The domain Ω is partitioned into three parts such that $\Omega = \overline{\Omega}_{\text{Ion}} \cup \overline{\Omega}_{\text{Sh}} \cup \Omega_{\text{Solv}}$, where Ω_{Ion} is the spherical domain occupied by the ion i, Ω_{Sh} is the hydration shell of the ion, and Ω_{Solv} is the rest of the solvent domain as shown in Fig. 1. The radii of Ω_{Ion} and the outer boundary of Ω_{Sh} are denoted by R_i^{Born} and R_i^{Sh}, respectively, whose values will be determined by experimental data. It is natural to choose the Born radius R_i^{Born} as the radius of Ω_{Ion} [12]. We consider both first and second shells of the ion [15,16]. The dielectric constants in $\overline{\Omega}_{\text{Ion}}$ and $\Omega \backslash \overline{\Omega}_{\text{Ion}}$ are denoted by ϵ_{ion} and ϵ_{w}, respectively.
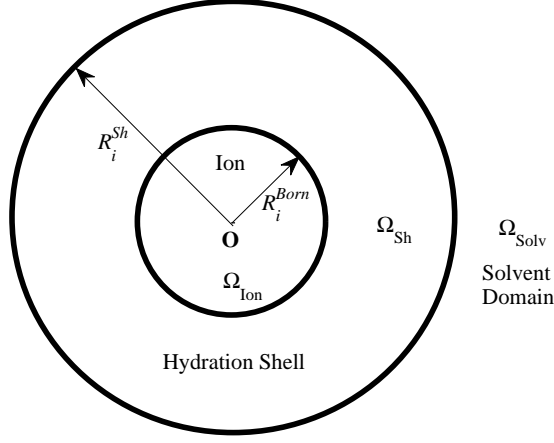
Fig. 1. The model domain Ω is partitioned into the ion domain Ω_{Ion} (with radius R_i^{Born}), shell domain Ω_{Sh} (with radius R_i^{Sh}), and solvent domain Ω_{Solv}.

The PF equation (5) then becomes

$$\epsilon \left(l_c^2 \nabla^2 - 1 \right) \nabla^2 \phi^{\text{PF}}(\mathbf{r}) = \rho(\mathbf{r}) = \begin{cases} q_i \delta(\mathbf{r} - \mathbf{0}) \text{ in } \overline{\Omega}_{\text{Ion}} \\ \sum_{j=1}^{K+1} q_j C_j(\mathbf{r}) \text{ in } \Omega \backslash \overline{\Omega}_{\text{Ion}}, \end{cases} \qquad (9)$$

where $\delta(\mathbf{r} - \mathbf{0})$ is the delta function at the origin, $l_c = 0$ in $\overline{\Omega}_{\text{Ion}}$, $l_c \neq 0$ in $\Omega \backslash \overline{\Omega}_{\text{Ion}}$, $\epsilon = \epsilon_{\text{ion}} \epsilon_0$ in $\overline{\Omega}_{\text{Ion}}$, and $\epsilon = \epsilon_s = \epsilon_{\text{w}} \epsilon_0$ in $\Omega \backslash \overline{\Omega}_{\text{Ion}}$. The shell radius R_i^{Sh} is determined by Eq. (6) as

$$S_{\text{Sh}}^{\text{trc}} = \ln \frac{V_{\text{Sh}} - v_{\text{w}} O_i^{\text{w}}}{V_{\text{Sh}} \Gamma^{\text{B}}} = \ln \frac{O_i^{\text{w}}}{V_{\text{Sh}} C_{\text{w}}^{\text{B}}} \Rightarrow V_{\text{Sh}} = \frac{\Gamma^{\text{B}}}{C_{\text{w}}^{\text{B}}} O_i^{\text{w}} + v_{\text{w}} O_i^{\text{w}}, \qquad (10)$$

where v_{w} is the volume of a water molecule and V_{Sh} is the volume of the hydration shell that depends on the bulk void fraction Γ^{B}, the bulk water density C_{w}^{B}, and the total number O_i^{w} (coordination number) of water molecules occupying the shell of the hydrated ion i. Note that the shell volume V_{Sh} varies with bulk ionic concentrations C_j^{B}. The occupancy number O_i^{w} is given by experimental data [15,16] and so is the shell volume that of course determines the shell radius R_i^{Sh}.

To deal with the singular problem of the delta function $\delta(\mathbf{r} - \mathbf{0})$ in Eq. (9), we use the numerical methods proposed in [6] to calculate $\phi^{\text{PF}}(\mathbf{r})$ as follows:

(i) Solve the Laplace equation $\nabla^2 \phi^{\text{L}}(\mathbf{r}) = 0$ in Ω_{Ion} with the boundary condition $\phi^{\text{L}}(\mathbf{r}) = \phi^*(\mathbf{r}) = q_i / (4\pi \epsilon_{\text{ion}} \epsilon_0 |\mathbf{r} - \mathbf{0}|)$ on $\partial \Omega_{\text{Ion}}$.
(ii) Solve the Poisson-Fermi equation (9) in $\Omega \backslash \overline{\Omega}_{\text{Ion}}$ with the jump condition

$$\left[\epsilon\nabla\phi^{\mathrm{PF}}(\mathbf{r})\cdot\mathbf{n}\right] = -\epsilon_{\mathrm{ion}}\epsilon_0\nabla(\phi^*(\mathbf{r})+\phi^{\mathrm{L}}(\mathbf{r}))\cdot\mathbf{n}$$ on $\partial\Omega_{\mathrm{Ion}}$ and the zero boundary condition $\phi^{\mathrm{PF}}(\mathbf{r}) = 0$ on $\partial\Omega$, where $[u]$ denotes the jump function across $\partial\Omega_{\mathrm{Ion}}$ [6].

The evaluation of the Green function $\phi^*(\mathbf{r})$ on $\partial\Omega_{\mathrm{Ion}}$ always yields finite numbers and thus avoids the singularity. Note that our model can be applied to electrolyte solutions at any temperature T having any arbitrary number (K) of ionic species with different size spheres and valences.

3   Results

Numerical values of model notations are given in Table 1, where the occupancy number $O_i^{\mathrm{w}} = 18$ is taken to be the experimental coordination number of the calcium ion Ca^{2+} given in [15] for all ions $i = \mathrm{Na}^+$, Ca^{2+}, and Cl^- since the electric potential produced by the solvated ion diminishes exponentially in the outer shell region in which a small variation of O_i^{w} for $i = \mathrm{Na}^+$ and Cl^- does not affect numerical approximations too much. Obviously the coordination number may be different for different types of ions and at different concentrations and so on. We were surprised that we can fit experimental data so well using a single experimentally determined occupancy number for all ions and conditions.

As discussed in [2], the solvation free energy of an ion i should vary with salt concentrations and can be expressed by a dielectric constant $\epsilon(C_i^{\mathrm{B}})$ that depends on the bulk concentration of the ion C_i^{B}. Following [2], we assume that

$$\epsilon(C_i^{\mathrm{B}}) = \epsilon_{\mathrm{w}} - \delta_i C_i^{\mathrm{B}} + \left(C_i^{\mathrm{B}}\right)^{3/2} \qquad (11)$$

with only one parameter δ_i, whose value is given in Table 1, instead of two in [2]. Note that $\epsilon(C_i^{\mathrm{B}})$ is a constant when the dimensionless C_i^{B} is given. It is not a function of a spatial variable $\mathbf{r}$ like $\widetilde{\epsilon}(\mathbf{r})$. The parameter δ_i represents the ratio of the factor of C_i^{B} to that of $\left(C_i^{\mathrm{B}}\right)^{3/2}$ in the original formula, where the factors of various electrolytes are taken from various sources of either theoretical or experimental data [2]. Our ratios δ_i in Table 1 are comparable with those given in [2].

The Born formula of the solvation energy can thus be modified as

$$\Delta G_i^{\mathrm{Born}}(C_i^{\mathrm{B}}) = \frac{q_i^2}{8\pi\epsilon_0\theta(C_i^{\mathrm{B}})R_i^0}\left(\frac{1}{\epsilon_{\mathrm{w}}} - 1\right), \quad \theta(C_i^{\mathrm{B}}) = \frac{\epsilon(C_i^{\mathrm{B}})\left(\epsilon_{\mathrm{w}} - 1\right)}{\epsilon_{\mathrm{w}}\left(\epsilon(C_i^{\mathrm{B}}) - 1\right)}, \qquad (12)$$

where R_i^0 is the Born radius when $C_i^{\mathrm{B}} = 0$ ($\theta(0) = 1$) and $R_i^{Born} = \theta(C_i^{\mathrm{B}})R_i^0$ is the concentration-dependent Born radius used to define Ω_{Ion} in Fig. 1 when $C_i^{\mathrm{B}} \neq 0$. The Born radii R_i^0 in Table 1 are cited from [2], which are computed from the experimental hydration Helmholtz free energies of these ions given in

[17]. All values in Table 1 are either physical or experimental data except that of δ_i, which is the only adjustable parameter in our model. All these values were kept fixed throughout calculations.

Table 1. Values of Model Notations

Symbol	Meaning	Value	Unit
k_B	Boltzmann constant	1.38×10^{-23}	J/K
T	temperature	298.15	K
e	proton charge	1.602×10^{-19}	C
ϵ_0	permittivity of vacuum	8.85×10^{-14}	F/cm
ϵ_{ion}, ϵ_{w}	dielectric constants	1, 78.45	
$l_c = 2a_i$	correlation length	$i = \text{Na}^+, \text{Ca}^{2+}, \text{Cl}^-$	Å
a_{Na^+}, $a_{\text{Ca}^{2+}}$	radii	0.95, 0.99	Å
a_{Cl^-}, $a_{\text{H}_2\text{O}}$	radii	1.81, 1.4	Å
$R^0_{\text{Na}^+}$, $R^0_{\text{Ca}^{2+}}$, $R^0_{\text{Cl}^-}$	Born radii in Eq. (12)	1.617, 1.706, 2.263	Å
δ_{Na^+}, $\delta_{\text{Ca}^{2+}}$, δ_{Cl^-}	in Eq. (11)	4.2, 5.1, 3.8	
O^{w}_i	in Eq. (10)	18	

The PF results of Na^+, Ca^{2+}, and Cl^- activity coefficients agree well with the experimental data [11] as shown in Figs. 2 and 3 for NaCl and CaCl_2 electrolytes, respectively, with various [NaCl] and [CaCl_2] from 0 to 2.5 M. In Fig. 4, we observe that the Debye-Hückel theory oversimplifies the Ca^{2+} activity coefficient to a straight line as frequently mentioned in physical chemistry texts [4] because the theory does not account for the steric and correlation effects of ions and water, let alone the atomic structure of the ion and its hydration shell as shown in Fig. 1. Both PB and PF results in Fig. 4 were obtained using the same atomic Fermi formula (10) for shell radii R^{Sh}_i in $\overline{\Omega}_{\text{Sh}}$ and the same concentration-dependent Born formula (12) for Born radii R^{Born}_i in $\overline{\Omega}_{\text{Ion}}$. Therefore, the only difference between PB and PF is in Ω_{Solv}, where $l_c = S^{\text{trc}} = 0$ for PB and $l_c \neq 0$ and $S^{\text{trc}} \neq 0$ for PF. Note that these two formulas are not present in previous PB models. Fig. 4 shows that the correlation and steric effects still play a significant role in the solvent domain Ω_{Solv} although the domain is $R^{Sh}_{\text{Ca}^{2+}} = 4.95$ Å (not shown) away from the center of the Ca^{2+} ion. The ion and shell domains are the most crucial region to study ionic activities. For example, Fraenkel's theory is entirely based on this region — the so-called smaller-ion shell region [1].

The PF model can provide more physical details near the solvated ion (Ca^{2+},
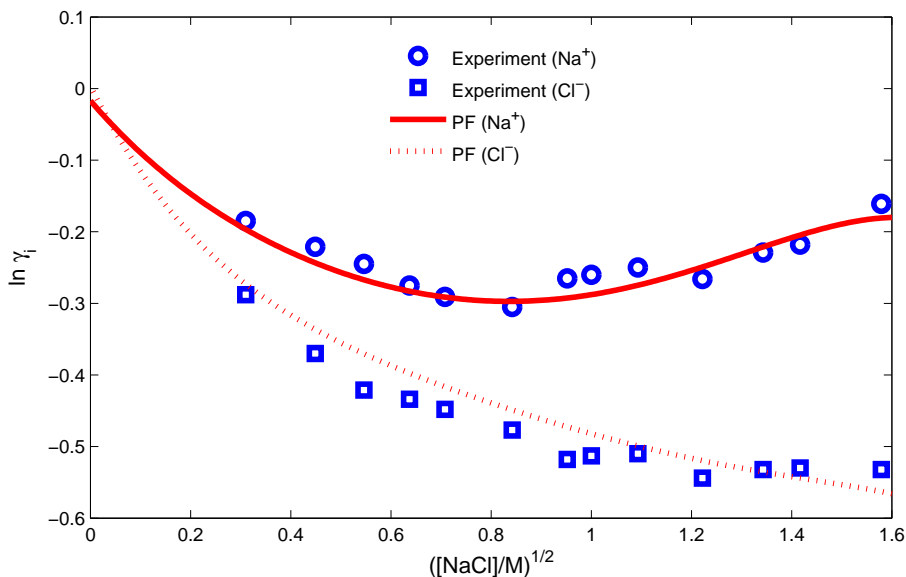
Fig. 2. Comparison of PF results with experimental data [11] on $i = $ Na$^+$ and Cl$^-$ activity coefficients γ_i in various [NaCl] from 0 to 2.5 M.
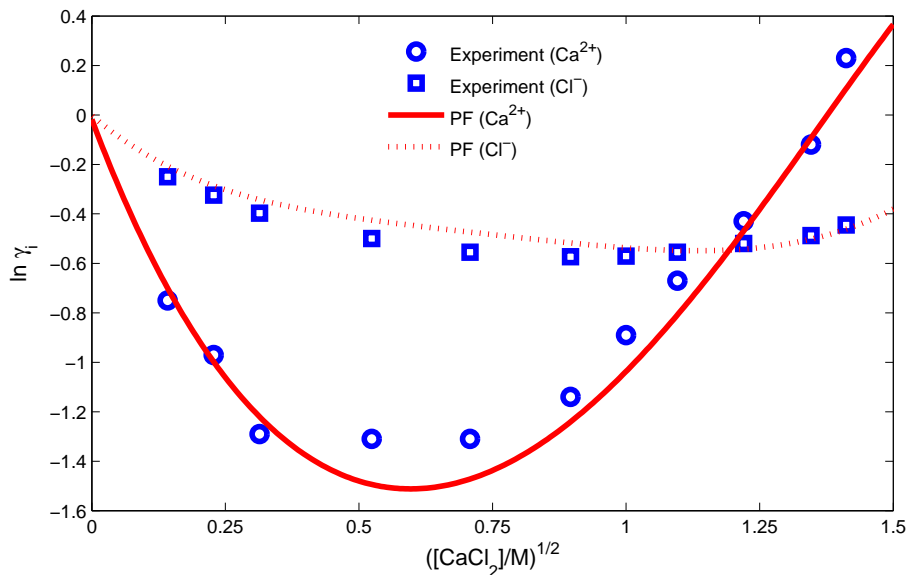

Fig. 3. Comparison of PF results with experimental data [11] on $i = $ Ca^{2+} and Cl$^-$ activity coefficients γ_i in various [CaCl$_2$] from 0 to 2 M.

for example) in a strong electrolyte ([CaCl$_2$] = 2 M) such as the dielectric function $\widetilde{\epsilon}(\mathbf{r})$ of varying permittivity (shown in Fig. 5), variable water density $C_{\mathrm{H_2O}}(\mathbf{r})$ (in Fig. 5), concentration of counterion ($C_{\mathrm{Cl}^-}(\mathbf{r})$ in Fig. 6), electric potential ($\phi^{\mathrm{PF}}(\mathbf{r})$ in Fig. 6), and the steric potential ($S^{\mathrm{trc}}(\mathbf{r})$ in Fig. 6). Note that the dielectric function $\widetilde{\epsilon}(\mathbf{r})$ is an output, not an input of the model. The steric effect is small because the configuration of particles (voids between
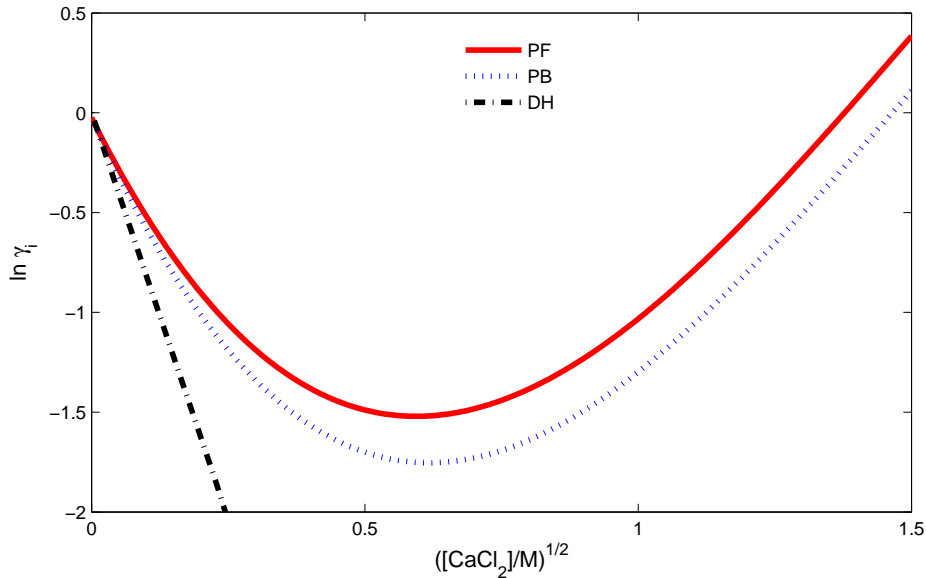
Fig. 4. Comparison of Poisson-Fermi (PF), Poisson-Boltzmann (PB), and Debye-Hückel (DH) results on $i = Ca^{2+}$ activity coefficients γ_i in various $[CaCl_2]$ from 0 to 2 M.

particles) does not vary too much from the solvated region to the bulk region. However, the variation of mean-field water densities $C_{H_2O}(\mathbf{r})$ has a significant effect on the dielectrics in the hydration region as shown by the dielectric function $\tilde{\epsilon}(\mathbf{r})$. The strong electric potential $\phi^{PF}(\mathbf{r})$ in the Born cavity $\overline{\Omega}_{Ion}$ and the water density $C_{H_2O}(\mathbf{r})$ in the hydration shell Ω_{Sh} are the most important factors leading the PF results to match the experimental data. PF theory deals well with the much more concentrated solutions in ion channels where void effects are important [9].

4 Conclusion

We have proposed a Poisson-Fermi model for studying activities of single ions in strong electrolyte solutions. The atomic structure of ionic cavity and hydration shells of a solvated ion is modeled by the Born theory and Fermi distribution using experimental data. The steric effect of ions and water of nonuniform sizes with interstitial voids and the correlation effect of ions are also considered in the model. With only one adjustable parameter in the model, it is shown that the atomic structure and these two effects play a crucial role to match experimental activity coefficients that vary nonmonotonically with salt concentrations.
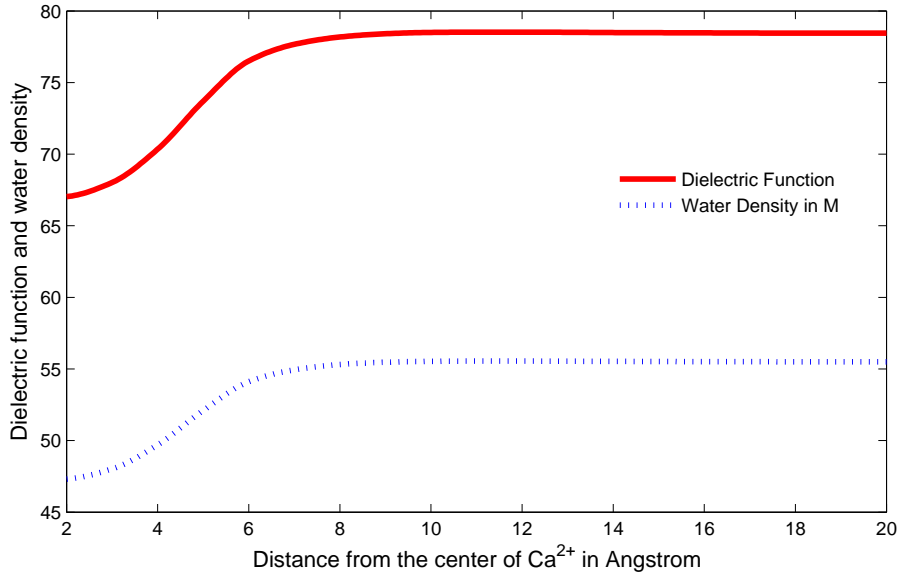
Fig. 5. Dielectric $\widetilde{\epsilon}(\mathbf{r})$ and water density $C_{\mathrm{H_2O}}(\mathbf{r})$ profiles near the solvated ion Ca^{2+} with $[\mathrm{CaCl_2}] = 2$ M.
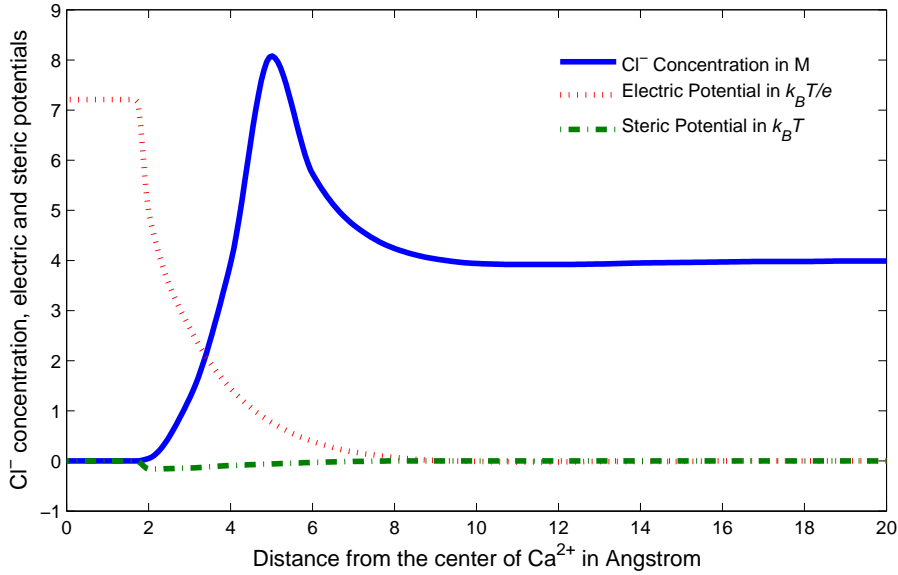

Fig. 6. Cl^- concentration $C_{\mathrm{Cl}^-}(\mathbf{r})$, electric potential $\phi^{\mathrm{PF}}(\mathbf{r})$, and steric potential $S^{\mathrm{trc}}(\mathbf{r})$ profiles near the solvated ion Ca^{2+} with $[\mathrm{CaCl_2}] = 2$ M.

5 Acknowledgements

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