

# Structural Analysis of Fluid Flow in Complex Biological Systems

Version 2

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## **Abstract**

Biology is about structure. Structures within structures. Organs within animals, tissues within organs, cells within tissues, and molecules, often proteins within cells. The structures are so complex that they can only be described by numbers. No numbers are of more importance than those that describe proteins. The numbers that describe coordinates of its atoms, often determined by Patterson functions (which are inverse Fourier Transforms of intensities) of crystal diffraction. Without these numbers, structural biology would hardly exist. Without numbers, engineering would not exist. Numbers are surely needed by the engineers who produce the x-rays diffracting from atoms of protein crystals. Devices of engineering have function. They are built to implement equations. Engineering devices use structures to implement equations, when power is supplied at the right places, that produces appropriate flows. Flows are the essence of life. Equilibrium means death in most living systems. Flows in biological structures are hard to analyze because we do not know input output equations in advance. Sometimes we do not know their function. Flows, forces, and structures of life (like those of engineering) are related by field equations of conservation laws, partial differential equations, constrained by location and properties of structures. Constraints are boundary conditions located on the complicated domain of biological structure. Dealing with this complexity is simplified if one systematically analyzes structure using the most general field theory known, electricity described by the Maxwell equations, without significant known error. Currents are involved because flows of biology usually involve migration of charges, convection of water and solutes, diffusion of ions that form the plasma of life, and their interactions. Interactions can dominate function. Here I show how a few complex structures can be understood in engineering detail. This approach may be useful in dealing with biological and medical issues in many other cases. In one limited case—the clearance of a toxic waste (potassium ions) from the optic nerve—this approach seems to have succeeded.

**1. Aim or Purpose:** To explain by example how structural analysis of fluid flow in complex biological systems.

**1.1 Fluid Flow.** A great deal of important biology involves fluid flow in large (cm) structures. The flows and biological function obviously depend on the structure, just as much as the work of an automobile engine depends on its structure. The flows of course depend on the atoms that flow and their properties. How could it not? But the forces that drive the flows and that allow biological control arise in proteins, as does so much of biological function. The central role of proteins is hardly surprising since the substrate of evolution, that creates biological systems, are genes, and genes make proteins! As a rule of thumb, it is much easier for evolution to mutate a protein, and select the mutation that produces more offspring, than to change a structure. And of course, as powerful as evolution is, it has a limited ability to change physical laws.

Proteins control flow in biology much as valves control flow in engineering. Channels and transporters are the transistors of life.(1-5) The transistors that make up our computers and smartphones are valves that control the flow of electrons and holes (6-9). The transistors are only helpful when they are embedded in integrated circuits (10, 11) that use the valves as switches (CMOS switches nowadays, for the most part). And the integrated circuits are embedded in groups that perform the main functions of the computers and smartphones they are in.

Life uses a hierarchy of systems much as engineers do (12). Channels (and transporters) are embedded in membranes. Membranes form organelles (e.g., mitochondria) and cells. Cells form tissues. And tissues form organs. Flows are important at all levels of the structural hierarchy that makes up biology. Function depends on the entire hierarchy of structures just as in a computer.

**1.2 Multiscale Flow.** Understanding fluid flow in biological structures thus involves a large range of separate systems, that involve many different forces and constraints, on a range of scales from atoms (literally, the ions of biological solutions), to membranes, organelles, cells, tissues and organs. The complexity of these systems has been the joy of classical anatomists and zoologists for centuries, as they assemble the parts list of the system and describe how they are connected. It is not such a joy to the physical scientists who prefer analysis and equations to descriptions, and simplicity of unstructured fields of force, to devices driven by power supplies.

The descriptive approach is as important in biology as it is in technology. No one could imagine understanding an automobile engine without a parts list, and drawings of its structure. A complete manual (usually oriented to what can go wrong, and so often called a repair manual) is almost a necessity nowadays.

**1.3 Complexity in Biology and Engineering.** Biological systems are often not much more complicated than an automobile engine, but we do not have a manual, and often we have only an incomplete set of drawings. We all seek a repair manual so we can take care of our loved ones in disease and decline as well as health and growth. Creating a manual is one of the main goals of modern medical and biological science.

The biological manual for organs and tissues that use fluid flow has to be built by scientists and clinicians from observations and experiments. We have to build the manual by reverse engineering. We look at outcomes and we try to infer mechanisms so we can predict other outcomes, and repair failures. These are the questions of inverse problems (13-17) and it will do a great deal of good if scientists confront that fact and the ambiguities and “ill posedness” (to use the jargon of the theory of inverse problems) inherent in reverse engineering and inverse problems. Mathematicians have shown in great detail that measurements of one type, even in a wide range of conditions, cannot determine underlying mechanisms in a robust way. No amount of measurement of the impedance of the black box can resolve that ambiguity. Think of the problem of measuring the values of resistors in a black box with a measured resistance of 10 ohms. That resistance could come from one resistor, or from two in parallel, or two in series, etc. etc.

**1.4 Resolving ambiguity. Inverse Problems.** Biologists know by training and intuition that the way to resolve ambiguity is often to make a different kind of measurement. To measure the total number of resistors, or better to determine the structure inside the box, with methods totally different from those used to measure the impedance of the box. Combining structural and functional measurements is usually needed to understand how biological systems work.

So, as we turn to our main issue of studying tissues and organs with fluid flow, we start with structure. We will do that in a few paragraphs, but we also must reiterate that we need to include the

physical properties of fluids and the biophysical properties of proteins, chiefly membrane proteins that are channels and transporters, before we can make much sense of the structures.

**1.5 Complex Fluids.** Biologists are fortunate that in the last decades physical scientists have learned to deal with flow in systems of this sort. The theory of complex fluids (18-26) has been developed to deal in a precise quantitative way with systems of fluids in which energy is stored, dissipated and drives flows on many scales in many ways.

It is only quite recently that workers have realized that any ionic fluid (like the intracellular and extracellular fluids of life) is a complex fluid (26-29). Traditionally those electrolytes have been treated as simple fluids (30-34), with some difficulties (35-40) as well known to leading experimental physical chemists (41). Indeed, it still is not widely realized that quasi crystalline biological systems (think skeletal and even cardiac muscle) are quite similar to liquid crystals (19, 42-46) that have been analyzed successfully with the theory of complex fluids for some decades now.

This review makes the case that a systematic approach to biological systems of complex structure is possible if it starts with structure, includes the structure (i.e., location and density) of channels and transporters, and uses the theory of complex fluids to analyze the flows and forces. (47-57)

## 2 Methods:

**2.1 Biophysics of Channels.** The analysis depends on the still evolving representation and understanding of the biophysics of flow in channels and transporters. The hope is that details of that biophysics may not always dominate the biological function and that the details can be described at technological resolution, as the Hodgkin Huxley model and equations describe channels and the current flows they produce.

It is reasonable to hope that physicists and mathematicians now entering the field will be able to use quantitative methods, inspired by the theory of complex fluids, to understand how these proteins work, at least well enough to understand their role in the transport of fluids. Structural and molecular biologists have made enormous progress in resolving the atomic details of channel and transporter function.

Sadly, models at atomic resolution, that use assumptions of periodicity and zero flow so atomic details can be computed in systems that function on biological time scales, cannot provide that understanding. Why? Because the biological systems are not periodic and depend on large flows for their function.

Computations that assume periodicity find it difficult to deal with irregular systems. Computations that deal with zero flows or nearly zero flows find it difficult to deal with systems that require large flows to function. It should be clearly understood that most of the devices of engineering depend on large flows. They need power supplies. Amplifiers without power supplies are no more worth studying than automobile engines with water in the gas tank. Most of biology requires flows.

Mathematics does not allow one to compute a finite value for something assumed to be zero. Ever. Approximations are possible of course, but they must be testable, either by mathematically defined error analysis, or by the scientific procedure of 'guess and check'. The approximation is the 'guess', and experimental work is 'the check' in most cases. When theories are made that are inconsistent with mathematics or ignore the need to check approximations,, confusion and debate are almost certain to result.

Methods that assume periodicity and zero flow are unlikely to allow the consensus needed for scientific progress or clinical application in systems that are not periodic or systems that depend on flow. Different research groups deal with the inconsistency (between periodic assumptions and irregular reality for example) in different ways and so get different results. Discussion of inconsistencies is more likely to produce emotional heat than intellectual light, given the reality that science is done by human beings with practical needs that are in the foreground of their thoughts, understandably enough.

We begin with the structures and move from their relative certainty to the uncertainty of complex fluids later on.

**2.2 Role of Structure.** The role of structure in the multiscale hierarchical analysis I will describe is best seen in the analysis of electrical properties started by Falk and Fatt (58) passed on to their graduate student (59) as they looked at the internal membranes of skeletal muscle. Falk and Fatt decided to measure the linear properties of cells, by applying currents that crossed the membrane (with an intracellular microelectrode) instead of by applying currents that avoided the membrane and flowed around cells. In that way, they could use the highly developed methods of linear circuit theory or linear systems analysis (60-70) to work out how current and potential spread in the complex membrane systems of skeletal muscle. Instead of assuming that spread, as cardiac electrophysiologists did, Falk and Fatt measured it. That way skeletal muscle was spared the metastasis of artifactual conductances and ion channels (71, 72) that produced a tangle of confusion in cardiology until single channel recording and the identification of channels as proteins cut through the tangle and brought the clarity of direct measurement to the field (73, 74).

The technical issues of measuring and analyzing linear circuits took some time to perfect but a variety of scientists (58, 59, 75-79) (reviewed by (80-99)) succeeded in making a powerful and general tool.

**2.1 Biophysical Invariants: membrane capacitance and internal resistance** A crucial factor in the success of the circuit analysis of complex tissues was the fortunate fact that two properties of cells were simple. The interior of cells is a pure resistance. The membrane capacitance is essentially a

constant independent of ionic conditions. Both statements are documented later in this paper. Thus, methods could be designed to exploit the constancy of these values (76-79, 84)

An essential step in analyzing the capacitance, and of course the circuit properties in general, was understanding what is intra and what is extracellular space. ***This seemingly trivial necessity needs to be stated explicitly when dealing with new systems that do not have a long history of biophysical analysis.*** Experiments are needed in those cases, not arguments. Localizing junction proteins is of course a help but nothing replaces direct measurements of accessible spaces with markers.

A variety of marker systems are available as a literature search will show (starting for example with (100, 101), but the horseradish peroxidase system seems to have been lost in that variety despite its significant advantages of black-and-white contrast yielding nearly 100% efficiency (102, 103) in chemically treated, fixed, and sectioned material.

**2.2 Morphometric Measurement of Area** Direct visualization of living material with modern microscopic techniques has the advantage of using native tissue without the artifacts of fixation. The advantage is profound when estimating volumes. ***But when determining electrical circuits, it is the area which is the main goal*** because the area is connected to the electrical parameter capacitance in a definite way, independent of ionic conditions. Measurements of area with light microscopy is dangerous because of its limited resolution, no matter what the wonders of modern techniques. The electron microscope will include fine structures (bends and wiggles) not apparent with the highest resolution light microscopy. These complexities can be significant, indeed very significant, both quantitatively and qualitatively.

It is important however that seeing not be believing. The visualization methods must be accompanied by objective validation before they can be trusted to give reliable results.

Making measurements of area is not easy. It is quite difficult to make such measurements with the obvious curve tracing methods, because of distortion and disruption in the sectioning methods used in most histology and because of the huge irreproducible systematic errors people make when asked to trace structures. There is also large systematic error in actually performing the tracing, even when the membrane is visible as morphometric calibrations against known systems will show. And then there are the frequent regions where structures are not clearly defined. What do you do when the membrane cannot be defined?

Another less known approach is the statistical sampling methods called 'stereology' (81, 104-107). A series of articles from Brenda Russell Eisenberg and her group (83, 107-116) shows the power of these methods in a biological and functional context.

Stereological methods exploit the fact that it is not necessary or even wise to extract all the structural information in a single micrograph or section. Many sections and micrographs must be measured if the tissue or organ is to be adequately sampled. The variance of estimation produced by using only part of the data from each micrograph is often not significant in this process. The sampling process turns out to be much less tedious than one might expect, particularly when aided by modern computer technology and many clinically interesting systems might well benefit from stereological analysis. A search for stereological software will be helpful.

Once the extracellular and intracellular volumes, and membrane areas are determined, the next step is the construction of a model. An important part of constructing a model are the properties of the components of the model. The intracellular medium enters into essentially every biological model, and its representation was the source of controversy years ago.(117, 118) The issue is now settled. In electrical models: the intracellular medium is a resistance.

The intracellular medium can safely be modeled as a resistance because current goes around obstacles, even very dense obstacles, see (119) that builds on the still useful 1873 analysis of Maxwell (120). Impedance measurements in which current flows around the structure are rarely useful.(121, 122). The membrane resistance serves to isolate and define the inside of a cell. The membrane makes a cell an obstacle to electricity. The membrane is such a high resistance (123-129) that current flows in a parallel path even if that path is only a 10 nm width of salt solution. It is only when intracellular objects are within a Debye length or so (typically less than a nanometer in biological salt solutions) that current flow in the intracellular solution is significantly impeded, beyond the reduction of cross sectional area.

This simple fact was not apparent to early workers in electrophysiology, some of whom insisted on attributing selectivity and other special properties to the cell cytoplasm (117, 118) instead of the membrane where we now know they reside. We now know that biology uses proteins in membranes (as it uses proteins in so many other places) to provide selectivity and otherwise control many cellular processes. Few if any biological processes are controlled by the cytoplasm in general, although local concentrations of specific chemicals—organics and  $\text{Ca}^{2+}$ —are often used for control.

The modern view is that complex electrical properties are not present even in the extraordinarily packed interior of skeletal muscle fibers.

**2.1 Measurements of Impedance.** Measurements were made of the impedance of skeletal muscle (130, 131) because they are the most extreme example of an organized dense cytoplasm. **If there is any system in which cytoplasm might produce significant complexities in current flow, it is skeletal muscle.**

Specifically, measurements were made of the longitudinal impedance down a very long length of a single muscle fiber. There is hardly any cell more solidly packed with proteins and structures than skeletal muscle, as a glance at its ultrastructure will show. If current is significantly impeded in its longitudinal flow in any tissue, it should be in skeletal muscle. The diameter of typical frog skeletal muscle fibers might be 50 micrometers and their inherent longitudinal periodicity is that of a sarcomere, say 2 or 3 micrometers (depending on stretch). Thus measurements over a centimeter or so of fiber length give ample opportunity for interference in resistive current flow by the organelles and proteins within each sarcomere.

The dissection of centimeters of single fibers in these experiments was a significant personal achievement of graduate student Joe Leung, aided by his appropriate use of an air delivered pharmacological agent to reduce his already minimal hand tremor.

The results were striking. The current and voltage down the length of a muscle fiber were related by a resistor with almost no phase change even though the muscle fiber appears entirely filled with contractile proteins and membranes, when viewed in the electron microscope. The resistance representation was precise within 1% in the cases with the largest error, and was unmeasurable in the great majority of muscle fibers, whether skinned or intact.

The next choice of circuit component that must be made is the representation of the membrane itself. For strictly electrical measurements, as used in linear impedance analysis, the membrane was represented as a capacitor, with some success in a range of papers studying lipid bilayers starting with the life's work of Denis Haydon (132-140), then moving to nerve axon (141-145), skeletal muscle, cardiac muscle, and the lens of the eye (84-99). Various epithelia (146-153) were analyzed this way in a literature starting with (146) using the methods described by Clausen and Eisenberg (76-78).

For analysis of more general properties, beyond linear circuit properties, a realistic representation of transport properties of membrane proteins is required, and a linear representation is inadequate. The actual nonlinear properties of channels and transporters must be known, of course, but that is not enough. The spatial distribution of membrane proteins is needed. And it should go without saying that the properties of the channels and pumps must be measured in solutions that are a reasonable approximation to their natural environment. It is not satisfactory, for example, to study a channel or transporter in millimolar concentrations of calcium ion on both sides of the transporter molecules, when in life they face intracellular calcium concentrations of some  $10^{-8}$  M that change the properties of the channel or transporter if the calcium concentration is changed.

### 3.0 Results:

Decades of work has shown biologically what is obvious physically: the type, properties, and location of transporters (and their density) determine the flows and thus the function of many tissues and organs of great biological importance. Kidneys and transporting epithelia have been extensively studied and are a major success of physiology and biophysics (154-158). The distribution of channels is also known from work on myelinated and unmyelinated nerve, and skeletal and cardiac muscle, to be dominant determinants of their function.

Unfortunately, many interesting tissues, of great clinical importance, have not been studied well enough experimentally to allow as specific prediction of flows as would be helpful (159-164). The types of transporters and channels and their locations must be measured in those tissues and organs and the input output properties of those proteins must be known before any theory can be expected to provide robust realistic results. For example, one must expect the location of transporters to be as important for the glymphatic hypothesis of flow in the brain (47, 50-53, 55, 56, 165-167) as the location of channels and transporters is in the kidney. (168-171)

Much can be done when that information is available as we shall see in studies of transport in the lens of the eye, and we will use that example of success in this paper.

**3.1 The lens of the eye.** The electrical properties of the lens of the eye were resolved by Rae and Eisenberg (172) who explained the substantial differences between measurements made with single microelectrodes and two microelectrodes. The single microelectrode measurements obscured the interesting membrane properties because of three dimensional effects (123) analyzed first by (173, 174) and later by methods of singular perturbation theory (124-129, 175-177). The misleading three dimensional effects can be easily removed once theory shows where they come from. Then single electrode and two electrode measurements are in agreement. Otherwise, they differ by orders of magnitude producing confusion and recrimination between laboratories using different methods.

The lens was described as a bidomain syncytium consisting of two domains, one intracellular and the other extracellular, connected by membranes very much in the spirit of the description of the transverse tubular system of skeletal muscle fibers, as reported (for example) in (58, 59, 75-79) and reviewed by (80, 81).

The models of the lens exploited the important fact that membranes were capacitors, in parallel with resistors (in impedance measurements), while cytoplasm was a resistance. A rather complete analysis of the tissue was possible leading to many useful results (84-99).

**It is of great importance that the models were built from defined structures with components with definite known properties.** Otherwise models degenerate into uncertainty, with approximations made in different laboratories in different ways, that are hard to reconcile. Such approximations lead to more heat than light in the interactions of scientists, who are after all human beings, particularly when they are forced to compete for resources.

The essential feature of the lens (86, 172, 178-184) that separates it from almost all other tissues is its transparency. That transparency means the lens cannot contain blood vessels which would seriously degrade optical performance. The lens is a living tissue that is maintained in its function for some 60 or 70 years in most humans. It must clear metabolic wastes and supply nutrients throughout its large (centimeters) cytoplasm. Other cells in the body are only a few micrometers from blood vessels. Indeed, as students at University College (London) Physiology and Biophysics were taught in the 1960s: animals are made of cells decorating their circulatory system. That is how extensive a circulatory system is needed so that convection (i.e., blood flow) can replace ineffective diffusion.

**The lens uses convection to supply and support its cytoplasm, but its convection is driven by an osmotic pump, not by the mechanical pump of the heart.** The electrical analysis of the lens is thus of limited interest and could only be a part of the understanding the lens itself when extended to include water flow by Richard (Rick) Mathias more than anyone else.

Mathias and his collaborators have extended the structural analysis of electrical properties of the lens to the analysis of the vital function of convection. It has been his life's work and is (in my view) a triumph of biophysics, theory and experiment. (95, 179, 180, 182, 183, 185-202). Indeed, Mathias resolved (to my satisfaction) several damaging controversies in epithelial physiology (154, 203-205) as he studied water flow in the lens and in a more general context (206-208).

**3.2 Two Domain Model.** Mathias built his analysis on the two domain approach of the electrical models but included convection and water flow using an engineering circuit approach (95, 185, 206, 207, 209, 210) written in the same spirit of approximation as the cable equation of Kelvin (211, 212) that he used to design the Atlantic cable—"a thread across the ocean"(213) . The Atlantic cable provided the first high speed (electrical) communication between Europe and the United States, cutting communication delays from weeks to seconds or minutes. The cable equation was used in biology, where it played a crucial role in our understanding of nerve conduction (214-216) and it was not thought necessary to derive it, because it seemed so obvious physically (with a few exceptions, important when cables had large diameters, or recording was done on or near point sources of current (123-129, 173-177, 217-220).

As satisfying as the engineering treatment of the lens was to me personally, it was not found convincing I fear by biologists and clinicians unfamiliar with cable theory or even elementary circuit methods. And to be fair, water flow is not current flow, and the possibility existed that the engineering model might prove less solid a foundation for water flow than for the flow of electricity.

**3.3 Fluid Flow Starling Equation** Huaxiong Huang (the leader of the project) at the Fields Institute (Toronto) and Shixin Xu (his immensely capable postdoctoral fellow and then faculty colleague) were interested in analyzing convection and flow in biological systems with significant structure and asked if I might help their graduate student Yi Zhu (at York University Toronto) choose and formulate a problem. We chose the lens as our initial project because of the magnificent experimental work of the Mathias group, that went from molecules to pressure measurements, with electrical structure worked out along the way, not knowing of course whether we could do this problem at all, let alone how it would relate to the Mathias engineering approach.

Our essential challenge was that we approached the problem as mathematicians unwilling to start with anything except the structure of the tissue, conservation laws, and simple known representations of the properties of channels and membranes. No one had solved such a complex system of partial differential equations, as far as we knew, in this context, although Weinstein had done something in this spirit in the specialized (and very important) context of the kidney (170, 221-225).

We wished to avoid simplified models using ordinary differential equations for mathematical, scientific, and human reasons. Mathematically, the ordinary differential equations of (chiefly) models lumped in space could not be derived without dealing with the full field equations and conservation laws. Scientifically, it was entirely unclear to us how to make such lumped models (or other simplified models) that would be valid with one set of unchanging (more or less) parameters over the range of conditions of clinical and physiological interest. Different conditions and different questions might well need different approximate (or lumped) models. After all, the Taylor expansions of nonlinear differential equations are notoriously complex and different ranges of parameters yield very different approximations. (I have never forgotten the graph and apparently simple ordinary differential equation that Julian Cole used to start his course on asymptotics Fig. 2.3.2 of (175) and the surrounding discussion.) Humanly, we had lived through controversies comparing simplified models in which investigators could not find a common ground for comparison, and found it easier to find fault than to solve problems.

We found to our surprise (and to my horror) that the fundamental transport law called the Starling equation in the physiology literature for more than a century (since 1896 (226)) did not have a mathematically satisfying derivation despite the best efforts of many (227-233).

The central issue, from our point of view, was the classical treatment of ionic solutions as simple liquids (nearly ideal gases, so well presented in (30-34). It seemed clear to us that ionic solutions had too many too complex forms of energy (and flow and dissipation) to fit comfortably in a tradition that was so thoroughly dependent on the theory of ideal (i.e., perfect) gases **at equilibrium (with zero flows** of all components) from which it grew (234).

Consider for example the simple issue of incompressibility. While it is certainly true that ionic solutions as a whole are nearly incompressible, as is pure water, this fact is highly misleading. The individual components of ionic solutions change number density enormously (i.e., concentration) as location and conditions change. Ionic solutions are mixtures of compressible fluids in which the 'entropic' energy stored by variable concentrations are major determinants of the behavior of these systems. These systems are non-ideal: **the finite size of ions (and many other effects) mean that the free energy of any one species depends on every other.**

In other words, everything in these systems is coupled. Surprisingly, it is an historical fact that distinguished biochemists and biophysical chemists like John Edsall were unaware as late as the 1970's that activity coefficients were functions of the type and concentration of all ions. They taught students like the author about nonideal solutions, with activity coefficients less than one. The reality that these activity coefficients depend on everything and thus properties of components of solutions are coupled to concentrations of other components was not taught. Indeed, it is absent in the 'independence principle used indiscriminately by Hodgkin and Huxley for ions in solution, ions in selective channels, and ions in non selective channels.

Even the compressibility of the total solution and water itself comes into play when nonideal solutions are considered. The small compressibility of water and ionic solutions turned out to be important later with Jinn Liang Liu and I tried to construct a theory of the free energy of ionic solutions in a series of some ten papers, reviewed in (38). If this compressibility were not included self-consistently, we could not compute properties of the so called simple primitive model in which water was spheres along with the spheres of ions. Perhaps others have had the same problem, although we are not sure. The practical consequences are important. Experimental properties of ionic solutions cannot be computed and in that sense cannot be understood, in my opinion, unless a selfconsistent theory can be built. See p. 351 of (41) for an experimental perspective and judgement as strong as my own.

**3.4 Fields cannot be constant.** Turning to more general issues, that require a more complex theory, we look at the central fact that is present in a wide range of theories and simulations over an enormous range of scales.

Force fields like electrical potential can never be assumed to have a specific shape (like constant fields) independent of conditions (2, 12, 15, 235-246). As conditions change, even a little bit, the shape of the electrical potential (i.e., the electrical force field) changes a lot. One suspects similar considerations apply to the force fields used in molecular dynamics, for example. Can one really expect the force fields for sodium ions in bulk solution with low concentration (say 200 millimolar) and tiny electric fields to be the same as those for sodium ions in an ionic channel with molar (or higher) concentration and fields of 100 mV in say 2 nanometers?

These changes in electric field are dramatically illustrated by semiconductors. The changes in field determine the properties of bipolar transistors in which the charge carriers are nearly ions, namely holes and electrons.

The central property of bipolar transistors is rectification (7, 247-255). Rectification is produced by changes in the shape of the field. The field is never constant. The field must always be computed from the charges, even in Brownian motion theory (238, 256) as Mott (257) quickly learned as he tried to understand real rectifiers. Mott quickly abandoned the idea of constant fields (258, 259). The constant field Goldman Hodgkin Katz GHK theory of selectivity (popularized by (262)) leaned heavily on Mott's prewar work, as Hodgkin and Cole (Goldman's Ph.D. supervisor) made clear to me in many conversations around 1962. But Mott abandoned that theory just as it was introduced to biophysics (260, 261) where it metastasized into nearly universal (mis)use (262). This strong language is warranted because constant field GHK does not even include the main properties of a channel protein that surely control permeability. It does not include charge, diameter of ions, diameter of a pore, and so on.

Simple liquid theory, let alone the theory of ideal gases, cannot deal with the reality of ions in devices, channels, transporters, enzymes, and near nucleic acids.

**3.5 Ions are often crowded where they are important.** The ions in ionic solutions are usually crowded where they are most important (263, 264) inside ion channels (265-267) or near electrodes (38-40, 268) (of batteries and electrochemical cells) or in active sites of enzymes (269) where number densities of more than 20 Molar are often found. (For comparison, solid NaCl is something like 40 Molar.) At these number densities, the ionic solutions cannot be described in any useful way as ideal gases, simple liquids or fluids. **A great deal of energy is involved in their nonideal properties.** And interactions are profoundly important. The activity coefficient (or if you prefer the excess free energy) of each ion depends on the number densities of every other ion in a solution. Sadly this simple fact was not known to me as a student or for many years and I do not believe was understood by many biophysical chemists who taught me (270, 271) although I of course could be mistaken in this supposition.

The systems of biology usually depend on flow for their actual function. These systems are not close to equilibrium in the Green Kubo sense (272). Indeed, when used in devices, these systems always have power supplies needed for their function. The biological device of the nerve axon for example does not exist as a device if there are no power supplies (here gradients of concentration of ions).

Power supplies require spatially nonuniform boundary conditions that drive substantial flow. An equilibrium treatment of such devices is silly and is likely to be fundamentally misleading as a study of amplifiers would be with zero volt power supplies, or the study of an automobile engine with water in the gas tank.

Flow produces friction (273) as is obvious when one remembers that ionic solutions are condensed phases with little empty space. Ideal gases have no velocity or flow or friction for that matter. Real atoms and molecules are not at all ideal. They collide (i.e., change direction) at enormous rates (think  $10^{15}$  times per second). Even four collisions of deterministic trajectories produces nearly random (Gaussian) noise. All motion in ionic solutions is dissipative and theories (262, 274) that do not include friction or other form of dissipation are as pointless as theories of reproduction of mammals that do not deal with two classes of structure (and function), male and female. These issues were discussed sometime ago (3, 275-284) and became clear to me as I learned of the thesis work of Kim Cooper (285) that was later extended in our work together (286-288).

For all these reasons, the classical approach based on the theory of ideal gases and simple liquids, that described ion membranes without mention of the charge or structure of membranes or proteins ((260, 261) popularized by (262)) needed to be replaced by something that was not so obviously false.

Fortunately, in the last decades, a theory of complex fluids has been developed that allows treatment of remarkably complex problems (289), in which energy is stored **and dissipated** as it flows far from equilibrium. The theory of complex fluids deals with many types of fields and flows consistently (i.e., the diffusion, migration, and bulk flow of the ions and water in electrolyte solutions), and is mathematically well defined, as a variational principle developed by Chun Liu, more than anyone else. The EnVarA approach (for Energy Variational Approach) was named (26) with the help of Fred Cohen and is described in the tutorial lectures (18) that yield well defined field theories—usually partial differential equations—in structures defined by boundary conditions (19). The partial differential equations are fully coupled field theories including dissipation in which “everything interacts with everything else” as it obviously must in the concentrated salt solutions so important to life, and our electrochemical technology(20-25). Ionic solutions need to be dealt with by mathematics appropriate for these physical realities (29, 38, 239, 263, 264, 290-295)

**3.6 Plasmas of Life are Complex Fluids.** The first task of our group (led by Huaxiong Huang, Shixin Xu and Zilong Song) was then to apply EnVarA to derive a transport equation like Starling had written, but one that included all interactions consistently. To do this required a reworking of the techniques of EnVarA. The usual methods had to be modified to accommodate membranes within the domain as found in the lens and most other syncytial tissues and not just in boundary conditions on the membranes defining the outer borders of the system. That took some time but the result was pleasing: we rederived the Starling equation, rather to our surprise.

It should be clearly understood, however, that we were not able to deal with nonideal ionic components. The full treatment of flow for ions with substantial excess (electro)chemical potentials (i.e., with activity coefficients much less than one) remains for the future. The essential difficulty is dealing with the dependence of the excess electrochemical potential (i.e., activity coefficient) on every ionic species in real solutions. The classical assumption that activity coefficients of an ion (say  $\text{Na}^+$ ) depend only on the  $\text{Na}^+$  concentration must be replaced by the reality that the activity coefficient depends on all ion concentrations, e.g., the activity coefficient of  $\text{Na}^+$  depends on the concentration of  $\text{K}^+$ . This cross dependence can have profound practical effects throughout chemical science as Wolfgang Nonner (266) discovered when we evaluated the activity coefficients of  $\text{Ca}^{2+}$  from tables of EGTA solutions used throughout chemistry and biochemistry (296-300), particularly the biochemistry of contractile proteins. The activity coefficient of  $\text{Ca}^{2+}$  was assumed to be the same in equimolar  $\text{Na}^+$  and  $\text{K}^+$  solutions. Our results showed it was not. And of course it could not be, given the different excess chemical potentials and activity coefficients of these ions.

**3.7 Structural model of the lens.** Using variational treatment of bulk solutions, the next task was to construct a structural model of the lens. This was an easy task, following in the footsteps of the Mathias group (op. cit.) whose structural model we essentially copied using their representation of the

anatomy and histology of the lens, and the distribution of connexins that connect lens fibers. We used a much more primitive spatial distribution of channels and that needs to be improved in future work.

The results have been recently published in a bidomain model of lens circulation (301) with even more details in the thesis of Yi Zhu (302).

**3.8 More Detail: derivation of the field equations.** It is tempting to just write down the physically intuitive field equations for the bidomain model (129, 177)—correct, or rather not incorrect, as that approach might be—using the general approach of the structural analysis of electrical properties ((79, 84, 123-125, 303-305) conjoined to a representation of fluid flow. An abundance of caution, and mis-steps using this approach in our several careers, motivated a complete derivation from conservation laws themselves, and indeed a derivation of the underlying fluid flow laws like Starling's equation (306). Such an approach also makes it more difficult to quibble about the best of multiple representations, since conservation laws are conservation laws, with representations as integral or differential equations that are not in dispute. The following derivation follows that of Yi et al closely, lest simplification lead to inadvertent ambiguities.

The derivation follows the approach used for the analysis of electrical structure (85, 88, 176) joined with the laws of fluid flow (306) using the variational approach to the theory of complex fluids (26) of Liu and collaborators (18-25). The conservation law for each species of ions is

$$\begin{cases} \int_{S_e} J_{ex}^i \cdot n_v dS + \int_{S_i} J_{in}^i \cdot n_v dS = 0, \\ \int_{S_e} J_{ex}^i \cdot n_v dS - \int_{S_m} J_m^i \cdot n_m dS = 0, \end{cases} \quad (1)$$

where we define symbols below. A physical discussion of the symbols and their meaning is found in the original paper (302).

$$\begin{aligned} J_l^i &= u_l C_l^i - C_l^i M^i \nabla \mu_l^i \quad l \in \{ex, in\}, \\ \mu_l^i &= k_B T \ln(C_l^i) + z^i e \phi_l, \\ J_m^i &= \alpha (j^i + a^i) n_m, \\ j^i &= g^i \frac{1}{e z_i} \left[ (\phi_{in} - \phi_{ex}) + \frac{k_B T}{e z_i} \ln \left( \frac{C_{ex}^i}{C_{in}^i} \right) \right]. \end{aligned} \quad (2)$$

The  $j^i$  are passive fluxes [4,5] driven only gradients of electrical and chemical potential of the ions. ATP hydrolysis is not involved.

Following classical derivations of conservation partial differential equations, and vector differential operators, we approximate surface integral on  $S_e$  and  $S_i$

$$\int_{S_e} J_{ex}^i \cdot n_v dS = \int_{S_e^x} J_{ex}^i \cdot \bar{dS} + \int_{S_e^y} J_{ex}^i \cdot \bar{dS} + \int_{S_e^z} J_{ex}^i \cdot \bar{dS}. \quad (3)$$

$$\begin{aligned} \int_{S_e^x} J_{ex}^i \cdot \bar{dS} &\approx \Delta V \left[ \frac{\mathcal{M}_{ex}^x (x + \frac{\Delta x}{2}) J_{ex}^i \cdot e_x (x + \frac{\Delta x}{2}, y, z) - \mathcal{M}_{ex}^x (x - \frac{\Delta x}{2}) J_{ex}^i \cdot e_x (x - \frac{\Delta x}{2}, y, z)}{\Delta x} \right] \\ &\approx \Delta V \frac{\partial (\mathcal{M}_{ex}^x J_{ex}^i \cdot e_x)}{\partial x}. \end{aligned} \quad (4)$$

By (2), we have the approximation

$$\int_{S_m} J_m^i \cdot n_s dS \approx \Delta V \mathcal{M}_v \alpha (j^i + a^i). \quad (5)$$

By (3), (4) and (5), we have

$$\begin{cases} \nabla \cdot (\mathcal{M}_{ex} J_{ex}^i) + \nabla \cdot (\mathcal{M}_{in} J_{in}^i) = 0, \\ \nabla \cdot (\mathcal{M}_{ex} J_{ex}^i) - \alpha \mathcal{M}_v (j^i + a^i) = 0, \end{cases} \quad (6)$$

where the  $\mathcal{M}_{ex}$  and  $\mathcal{M}_{in}$  are both diagonal matrices with

$$\mathcal{M}_l = \begin{bmatrix} \mathcal{M}_l^x(x), & 0, & 0 \\ 0, & \mathcal{M}_l^y(y), & 0 \\ 0, & 0, & \mathcal{M}_l^z(z) \end{bmatrix}, \quad l \text{ is either } ex \text{ or } in\}.$$

Mass conservation law gives

$$\begin{cases} \int_{S_e} u_{ex} \cdot n_v dS + \int_{S_i} u_{in} \cdot n_v dS = 0, \\ \int_{S_e} u_{ex} \cdot n_v dS - \int_{S_m} u_{ex} \cdot n_m dS = 0. \end{cases} \quad (7)$$

Trans-membrane velocity  $u_m$  is given by

$$\begin{aligned} u_m \cdot n_m &= -L_p(P_{ex} - P_{in}) + L_p \sigma RT(O_{ex} - O_{in}). \\ O_l &= \sum_i C_l^i, \quad l \in \{ex, in\}. \end{aligned} \quad (8)$$

Similarly, (7) is approximated by

$$\begin{cases} \nabla \cdot (\mathcal{M}_{ex} u_{ex}) + \nabla \cdot (\mathcal{M}_{in} u_{in}) = 0, \\ \nabla \cdot (\mathcal{M}_{ex} u_{ex}) - \mathcal{M}_v [-L_p(P_{ex} - P_{in}) + L_p \sigma RT(O_{ex} - O_{in})] = 0. \end{cases} \quad (9)$$

Darcy's law (307) is used to eliminate  $u_{ex}$  and  $u_{in}$  in (6) and (9).

$$\begin{cases} u_{ex} = -\frac{\kappa_{ex}}{\mu} \left[ \nabla P_{ex} + e \nabla \varphi_{ex} \left( \sum_i z^i C_{ex}^i \right) \right], \\ u_{in} = -\frac{\kappa_{in}}{\mu} \left[ \nabla P_{in} + e \nabla \varphi_{in} \left( \sum_i z^i C_{in}^i \right) \right]. \end{cases} \quad (10)$$

Thee potential  $\varphi_{ex}$  and  $\varphi_{in}$  is determined from (88, 176)

$$\begin{cases} \nabla \cdot (g_{ex} \nabla \varphi_{ex}) + \nabla \cdot (g_{in} \nabla \varphi_{in}) = 0, \\ \nabla \cdot (g_{in} \nabla \varphi_{in}) - \mathcal{M}_v Y_m (\varphi_{in} - \varphi_{ex}) = 0, \end{cases} \quad (11)$$

where

$$g_{ex} = \begin{bmatrix} \tau_{ex}^x \sigma_{ex}^x \mathcal{M}_{ex}^x, & 0, & 0 \\ 0, & \tau_{ex}^y \sigma_{ex}^y \mathcal{M}_{ex}^y, & 0 \\ 0, & 0, & \tau_{ex}^z \sigma_{ex}^z \mathcal{M}_{ex}^z \end{bmatrix}, \quad (12)$$

$$g_{in} = \begin{bmatrix} g_c^x + \tau_{in}^x \sigma_{in}^x \mathcal{M}_{in}^x, & 0, & 0 \\ 0, & g_c^y + \tau_{in}^y \sigma_{in}^y \mathcal{M}_{in}^y, & 0 \\ 0, & 0, & g_c^z + \tau_{in}^z \sigma_{in}^z \mathcal{M}_{in}^z \end{bmatrix}.$$

We combine (6), (8), (9) and (11), to produce

## Bidomain Model

$$\left\{ \begin{array}{l} \nabla \cdot (\mathcal{M}_{ex} J_{ex}^i) + \nabla \cdot (\mathcal{M}_{in} J_{in}^i) = 0, \\ \nabla \cdot (\mathcal{M}_{ex} J_{ex}^i) - \alpha \mathcal{M}_v (j^i + a^i) = 0, \\ \nabla \cdot \left[ \frac{\kappa_{ex}}{\mu} \mathcal{M}_{ex} \left( \nabla P_{ex} + e \nabla \varphi_{ex} \left( \sum_i z^i C_{ex}^i \right) \right) \right] + \nabla \cdot \left[ \frac{\kappa_{in}}{\mu} \mathcal{M}_{in} \left( \nabla P_{in} + e \nabla \varphi_{in} \left( \sum_i z^i C_{in}^i \right) \right) \right] = 0, \\ \nabla \cdot \left[ \frac{\kappa_{ex}}{\mu} \mathcal{M}_{ex} \left( \nabla P_{ex} + e \nabla \varphi_{ex} \left( \sum_i z^i C_{ex}^i \right) \right) \right] + \mathcal{M}_v \left[ -L_p (P_{ex} - P_{in}) + L_p \sigma RT (O_{ex} - O_{in}) \right] = 0, \\ \nabla \cdot (g_{ex} \nabla \varphi_{ex}) + \nabla \cdot (g_{in} \nabla \varphi_{in}) = 0, \\ \nabla \cdot (g_{in} \nabla \varphi_{in}) - \mathcal{M}_v Y_m (\varphi_{in} - \varphi_{ex}) = 0. \end{array} \right. \quad (13)$$

where

$$J_l^i = -C_l^i \frac{\kappa_l}{\mu} \left[ \nabla P_l + e \nabla \varphi_l \left( \sum_i z^i C_l^i \right) \right] - C_l^i M^i \nabla \mu_l^i, \quad l \in \{ex, in\}. \quad (14)$$

The bidomain model is complex because the underlying tissue is complex. Biology imposes a structure. Physics and biology impose boundary conditions on the structure, by the properties of the channels and lipids of the structure. Biophysics imposes the specific values of these properties. The specific values matter. Just as engineering devices behave properly and perform their function only in a specific environment of power supply voltages and so on, biological devices only function under specific conditions. They, and engineering devices, 'die', i.e., cease to function in other conditions. We are interested in functioning bidomain models, so we present the parameters under which the model has been shown to function properly.

## Parameter Values

Parameter	Value	Reference	Parameter	Value	Reference
$M_v$	$6 \times 10^1 m$	[3]	$e$	$1.6 \times 10^{-19} C$	
$R_{ex}$	$4.85 \times 10^6 ohm \cdot m$	[3]	$k_B$	$1.38 \times 10^{-23} J \cdot K^{-1}$	
$R_{in}$	$6.25 \times 10^4 ohm \cdot m$	[3]	$R$	$8.314 J \cdot mol^{-1} \cdot K$	
$\mu$	$7 \times 10^{-4} Pa \cdot s$	[9]	$T$	$310 K$	
$\tau_{ex}$	$0.16$	[8]	$M^{Na}$	$0.3249 m^2 \cdot s^{-1} \cdot volt^{-1} \cdot C^{-1}$	[6]
$\tau_{in}$	$1$	[8]	$M^K$	$0.4769 m^2 \cdot s^{-1} \cdot volt^{-1} \cdot C^{-1}$	[6]
$\sigma_{ex}$	$1/(4.85 \times 10^6 \times 0.16 \times S_{ex}/S_T)$	[1]	$M^{Cl}$	$0.4956 m^2 \cdot s^{-1} \cdot volt^{-1} \cdot C^{-1}$	[6]
$\sigma_{in}$	$1/(6.25 \times 4 \times 1 \times S_{in}/S_T)$	[1]	$g^{Na}$	$2.2 \times 10^{-3} S \cdot m^{-2}$	[8]
$\sigma$	$1$	[9]	$g^{Cl}$	$2.2 \times 10^{-3} S \cdot m^{-2}$	[8]
$L_p$	$1.34 \times 10^{-13} m^3 \cdot N^{-1} \cdot s^{-1}$	[7]	$\kappa_{in}$	$5.159 \times 10^{-13} m^2$	[9]
$C_m$	$7.9 \times 10^{-3} F \cdot m^{-2}$	[1]	$\kappa_{ex}$	$1.33 \times 10^{-16} m^2$	[9]
$G_m$	$4.38 \times 10^{-3} ohm^{-1} \cdot m^{-2}$	[1]			

**3.9 More Detail:** The full model could be computed with little difficulty using a combination of home grown programs and library programs in **MATLAB**. Parameters could be robustly determined by the extensive experimentation available. Fits to experimental data required no further adjustment of parameters and were very good, almost embarrassingly so as far as at least one of the authors was concerned (however, good structural models are often found to fit good data well, e.g. (59, 76, 78, 88)).

The perturbation properties of the full model could be studied. That is to say, classical Taylor expansions (with occasional consideration of singular situations as done by (59, 123) a long time ago) allow construction of a simplified model. The errors in the simplified model were determined by direct comparison with the full computed solution, over a range of parameters and conditions although all pertinent parameters and conditions could not be tested of course. Thus, this computing of errors remains an unbounded iterative process. We are always confronted with new conditions, new experiments, new testing of the reduced models. I point out this process because the reduced

perturbation models give important physical and biological insight that is easily lost amongst the complete calculation of the model, which correctly determine everything (within the accuracy of the model specification itself, of course). But everything is such a large set that nothing may be understood as a result (of our human limitations).

The perturbation model allowed derivation of the Mathias engineering model. Our simplified model (determined I repeat by actual perturbation and Taylor expansion not by guessing, however inspired) was a slight but significant generalization of the Mathias engineering model.

We could study the effects of the effective resistance of connexins, showing an interesting qualitative bifurcation in quantitative properties predicted (and measured). The intracellular pressure changed dramatically as the effective connexin conductance was changed but many variables were essentially unchanged. In our model, all variables are related to the potential of the extracellular space within the lens, except the hydrostatic pressure. The potential of the extracellular phase within the lens hardly changes with effective resistance of the connexins, and so only the hydrostatic pressure changes significantly as the effective resistance of the connexins is changed.

The perturbation structure of the problem shows the mathematical reason the potential of the extracellular phase within the lens does not vary with connexin density. Physically and physiologically it seems obvious that changes in the intracellular domain would have only second order effects on the potential of the extracellular phase within the lens because they are separated by a high resistance membrane that buffers extracellular from the intracellular region (see similar effects in other systems, see qualitative general discussion in (123) and quantitative special cases computed in (125-128).

The insensitivity of measurements to changes in connexin effective resistance has been a cause of concern in the lens experimental literature as I read it (183, 198). It seems likely that much of the experimental literature of the lens (sampled by the references (184, 200, 201, 308-314)) will benefit from an understanding of the role of the potential of the extracellular phase within the lens and its insensitivity to many parameters.

**4.0 Optic nerve of the amphibian salamander** We turn now to the generalization of the bidomain model of flow in the lens to the tridomain model of the optic nerve of the salamander *Necturus*.

The approach to tissue complexity described and advocated here depends on careful connection of structural and experimental work, emphasizing known biophysical properties of membranes, channels, and transporters. It grew out of the tradition of Cambridge (England) Physiology, driven by Alan Hodgkin more than anyone else, of choosing preparations given us by evolution in which complexity is not overwhelming, and physiological function is evident.

The history is evident in the study of the nerve impulse that carries information throughout the nervous system. The squid axon (315-317) was chosen as a model preparation (318-321) after Hodgkin, as a student, showed that the nerve impulse was conducted by electric current (322, 323) as opposed to the chemical hypothesis of the then leader of English Biophysics, the Nobel Laureate AV Hill (324). The squid axon was large enough to allow control of potential using the voltage clamp method of KS Cole (318-321, 325) but results were difficult to interpret because of the fear that concentrations of ions would change (in a process quaintly called 'concentration polarization' then and still today (326-328)). Hodgkin, Huxley and Katz seemed to have always left out the word 'concentration' thus creating singular confusion between their 'concentration polarization' (which was a concentration change driven by current flow) and the dielectric (and vacuum) polarization of Maxwell, Heaviside and followers (i.e., all physicists) for whom polarization did not include the translation of ions (rather for them polarization was the rapid small reversible change in distribution of charge in solids, molecules, and atoms and the mysterious displacement current found everywhere including a vacuum). Wang (329) has recently shown how the energetic variational approach EnVarA (26) of Liu and collaborators (18-25) can be used to reunite both meanings of polarization, in a way that would have brought a smile to Hodgkin's face, more likely than not.

Once a dual electrode system was placed inside the squid axon, making what is usually called a four electrode recording system, the artifactual concentration polarization around the current electrode was of no importance, but Hodgkin, Huxley and Katz were surprised to find a slow (say 4 msec) change in current (in their voltage clamped axon) that might represent the change of concentration of ions near the squid membrane. It did not for the most part, but Hodgkin and Huxley were careful to leave open the possibility later demonstrated by Frankenhaeuser and Hodgkin (330) that some of the slower changes were the result of accumulation of potassium in the narrow space outside of the nerve membrane, a result made even more plausible by the theory of Taylor and Bezanilla (331). The

arguments for potassium accumulation were however somewhat indirect, and required a greater knowledge of channel biophysics than was widespread in the 1950's, so it was desirable to find a way to measure the potassium concentration in the narrow extracellular space outside the nerve directly, or to find an optical way to measure the membrane potential that depended on potassium flow in a different way (332).

Following the tradition of choosing the right preparation, Kuffler investigated the accumulation of potassium (333, 334) in the optic nerve of an amphibian salamander the mudpuppy *Necturus* found in the wetlands of Eastern North Carolina. Kuffler had studied the optic nerve (335-337) because it, as a part of the central nervous system, was made of three domains, nerve cell, extracellular space, and glia. Kuffler had shown that the glia were an electrical syncytium, so all three domains were regions of salt solutions of relatively small longitudinal and three dimensional effective resistivity. The glia had a membrane potential (i.e., measured at very low frequencies) that responded to potassium. So Kuffler could use the membrane potential recorded inside the glia to measure the potassium concentration immediately outside the nerve membrane.

I present here the precise specification of the electrical part of the tridomain model (338) lest an important detail be inadvertently omitted. Presenting the precise specification of the flow part of the tridomain would be too much for this paper. It is also not necessary: the material is presented in complete detail in the overlapping publications (167, 302, 338, 339). The many pages devoted to Validation and Calibration are of particular importance although not presented here because of space limitations.

**2.2.2 Ion Transport.** We define the regions:

- 1)  $\Omega_{OP}$  consist of axon compartment
- 2)  $\Omega_{ax}$  axonal interior
- 3)  $\Omega_{gl}$  glial compartment
- 4)  $\Omega_{ex}^{OP}$  extracellular space.

In addition there are regions. For example, the subarachnoid space  $\Omega_{SAS}$  that has only extracellular space.

The conservation of chemical species implies the following system of partial differential equations to describe the dynamics of ions in each of the three regions, for  $i = Na^+, K^+, Cl^-$

$$\frac{\partial(\eta_{gl}c_{gl}^i)}{\partial t} + \mathcal{M}_{gl}J_{gl}^{m,i} + \nabla \cdot (\eta_{gl}\mathbf{j}_{gl}^i) = 0, \text{ in } \Omega_{OP}, \quad (15)$$

$$\frac{\partial(\eta_{ax}c_{ax}^i)}{\partial t} + \mathcal{M}_{ax}J_{ax}^{m,i} + \frac{\partial}{\partial z}(\eta_{ax}j_{ax,z}^i) = 0, \text{ in } \Omega_{OP}, \quad (16)$$

$$\frac{\partial(\eta_{ex}c_{ex}^i)}{\partial t} - \mathcal{M}_{ax}J_{ax}^{m,i} - \mathcal{M}_{gl}J_{gl}^{m,i} + \nabla \cdot (\eta_{ex}\mathbf{j}_{ex}^i) = 0, \text{ in } \Omega_{OP}, \quad (17)$$

where the last equation reduces to the following in the subarachnoid  $\Omega_{SAS}$  region,

$$\frac{\partial c_{ex}^{i,SAS}}{\partial t} + \nabla \cdot \mathbf{j}_{ex}^{i,SAS} = 0. \quad (18)$$

The transmembrane ion flux  $J_k^{m,i}$  ( $k = gl, ax$ ) consists of active ion pump source  $J_{p,k}^i$  and passive ion channel source  $J_{c,k}^i$  on the  $k$  membrane,

$$J_k^{m,i} = J_{p,k}^i + J_{c,k}^i, \quad k = gl, ax, \quad i = Na^+, K^+, Cl^-.$$

On the glial cell membranes,  $J_{c,gl}^i$  is defined as

$$J_{c,gl}^i = \frac{g_{gl}^i}{z^i e} (\phi_{gl} - \phi_{ex} - E_{gl}^i), \quad i = Na^+, K^+, Cl^-, \quad (19)$$

where the Nernst potential is used to describe the gradient of chemical potential  $E_{gl}^i = \frac{k_B T}{e z^i} \log\left(\frac{c_{ex}^i}{c_{gl}^i}\right)$  for  $i$ th ion species in the glial membrane and the conductance  $g_{gl}^i$  is a fixed constant independent of voltage and time. On the axon's membrane  $J_{c,ax}^i$  is defined as

$$J_{c,ax}^i = \frac{g_{ax}^i}{z^i e} (\phi_{ax} - \phi_{ex} - E_{ax}^i), \quad i = Na^+, K^+, Cl^-,$$

where

$$g_{ax}^{Na} = \bar{g}^{Na} m^3 h + g_{leak}^{Na}, \quad g_{ax}^K = \bar{g}^K n^4 + g_{leak}^K, \quad g_{ax}^{Cl} = g_{leak}^{Cl}.$$

The time dependent dynamic of open probability, often loosely called 'gating' is governed by the Hodgkin-Huxley model (319, 321, 325, 340, 341)

$$\begin{aligned} \frac{dn}{dt} &= \alpha_n(1-n) - \beta_n n, \\ \frac{dm}{dt} &= \alpha_m(1-m) - \beta_m m, \\ \frac{dh}{dt} &= \alpha_h(1-h) - \beta_h h, \end{aligned} \quad (20)$$

Here,  $n$  is the open probability of the  $K^+$  channel.  $m$  is the open probability of the  $Na^+$  activation gate, and  $h$  is the open probability of the  $Na^+$  inactivation gate.

We assume that the only pump is the Na/K active transporter. We are more than aware that other active transport systems can and likely do move ions and thus water in this system. They will be included as experimental information becomes available.

In the case of the Na/K pump  $J_{p,k}^i$  ( $k = ax, gl$ ), the strength of the pump  $I_k$  depends on the concentration in the intracellular and extracellular space (342), i.e.

$$J_{p,k}^{Na} = \frac{3I_k}{e}, \quad J_{p,k}^K = -\frac{2I_k}{e}, \quad J_{p,k}^{Cl} = 0, \quad k = gl, ax, \quad (21)$$

where

$$\begin{aligned} I_k &= I_{k,1} \left( \frac{c_k^{Na}}{c_k^{Na} + K_{Na1}} \right)^3 \left( \frac{c_{ex}^K}{c_{ex}^K + K_{K1}} \right)^2 \\ &+ I_{k,2} \left( \frac{c_k^{Na}}{c_k^{Na} + K_{Na2}} \right)^3 \left( \frac{c_{ex}^K}{c_{ex}^K + K_{K2}} \right)^2, \quad k = ax, gl. \end{aligned} \quad (22)$$

$I_{k,1}$  and  $I_{k,2}$  are related to the maximum current of  $\alpha_1$  - and  $\alpha_2$  - isoform of Na/K pump on the glial membrane ( $k = gl$ ) or axon membrane ( $k = ax$ ).

The definitions of ion flux in each domain are as follows, for  $i = Na^+, K^+, Cl^-$ ,

$$\begin{aligned} \mathbf{j}_l^i &= c_l^i \mathbf{u}_l - D_l^i \tau_l \left( \nabla c_l^i + \frac{z^i e}{k_B T} c_l^i \nabla \phi_l \right), \quad l = gl, ex, \\ j_{ax,z}^i &= c_{ax}^i u_{ax}^z - D_{ax}^i \left( \frac{\partial c_{ax}^i}{\partial z} + \frac{z^i e}{k_B T} c_{ax}^i \frac{\partial \phi_{ax}}{\partial z} \right). \end{aligned}$$

For the axon compartment and glial compartment boundary condition, we have

$$c_{ax}^i = c_{ax}^{i,re}, \quad \text{on } \Gamma_2 \cup \Gamma_6, \quad (23)$$

and

$$\begin{cases} \mathbf{j}_{gl}^i \cdot \hat{\mathbf{r}} = 0, & \text{on } \Gamma_1, \\ c_{gl}^i = c_{gl}^{i,re}, & \text{on } \Gamma_2 \cup \Gamma_6, \\ \mathbf{j}_{gl}^i \cdot \hat{\mathbf{r}} = 0, & \text{on } \Gamma_7, \end{cases} \quad (24)$$

where the Dirichlet boundary conditions are used at locations  $\Gamma_2 \cup \Gamma_6$ .

for axons and glial cell, and a non-flux boundary condition is used for glial cells, ions, flux, and pia matter. For the extracellular space boundary condition, similar boundary conditions are imposed except on the pia matter  $\Gamma_7$ . The flux across the pia matter is assumed continuous and Ohm's law is used (338). Additionally, a non-permeable boundary condition is used at location  $\Gamma_5$ , and a homogeneous Neumann boundary condition is applied at the location of the dura matter  $\Gamma_4$ .

$$\begin{cases} \mathbf{j}_{ex}^i \cdot \hat{\mathbf{r}} = 0, & \text{on } \Gamma_1, \\ c_{ex}^i = c_{csf}^i, & \text{on } \Gamma_2 \cup \Gamma_3, \\ \nabla c_{ex}^i \cdot \hat{\mathbf{r}} = 0, & \text{on } \Gamma_4, \\ \mathbf{j}_{ex}^i \cdot \hat{\mathbf{z}} = 0, & \text{on } \Gamma_5, \\ c_{ex}^i = c_{eye}^i, & \text{on } \Gamma_6, \\ \mathbf{j}_{ex}^{i,OP} \cdot \hat{\mathbf{r}} = \mathbf{j}_{ex}^{i,SAS} \cdot \hat{\mathbf{r}} = \frac{c_{pia}^i}{z^i e} (\phi_{ex}^{OP} - \phi_{ex}^{SAS} - E_{pia}^i), & \text{on } \Gamma_7. \end{cases} \quad (26)$$

We define

$$\delta_{7,l}^i = \frac{\Delta c_l^{i*}}{c_l^{i*}}, \quad i = \text{Na}^+, \text{K}^+, \text{Cl}^+, \quad l = ax, gl, ex.$$

so the ion fluxes can be written as

$$\begin{aligned} \tilde{\mathbf{j}}_l^i &= P e_l^i \delta_{7,l}^i \tilde{c}_l^i \tilde{\mathbf{u}}_l - \left( \delta_{7,l}^i \tilde{\nabla} \tilde{c}_l^i + z^i \tilde{c}_l^i \tilde{\nabla} \tilde{\phi}_l \right), \quad l = gl, ex, \\ \tilde{j}_{ax,z}^i &= P e_{ax}^i \delta_{7,l}^i \tilde{c}_l^i \tilde{u}_{ax}^z - \left( \delta_{7,l}^i \frac{\partial \tilde{c}_l^i}{\partial z} + z^i \tilde{c}_l^i \frac{\partial \tilde{\phi}_l}{\partial z} \right), \end{aligned}$$

with Peclet numbers

$$P e_{ax}^i = \frac{u_{ax}^* z^* c_{ax}^{i*}}{D_{ax}^i \Delta c_{ax}^{i*}}, \quad P e_l^i = \frac{u_l^* r^* c_l^{i*}}{D_l^i \tau_l \Delta c_l^{i*}}, \quad l = gl, ex. \quad (27)$$

If we let  $g_l^*, l = ax, gl$  be the characteristic membrane conductance,  $\frac{k_B T}{e}$  be the characteristic electric potential, the dimensionless form of transmembrane flux is

$$\tilde{j}_l^{m,i} = \tilde{j}_{c,l}^i + \tilde{j}_{p,l}^i,$$

where for  $i = \text{Na}^+, \text{K}^+, \text{Cl}^-, l = gl, ax$ ,

$$\tilde{j}_{c,l}^i = \frac{\tilde{g}_l^i}{z^i} (\tilde{\phi}_k - \tilde{\phi}_{ex} - \tilde{E}_{gl}^i), \quad \tilde{j}_{p,l}^i = \frac{j_{p,l}^i e^2}{k_B T g_l^*}.$$

The governing equations for ions become

$$\frac{\partial(\eta_{gl} \tilde{c}_{gl}^i)}{\partial \tilde{t}} + \delta_8^i \tilde{j}_{gl}^{m,i} + \delta_9^i \tilde{\nabla} \cdot (\eta_{gl} \tilde{\mathbf{j}}_{gl}^i) = 0, \quad (28)$$

$$\frac{\partial(\eta_{ax} \tilde{c}_{ax}^i)}{\partial \tilde{t}} + \delta_{10}^i \tilde{j}_{ax}^{m,i} + \delta_{11}^i \frac{\partial}{\partial z} (\eta_{ax} \tilde{j}_{ax,z}^i) = 0, \quad (29)$$

$$\frac{\partial(\eta_{ex} \tilde{c}_{ex}^i)}{\partial \tilde{t}} - \delta_{12}^i \delta_{10}^i \tilde{j}_{ax}^{m,i} - \delta_{13}^i \delta_8^i \tilde{j}_{gl}^{m,i} + \delta_{14}^i \tilde{\nabla} \cdot (\eta_{ex} \tilde{\mathbf{j}}_{ex}^i) = 0, \quad (30)$$

where

$$\tilde{\nabla} \cdot (\eta_l \tilde{\mathbf{j}}_l^i) = \frac{1}{\tilde{r}} \frac{\partial(\tilde{r} \eta_l \tilde{j}_l^{r,i})}{\partial \tilde{r}} + (\delta_0^i)^2 \frac{\partial(\eta_l \tilde{j}_l^{z,i})}{\partial z}, \quad l = gl, ex,$$

$$\delta_8^i = \frac{t^* M_{gl} g_{gl}^* k_B T}{c_{gl}^{i*} e^2}, \quad \delta_9^i = \frac{D_{gl}^i \tau_{gl} t^*}{(r^*)^2},$$

$$\delta_{10}^i = \frac{t^* M_{ax} g_{ax}^* k_B T}{c_{ax}^{i*} e^2}, \quad \delta_{11}^i = \frac{D_{ax}^i t^*}{(z^*)^2},$$

$$\delta_{12}^i = \frac{c_{ax}^{i*}}{c_{ex}^{i*}}, \quad \delta_{13}^i = \frac{c_{gl}^{i*}}{c_{ex}^{i*}}, \quad \delta_{14}^i = \frac{D_{ex}^i \tau_{ex} t^*}{(r^*)^2}.$$

The symbol  $\Delta f$  is used to denote the variation of the variable  $f$  from its resting state. Multiplying the equations (23-25) with  $z_i e$  respectively, summing up, and using the charge neutrality condition, we have the following system for the electric fields in  $ax, gl, ex$ ,

$$\sum_i z^i e \mathcal{M}_{gl} J_{gl}^{m,i} + \sum_i \nabla \cdot (z^i e \eta_{gl} \tilde{\mathbf{j}}_{gl}^i) = 0, \quad (31)$$

$$\sum_i z^i e \mathcal{M}_{ax} J_{ax}^{m,i} + \sum_i \frac{\partial}{\partial z} (z^i e \eta_{ax} \tilde{j}_{ax,z}^i) = 0, \quad (32)$$

$$\sum_i z^i e \nabla \cdot (\eta_{gl} \tilde{\mathbf{j}}_{gl}^i) + \sum_i \frac{\partial}{\partial z} (z^i e \eta_{ax} \tilde{j}_{ax,z}^i) + \sum_i \nabla \cdot (z^i e \eta_{ex} \tilde{\mathbf{j}}_{ex}^i) = 0, \quad (33)$$

In the subarachnoid space  $\Omega_{SAS}$ , the extracellular equations reduce to

$$\sum_i \nabla \cdot (z^i e \sum_i \tilde{\mathbf{j}}_{ex}^{i,SAS}) = 0. \quad (34)$$

The boundary conditions for electric fields  $\phi_{ax}, \phi_{gl}$  and  $\phi_{ex}$  are given below.

In the axon compartment:

$$\begin{cases} \nabla\phi_{ax} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_2, \\ \nabla\phi_{ax} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_6, \end{cases} \quad (35)$$

In the glial compartment:

$$\begin{cases} \nabla\phi_{gl} \cdot \hat{\mathbf{r}} = 0, & \text{on}\Gamma_1, \\ \nabla\phi_{gl} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_2, \\ \nabla\phi_{gl} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_6, \\ \nabla\phi_{gl} \cdot \hat{\mathbf{r}} = 0, & \text{on}\Gamma_7, \end{cases} \quad (36)$$

and in the extracellular space:

$$\begin{cases} \nabla\phi_{ex} \cdot \hat{\mathbf{r}} = 0, & \text{on}\Gamma_1, \\ \nabla\phi_{ex} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_2 \cup \Gamma_3, \\ \nabla\phi_{ex} \cdot \hat{\mathbf{r}} = 0, & \text{on}\Gamma_4, \\ \nabla\phi_{ex} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_5, \\ \nabla\phi_{ex} \cdot \hat{\mathbf{z}} = 0, & \text{on}\Gamma_6, \\ \sum_i z^i e_j^{i,OP} \cdot \hat{\mathbf{r}} = \sum_i z^i e_j^{i,SAS} \cdot \hat{\mathbf{r}} \\ = \sum_i G_{pia}^i (\phi_{ex}^{OP} - \phi_{ex}^{SAS} - E_{pia}^i), & \text{on}\Gamma_7. \end{cases} \quad (37)$$

This concludes the formal presentation of this component of the tridomain model. The rest is found in the original overlapping papers (167, 302, 338, 339) and the entire model, and its components, are validated and calibrated at length in those publications. To repeat: the validation and calibration are at least as important as the derivation.

#### 4.1 Implications. Potassium ions are a Toxic Waste.

I turn now to the implications of this analysis. It should be clearly understood that accumulated potassium is a toxic waste, a hazard to both the signaling function of neurons and their survival as cells. Signaling is blocked by high potassium (because of inactivation of sodium channels more than anything else). Cell volume changes when potassium is elevated and with enough external potassium, cells will burst and die.

Potassium accumulation is important in many biological and clinical applications. Potassium accumulation and flow in the extracellular space have been shown to have important roles in aging, Alzheimers disease, anesthesia, dementia, diabetes, epilepsy, migraine, sleep, stroke and traumatic brain injury [alphabetical order], as well as an important role in the biology of the central nervous system (47, 49, 50, 52, 53, 55, 56, 100, 101, 343-362). Indeed, Filipidis et al (363) have identified many such systems in biology containing three intercalated sets of tissues, including pleura, peritoneum, pericardium, fetal membranes and leptomeninges citing the significant references [261-290(364-393)]. Each of these tissues likely to include intercalated syncytia and thus require multidomain models of the type we have studied. Hopefully, a structural analysis, using conservation laws and stereological estimates of structure, along with the biophysics of the type and distribution of channels and transporters will help understand each of these tissues in health and disease.

While I was aware of the classical physiological literature on potassium accumulation (as the many words of this review illustrate), I was insensitive, and mostly unaware of the important literature on other related systems. I apologize for my inadvertent neglect of appropriate references.

The reader will benefit from reading different approaches from ours in the work of several laboratories, including those of Mori (55, 354, 394-404), Ellingsrud (405-410), Sacco (411) and the many groups interested in glymphatics, sampled in (47-57). While few if any of these papers deal with potassium accumulation in tridomain systems, no doubt their methods could be usefully applied to those issues. Sadly, I am so distracted by other work—on the Maxwell equations, models of polarization as a stress strain relation of charge density, the voltage sensor of sodium and potassium nerve channels, and most recently (and speculatively) the stochastic analysis of the mechanics of hydrogen bonds—that I have been unable to focus on those other methods.

Huaxiong Huang led our efforts (with his graduate student Yi Zhu and postdoctoral fellow Shixin Xu) to create a tridomain model able to deal with the Orkand data on potassium accumulation in optic nerve (333, 334) as he had led the team analyzing the lens. See Acknowledgement.

**4.2 Clearance of Potassium ions from the extracellular space.** Turning to the results of our analysis and simulations (338, 339, 412), I discuss the clearance of potassium from the extracellular

space. Only in the idealized world of the classical squid axon experiments is a single action potential enough to study. In the real world, trains of action potentials are frequent. Potassium accumulated after these many action potentials acts as a toxic waste. As it accumulates potassium will depolarize the nerve membrane (i.e., make the resting potential more positive) and that in turn will inactivate sodium channels and so block the action potential. Extracellular potassium poisons nerve signals this way. Extracellular potassium can also kill the nerve fiber itself (as well as its signal). When potassium accumulates sufficiently it will diffuse into the nerve fiber, along with water, and that diffusion can overwhelm the pumping mechanisms that maintain homeostasis of cell volume. In plain English, the potassium swells the nerve until it bursts, killing the nerve.

The mechanism of potassium clearance thus is of particular biological interest since it is needed to maintain nerve life, and thus animal life, as well as the nerve signal.

To determine that mechanism, we built a model (338, 339, 412) of all three of the domains of the optic nerve including the coupled migration (in the electric field), diffusion (in the concentration field), and convection (in the diffusion field of water) of sodium, potassium and chloride ions, and water itself. We could not predict ahead of time which of the flows in which of the domains would be used by evolution for potassium clearance. And we hasten to say that we do not know which of the flows (and the channels and transporters that control the flow) will be most important for other functions of tridomain systems in the brain, and elsewhere.

We could not build a simplified transport model, or a lumped model in which compartments had a single concentration of ions and water independent of location, before we analyzed the entire system. And we were convinced from study of similar systems, that if we did guess a compartment model, it would not be the same as that of other investigators, with just as good intuition and probably greater knowledge than ours.

Confronted with different models of the same results, it seemed unlikely that progress would be rapid. The unavoidable competition of different research groups for resources (and recognition) and the absence of an agreed upon universal model has slowed research in other epithelia for decades (413-415), and it seemed it would do so here as well. So ***we constructed what seemed to us to be a complete model of twenty one partial differential equations and three ordinary differential equations***. We used this model to describe the structures actually used by the optic nerve to clear potassium from the extracellular space during a train of action potentials.

This much structural detail is not arbitrary. It is needed because that is how evolution has built its system of potassium clearance. There is some arbitrariness, particularly in the choice of channels open at long times and the choice and distribution of transporters, but our results were robust enough to allow some conclusions. Experimental data will resolve these issues and be easy to incorporate into our model and its implementations. It is important to note that other functions may depend critically on the distribution and properties of channels and pumps and so the experimental measurement of those properties and distributions will be critical, in my opinion, in the application of these methods to crucial biological and clinical problems, from waste clearance in sleep to glaucoma.

When the differential equations were solved (including the full computation of the Hodgkin Huxley nerve model on the time scale of the action potential) clearance was found to greatly enhance the syncytial nature of the glial domain. The syncytium allowed the glia to function as a pipe to clear waste, with the flow in the pipe driven by the diffusion of water, more than anything else. The diffusion of water was in turn the result of the difference in sodium and potassium movements across the glial membrane.

We had not foreseen this mechanism, and hasten to add that we cannot conclude that the same mechanism occurs in other related systems, as clear as the mechanism of waste clearance was in this optic nerve. For example, if the glia are connected to blood vessels through aquaporins, it is possible (although unlikely in my opinion) that qualitative properties will change significantly.

A calculation of this sort must be extensively checked to see how sensitive its results are to its complexity. We were surprised to find how sensitive flows were to the spatial distribution of transporters and channels. When these were varied stochastically (in space) we found substantial effects.

Thus, we are led to the unavoidable reality. While we can understand quite well the mechanisms of waste clearance in the optic nerve of *Necturus*, we cannot be sure the same mechanisms dominate in the brain in general.

**We can however be reasonably certain that the model we have used can be modified to accommodate flow into blood vessels.** We can be sure it can be modified to include other transporters and channels, in other distributions, because we have actually shown that by building such models. We can be reasonably certain that a full structural model is feasible and almost certainly needed to understand the structural and molecular basis of flow in cells and tissues in general, including the brain, including in sleep.

**4.3 Conclusions and future perspectives:** It is tempting to reach sweeping conclusions from the powerful tridomain theory and the successful analysis of the optic nerve of a salamander. But it is after all only a salamander that we have studied, and its optic nerve does not contain blood vessels and so is not a wise preparation in which to study flow into blood vessels, as occurs in the mammalian brain.

What does seem clear is that the approach and model we have constructed can be generally applied, because it is based on observed biological structure, measured biophysical properties of those structures and of the protein transporters and channels embedded in those structures, using conservation laws likely to be as true in biology as in the rest of the world.

Our approach is as faithful as it can be to the details of that structure because we know from long experience that details matter in physiological systems as they do in engineering systems. What seem small inconsequential details often wind up as controls for the entire system. After all the action potential is a big system involving meters of nerve fibers, but it is controlled by a few protein channels in the cells that initiate the action potential. Thus, I do not speculate. I advocate real work.

It seems clear that the path to understanding (and clinical control) of fluid flow in brain function requires

- 1) Construction of a structural model based on conservation laws and the structure(s) of clinical and biological interest. For example, studying the glymphatic hypothesis (the literature is sampled in in (47-57)) is likely to require construction of a four domain model involving blood vessels (of variable size and shape and properties) along with glia, extracellular space and axon. We see no sign that there will be serious theoretical, mathematical, or numerical obstacles to building such a model, although the work involved should not be underestimated, nor the skill level needed to construct its equations, and compute them, with appropriate perturbation expansions for understanding, as Shixin Xu and Yi Zhu and Zilong Song have done, with the help and leadership of Huaxiong Huang.
- 2) The identification of the types and locations (and densities) of the various pumps and channels used by the genome to control function. Experiments to acquire this data can be done but they are as tedious as they are necessary. Frankly, it is not clear they will be done until the community of biologists and clinicians are convinced that they are needed. **One of the main roles of a complete structural model is to convince biologists to make the measurements needed to understand function.**
- 3) The calculation of the biological function. Is the waste actually cleared? Can the spatial distribution of transporters (for example the sodium potassium pump) or its up regulation in sleep actually explain waste clearance in the brain, thus supporting the glymphatic hypothesis in a more or less quantitative way.

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## 6. References

1. Eisenberg B. Living Transistors: a Physicist's View of Ion Channels (version 2). <http://arxiv.org/q-bio>BM: arXiv:q-bio/0506016v2 2005.
2. Eisenberg B. Ions in Fluctuating Channels: Transistors Alive. *Fluctuation and Noise Letters*. 2012;11(01):1240001 available on arXiv.org with Paper ID arXiv:q-bio/0506016v3.
3. Eisenberg B. Engineering channels: Atomic biology. *Proceedings of the National Academy of Sciences*. 2008;105(17):6211-2.
4. Eisenberg RS. Look at biological systems through an engineer's eyes. *Nature*. 2007;447(7143):376.
5. Miedema H, Vrouwenraets M, Wierenga J, Meijberg W, Robillard G, Eisenberg B. A Biological Porin Engineered into a Molecular, Nanofluidic Diode. *Nano Lett*. 2007;7(9):2886-91.
6. Shur M. *Physics of Semiconductor Devices*. New York: Prentice Hall; 1990. 680 p.
7. Pierret RF. *Semiconductor Device Fundamentals*. New York: Addison Wesley; 1996.
8. Sze SM. *Semiconductor devices: physics and technology*: John Wiley & Sons; 2008.
9. Laux SE, Hess K. Revisiting the analytic theory of pn junction impedance: Improvements guided by computer simulation leading to a new equivalent circuit. *IEEE Transactions on Electron Devices*. 1999;46(2):396-412.
10. Cressler JD. *Silicon Heterostructure Handbook: Materials, Fabrication, Devices, Circuits and Applications of SiGe and Si Strained-Layer Epitaxy*. Boca Raton, FL: CRC 2005. 1248 pages p.
11. Muller RS, Chan M, Kamins TI. *Device Electronics For Integrated Circuits*, 3rd Ed: Wiley India Pvt. Limited; 2003.
12. Eisenberg B. Living Devices: The Physiological Point of View. Available on arXiv as <http://arxiv.org/abs/12066490>. 2012.
13. Engl HW, Hanke M, Neubauer A. *Regularization of Inverse Problems* Dordrecht, The Netherlands: Kluwer; 2000.
14. Kaipio J, Somersalo E. *Statistical and Computational Inverse Problems* New York: Springer; 2005. 344 pages p.
15. Burger M. Inverse problems in ion channel modelling. *Inverse Problems*. 2011;27(8):083001.
16. Burger M, Eisenberg RS, Engl H. Inverse Problems Related to Ion Channel Selectivity. *SIAM J Applied Math*. 2007;67(4):960-89
17. Keller JB. *Inverse Problems*. *The American Mathematical Monthly*. 1976;83(2):107-18.
18. Forster J. *Mathematical Modeling of Complex Fluids [Master's]*. Wurzburg, Germany: University of Wurzburg; 2013.
19. Giga M-H, Kirshtein A, Liu C. Variational Modeling and Complex Fluids. In: Giga Y, Novotny A, editors. *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*. Cham: Springer International Publishing; 2017. p. 1-41.
20. Xu S, Sheng P, Liu C. An energetic variational approach for ion transport. arXiv preprint arXiv:14084114. 2014.
21. Hyon Y, Eisenberg B, Liu C. An energetic variational approach to ion channel dynamics. *Mathematical Methods in the Applied Sciences*. 2014;37(7):952-61.
22. Hyon Y, Fonseca JE, Eisenberg B, Liu C. Energy variational approach to study charge inversion (layering) near charged walls. *Discrete and Continuous Dynamical Systems Series B (DCDS-B)*. 2012;17(8):2725 - 43.
23. Hyon Y, Kwak DY, Liu C. Energetic Variational Approach in Complex Fluids : Maximum Dissipation Principle. available at URL: <http://www.wimaum.edu> as IMA Preprint Series # 2228. 2010;26(4: April):1291 - 304.
24. Ryham R, Liu C, Wang ZQ. On electro-kinetic fluids: One dimensional configurations. *Discrete Contin Dyn Syst-Ser B*. 2006;6(2):357-71.
25. Ryham RJ. *An Energetic Variational Approach to Mathematical Modeling of Charged Fluids, Charge Phases, Simulation and Well Posedness*, Ph.D. Thesis [Ph.D.]. State College: The Pennsylvania State University; 2006.
26. Eisenberg B, Hyon Y, Liu C. Energy Variational Analysis EnVarA of Ions in Water and Channels: Field Theory for Primitive Models of Complex Ionic Fluids. *Journal of Chemical Physics*. 2010;133(10):104104.
27. Eisenberg B. Life's Solutions are Complex Fluids. A Mathematical Challenge. arXiv preprint arXiv:12074737. 2012.
28. Eisenberg RS. Complexities in solution. *Trends Biochem Sci*. 1990;15(2):51-2.

29. Ionic Selectivity in Channels: complex biology created by the balance of simple physics [Internet]. Nanohub Purdue University. <http://www.nanohub.org/resources/4726/> 2008 [cited Jun]. Available from: <http://www.nanohub.org/resources/4726/>
30. Rice SA, Gray P. The Statistical Mechanics of Simple Fluids. New York: Interscience (Wiley); 1965. 582 p.
31. Barratt J-L, Hansen J-P. Basic concepts for simple and complex liquids: Cambridge University Press; 2003. 296 p.
32. Hansen J-P, McDonald IR. Theory of Simple Liquids. First Edition ed. New York: Academic Press; 1986. 395 pages p.
33. Hansen J-P, McDonald IR. Theory of Simple Liquids. Third Edition ed. New York: Academic Press; 2006. 428 p.
34. Hansen JP, Bellissent-Funel MC, Neilson GW. The Physics and Chemistry of Aqueous Ionic Solutions 1987. 1 p.
35. Kontogeorgis GM, Folas GK. Thermodynamic Models for Industrial Applications: From Classical and Advanced Mixing Rules to Association Theories: John Wiley & Sons; 2009. 721 p.
36. Kunz W. Specific Ion Effects. Singapore: World Scientific 2009. 348 p.
37. Kunz W, Neueder R. An Attempt at an Overview. In: Kunz W, editor. Specific Ion Effects. Singapore: World Scientific 2009. p. 11-54.
38. Liu JL, Eisenberg B. Molecular Mean-Field Theory of Ionic Solutions: a Poisson-Nernst-Planck-Bikerman Model. Entropy 2020;22:550 Preprint available at <https://arxiv.org/abs/2004.10300>.
39. Li C-L, Liu J-L. Generalized Debye--Hückel Equation From Poisson--Bikerman Theory. SIAM Journal on Applied Mathematics. 2020;80(5):2003-23.
40. Liu J-L, Li C-L. A generalized Debye-Hückel theory of electrolyte solutions. AIP Advances. 2019;9(1):015214.
41. Vera JH, Wilczek-Vera G. Classical Thermodynamics of Fluid Systems: Principles and Applications: Crc Press; 2016.
42. Haase W. Crystals that flow. Classic papers from the history of liquid crystals. Compiled with translation and commentary by Tim J. Sluckin, David A. Dunmur and Horst Stegemeyer. Liquid Crystals Book Series, edited by GW Gray, JW Goodby and A. Fukuda. Pp. xxiii+ 738. London: Taylor & Francis, 2004. Price (hardback) GBP 99. ISBN 0-415-25789-1. Journal of Applied Crystallography. 2005;38(3):579-.
43. Gennes P-Gd, Prost J. The Physics of Liquid Crystals. New York: Oxford University Press; 1993. 616 p.
44. Ericksen J. Conservation laws for liquid crystals. Trans Soc Rheol. 1961;5:22---34.
45. Lin P, Liu C, Zhang H. An Energy Law Preserving  $\mathcal{SC}^0$  Finite Element Scheme for Simulating the Kinematic Effects in Liquid Crystal Flow Dynamics. J Comput Phys,. 2007;227:1411-27.
46. Brannick J, Kirshtein A, Liu C. Dynamics of multi-component flows: diffusive interface methods with energetic variational approaches. In: Hashmi S, Buggy M, editors. Reference Module in Materials Science and Materials Engineering. Reference Module in Materials Science and Materials Engineering: Elsevier; 2016. p. 1-7.
47. Mestre H, Mori Y, Nedergaard M. The Brain's Glymphatic System: Current Controversies. Trends in neurosciences.
48. Nagelhus EA, Ottersen OP. Physiological Roles of Aquaporin-4 in Brain. Physiological reviews. 2013;93(4):1543-62.
49. Xie L, Kang H, Xu Q, Chen MJ, Liao Y, Thiyagarajan M, et al. Sleep Drives Metabolite Clearance from the Adult Brain. Science. 2013;342(6156):373-7.
50. Jessen NA, Munk ASF, Lundgaard I, Nedergaard M. The glymphatic system: a beginner's guide. Neurochemical research. 2015;40(12):2583-99.
51. Jiang Q, Zhang L, Ding G, Davoodi-Bojd E, Li Q, Li L, et al. Impairment of the glymphatic system after diabetes. Journal of Cerebral Blood Flow & Metabolism. 2017;37(4):1326-37.
52. Abbott NJ, Pizzo ME, Preston JE, Janigro D, Thorne RG. The role of brain barriers in fluid movement in the CNS: is there a 'glymphatic' system? Acta neuropathologica. 2018;135(3):387-407.
53. Gakuba C, Gaberel T, Goursaud S, Bourges J, Di Palma C, Quenault A, et al. General anesthesia inhibits the activity of the "glymphatic system". Theranostics. 2018;8(3):710.
54. Fultz NE, Bonmassar G, Setsompop K, Stickgold RA, Rosen BR, Polimeni JR, et al. Coupled electrophysiological, hemodynamic, and cerebrospinal fluid oscillations in human sleep. Science. 2019;366(6465):628-31.
55. Mestre H, Du T, Sweeney AM, Liu G, Samson AJ, Peng W, et al. Cerebrospinal fluid influx drives acute ischemic tissue swelling. Science. 2020;367(6483):eaax7171.

56. Nedergaard M, Goldman SA. Glymphatic failure as a final common pathway to dementia. *Science*. 2020;370(6512):50.
57. Vilcaes AA, Chanaday NL, Kavalali ET. Interneuronal exchange and functional integration of synaptobrevin via extracellular vesicles. *Neuron*. 2021.
58. Falk G, Fatt P. Linear Electrical Properties of Striated Muscle Fibres Observed with Intracellular Electrodes. *Proceedings of the Royal Society of London Series B, Containing papers of a Biological character*. 1964;160:69-123.
59. Eisenberg RS. The equivalent circuit of single crab muscle fibers as determined by impedance measurements with intracellular electrodes. *J Gen Physiol*. 1967;50(6):1785-806.
60. Sun HH. *Synthesis of RC Networks*: Hayden Book Publishers; 1967. 150 p.
61. Tuttle DF. *Network synthesis*: Wiley; 1958.
62. VanValkenburg ME. *Introduction to Modern Network Synthesis*: CBL Publishers; 1991.
63. Weinberg L. *Network analysis and synthesis*: Krieger Pub. Co.; 1975.
64. Balabanian N, Bickart TA. *Electrical network theory*: Wiley; 1969.
65. Ghausi MS, Kelly JJ. *Introduction to distributed-parameter networks: with application to integrated circuits*: Holt, Rinehart and Winston; 1968.
66. Guillemin EA. *Communications Networks Vol. 1 The Classical Theory of Lumped Constant Networks*: John Wiley; 1931.
67. LePage WR, Seely S. *General Network Analysis*: McGraw-Hill; 1952.
68. Zemanian AH. *Infinite electrical networks*: Cambridge University Press; 1991.
69. Bush V, Wiener N. *Operational Circuit Analysis: With an Appendix by Norbert Wiener*: Chapman & Hall; 1929.
70. Guillemin EA. *Introductory Circuit Theory*: Wiley; 1958.
71. McAllister RE, Noble D, Tsien R. Reconstruction of the electrical activity of cardiac Purkinje fibres. *The Journal of physiology*. 1975;251(1):1-59.
72. Fozzard HA, Beeler Jr GW. The voltage clamp and cardiac electrophysiology. *Circulation research*. 1975;37(4):403-13.
73. Bartos DC, Grandi E, Ripplinger CM. Ion Channels in the Heart. *Comprehensive Physiology*. 2015:1423-64.
74. Priest BT, McDermott JS. Cardiac ion channels. *Channels*. 2015;9(6):352-9.
75. Schneider MF. Linear electrical properties of the transverse tubules and surface membrane of skeletal muscle fibers. *J Gen Physiol*. 1970;56(5):640-71.
76. Valdiosera R, Clausen C, Eisenberg RS. Circuit models of the passive electrical properties of frog skeletal muscle fibers. *J Gen Physiol*. 1974; 63:432-59.
77. Valdiosera R, Clausen C, Eisenberg RS. Measurement of the impedance of frog skeletal muscle fibers. *Biophys J*. 1974;14:295-314.
78. Valdiosera R, Clausen C, Eisenberg RS. Impedance of frog skeletal muscle fibers in various solutions. *J Gen Physiol*. 1974;63(4):460-91.
79. Mathias RT, Eisenberg RS, Valdiosera R. Electrical properties of frog skeletal muscle fibers interpreted with a mesh model of the tubular system. *Biophys J*. 1977;17(1):57-93.
80. Eisenberg Robert S. Impedance Measurement of the Electrical Structure of Skeletal Muscle. *Comprehensive Physiology*, Republished by the American Physiological Society, as Volume 1, Supplement 27 of *Handbook of Physiology*, part of *Comprehensive Physiology*, SSN: 20404603 Online ISBN: 9780470650714. 1, Supplement 27, *Handbook of Physiology*: American Physiological Society and Wiley OnLine; 2011.
81. Eisenberg B. Electrical Structure of Biological Cells and Tissues: impedance spectroscopy, stereology, and singular perturbation theory. In: Barsoukov E, Macdonald JR, editors. *Impedance Spectroscopy: Theory, Experiment, and Applications Third Edition*. Third ed. New York: Wiley-Interscience; 2018. p. 472-8 Available on arXiv at <https://arxiv.org/abs/1511.01339> as arXiv:1511.
82. Milton RL, Mathias RT, Eisenberg RS. Electrical properties of the myotendon region of frog twitch muscle fibers measured in the frequency domain. *Biophys J*. 1985;48(2):253-67.
83. Eisenberg BR, Milton RL. Muscle fiber termination at the tendon in the frog's sartorius: a stereological study. *American journal of anatomy*. 1984;171(3):273-84.
84. Eisenberg RS, Mathias RT, Rae JS. Measurement, modeling, and analysis of the linear electrical properties of cells. *Annals of the New York Academy of Sciences*. 1977;303:342-54.
85. Mathias RT, Rae JL, Eisenberg RS. Electrical properties of structural components of the crystalline lens. *Biophys J*. 1979;25(1):181-201.
86. Rae JL. The electrophysiology of the crystalline lens. *Curr Top Eye Res*. 1979;1:37-90.

87. Rae JL, Stacey T. Lanthanum and procion yellow as extracellular markers in the crystalline lens of the rat. *Exp Eye Res.* 1979;28(1):1-21.
88. Mathias RT, Rae JL, Eisenberg RS. The lens as a nonuniform spherical syncytium. *Biophys J.* 1981;34(1):61-83.
89. Kuszak JR, Rae JL. Scanning electron microscopy of the frog lens. *Exp Eye Res.* 1982;35(5):499-519.
90. Rae JL, Mathias RT, Eisenberg RS. Physiological role of the membranes and extracellular space with the ocular lens. *Exp Eye Res.* 1982;35(5):471-89.
91. Rae JL, Thomson RD, Eisenberg RS. The effect of 2-4 dinitrophenol on cell to cell communication in the frog lens. *Exp Eye Res.* 1982;35(6):597-609.
92. Rae JL, Kuszak JR. The electrical coupling of epithelium and fibers in the frog lens. *Exp Eye Res.* 1983;36(3):317-26.
93. Rae JL, Truitt KD, Kuszak JR. The use of procion dyes for light microscopy of the frog lens. *Invest Ophthalmol Vis Sci.* 1983;24(9):1167-71.
94. Mathias RT, Rae JL. Steady state voltages in the frog lens. *Curr Eye Res.* 1985;4(4):421-30.
95. Mathias RT, Rae JL, Ebihara L, McCarthy RT. The localization of transport properties in the frog lens. *Biophys J.* 1985;48(3):423-34.
96. Baldo GJ, Mathias RT. Spatial variations in membrane properties in the intact rat lens. *Biophys J.* 1992;63(2):518-29.
97. Donaldson PJ, Roos M, Evans C, Beyer E, Kistler J. Electrical properties of mammalian lens epithelial gap junction channels. *Invest Ophthalmol Vis Sci.* 1994;35(9):3422-8.
98. Donaldson PJ, Dong Y, Roos M, Green C, Goodenough DA, Kistler J. Changes in lens connexin expression lead to increased gap junctional voltage dependence and conductance. *The American journal of physiology.* 1995;269(3 Pt 1):C590-600.
99. Rae JL, Bartling C, Rae J, Mathias RT. Dye transfer between cells of the lens. *The Journal of membrane biology.* 1996;150(1):89-103.
100. Nicholson C, Hrabětová S. Brain Extracellular Space: The Final Frontier. *Biophysical Journal.* 2017;113(10):2133-42.
101. Sykova E, Nicholson C. Diffusion in brain extracellular space. *Physiological reviews.* 2008;88(4):1277-340.
102. Eisenberg B, Eisenberg RS. Selective disruption of the sarcotubular system in frog sartorius muscle. A quantitative study with exogenous peroxidase as a marker. *The Journal of cell biology.* 1968;39(2):451-67.
103. Eisenberg BR, Mathias RT, Gilai A. Intracellular localization of markers within injected or cut frog muscle fibers. *The American journal of physiology.* 1979;237(1):C50-5.
104. Baddeley A, Jensen EBV. *Stereology for statisticians: Chapman and Hall/CRC; 2004.*
105. Howard V, Reed M. *Unbiased Stereology: Three-Dimensional Measurement in Microscopy: CRC Press; 2004.*
106. Mouton PR. *Principles and Practices of Unbiased Stereology: An Introduction for Bioscientists: Johns Hopkins University Press; 2002.*
107. Eisenberg BR. Skeletal muscle fibers: stereology applied to anisotropic and periodic structures. In: Weibel E, editor. *Stereological Methods: Practical methods for biological morphometry.* 1979. p. 274-84.
108. Eisenberg BR, Kuda AM, Peter JB. Stereological analysis of mammalian skeletal muscle: I. Soleus muscle of the adult guinea pig. *The Journal of cell biology.* 1974;60(3):732-54.
109. Eisenberg BR, Kuda AM. Stereological analysis of mammalian skeletal muscle: II. White vastus muscle of the adult guinea pig. *Journal of ultrastructure research.* 1975;51(2):176-87.
110. Eisenberg BR, Mobley BA. Size changes in single muscle fibers during fixation and embedding. *Tissue and Cell.* 1975;7:383-7.
111. Mobley BA, Eisenberg BR. Sizes of components in frog skeletal muscle measured by methods of stereology. *The Journal of general physiology.* 1975;66(1):31-45.
112. Eisenberg BR, Kuda AM. Discrimination between fiber populations in mammalian skeletal muscle by using ultrastructural parameters. *Journal of ultrastructure research.* 1976;54(1):76-88.
113. Eisenberg BR, Salmons S. The reorganization of subcellular structure in muscle undergoing fast-to-slow type transformation. *Cell and tissue research.* 1981;220(3):449-71.
114. Eisenberg BR. Quantitative ultrastructure of mammalian skeletal muscle. In: Peachey LD, Adrian RH, editors. *Handbook of Physiology. 10: American Physiological Society Bethesda MD; 1983.* p. 73-112.
115. Eisenberg BR, Brown J, Salmons S. Restoration of fast muscle characteristics following cessation of chronic stimulation. *Cell and tissue research.* 1984;238(2):221-30.

116. Eisenberg BR. Adaptability of ultrastructure in the mammalian muscle. *Journal of experimental biology*. 1985;115(1):55-68.
117. Ling GN. *A Physical Theory of the Living State: The Association-induction Hypothesis; with Considerations of the Mechanisms Involved in Ionic Specificity*: Blaisdell Publishing Company; 1962.
118. Pollack GH. *The Fourth Phase of Water: Beyond Solid, Liquid, and Vapor*: Ebner & Sons; 2013.
119. Sevostianov I, Mogilevskaya SG, Kushch VI. Maxwell's methodology of estimating effective properties: Alive and well. *International Journal of Engineering Science*. 2019;140:35-88.
120. Maxwell JC. *A Treatise on Electricity and Magnetism* (reprinted 1954). Third ed. New York: Dover Publications; 1865.
121. Schanne OF, Ceretti ERP. *Impedance Measurements in Biological Cells*: Wiley; 1978.
122. Mobley BA, Eidt G. Transverse impedance of single frog skeletal muscle fibers. *Biophysical Journal*. 1982;40(1):51-9.
123. Eisenberg RS, Johnson EA. Three dimensional electrical field problem in physiology. *Prog Biophys Mol Biol* 1970;20:1-65
124. Barcilon V, Cole J, Eisenberg RS. A singular perturbation analysis of induced electric fields in nerve cells. *SIAM J Appl Math*. 1971;21(2):339-54.
125. Peskoff A, Eisenberg RS. Interpretation of some microelectrode measurements of electrical properties of cells. *Annual review of biophysics and bioengineering*. 1973;2:65-79.
126. Peskoff A, Eisenberg RS. The time-dependent potential in a spherical cell using matched asymptotic expansions. *Journal of Mathematical Biology*. 1975;2(The time-dependent potential in a spherical cell using matched asymptotic expansions):277-300.
127. Peskoff A, Ramirez DM. Potential induced in a spherical cell by an intracellular point source and an extracellular point sink. *Journal of Mathematical Biology*. 1975;2(4):301-16.
128. Peskoff A, Eisenberg RS, Cole JD. Matched asymptotic expansions of the Green's function for the electric potential in an infinite cylindrical cell. *SIAM J Appl Math* 1976;30(2):222-39.
129. Peskoff A. Electric potential in three-dimensional electrically syncytial tissues. *Bulletin of mathematical biology*. 1979;41(2):163-81.
130. Mobley BA, Leung J, Eisenberg RS. Longitudinal impedance of skinned frog muscle fibers. *J Gen Physiol*. 1974;63(5):625-37.
131. Mobley BA, Leung J, Eisenberg RS. Longitudinal impedance of single frog muscle fibers. *J Gen Physiol*. 1975;65(1):97-113.
132. Hanai T, Haydon DA, Taylor J. The variation of capacitance and conductance of bimolecular lipid membranes with area. *Journal of theoretical biology*. 1965;9(3):433-43.
133. Hanai T, Haydon DA, Taylor J. The influence of lipid composition and of some adsorbed proteins on the capacitance of black hydrocarbon membranes. *Journal of theoretical biology*. 1965;9(3):422-32.
134. Everitt CT, Haydon DA. Electrical capacitance of a lipid membrane separating two aqueous phases. *Journal of theoretical biology*. 1968;18(3):371-9.
135. Haydon DA. Properties of lipid bilayers at a water-water interface. *J Am Oil Chem Soc*. 1968;45(4):230-40.
136. Fettiplace R, Andrews DM, Haydon DA. The thickness, composition and structure of some lipid bilayers and natural membranes. *The Journal of membrane biology*. 1971;5(3):277-96.
137. Haydon DA, Myers VB. Surface charge, surface dipoles and membrane conductance. *Biochim Biophys Acta*. 1973;307(3):429-43.
138. Haydon DA, Kimura J, Requena J, Urban BW. Impedance measurement as an indication of membrane thickness change in the squid giant axon [proceedings]. *J Physiol*. 1979;287:2P.
139. Brosseau C, Sabri E. Resistor-capacitor modeling of the cell membrane: A multiphysics analysis. *Journal of Applied Physics*. 2021;129(1):011101.
140. Ziebert F, Lacoste D. A planar lipid bilayer in an electric field: membrane instability, flow field, and electrical impedance. *Advances in planar lipid bilayers and liposomes*. 14: Elsevier; 2011. p. 63-95.
141. Fernandez J, Bezanilla F, Taylor R. Distribution and kinetics of membrane dielectric polarization. II. Frequency domain studies of gating currents. *J Gen Physiol*. 1982;79(1):41-67.
142. Taylor RE, Bezanilla F. Comments on the measurement of gating currents in the frequency domain. *Biophys J*. 1979;26(2):338-40.
143. Bezanilla F, Taylor RE, Fernandez JM. Distribution and kinetics of membrane dielectric polarization. 1. Long-term inactivation of gating currents. *J Gen Physiol*. 1982;79(1):21-40.
144. Fernandez JM, Taylor RE, Bezanilla F. Induced capacitance in the squid giant axon. Lipophilic ion displacement currents. *J Gen Physiol*. 1983;82(3):331-46.

145. Taylor RE. Impedance of the squid axon membrane. *Journal of Cellular Physiology*. 1965;66(S2):21-5.
146. Clausen C, Lewis SA, Diamond JM, Eisenberg RS. Electrical circuit analysis of tight epithelia by alternating current techniques. *Biophys J* 1976;16:131a.
147. Clausen C, Fernandez JM. A low-cost method for rapid transfer function measurements with direct application to biological impedance analysis. *Pflügers Archiv*. 1981;390(3):290-5.
148. Clausen C. [32] Impedance analysis in tight epithelia. *Methods in enzymology*. 171: Elsevier; 1989. p. 628-42.
149. Clausen C, Machen TE, Diamond JM. Use of AC impedance analysis to study membrane changes related to acid secretion in amphibian gastric mucosa. *Biophysical Journal*. 1983;41(2):167-78.
150. Clausen C, Reinach PS, Marcus DC. Membrane transport parameters in frog corneal epithelium measured using impedance analysis techniques. *The Journal of membrane biology*. 1986;91(3):213-25.
151. Lewis SA, Clausen C, Wills NK. Impedance analysis of epithelia. *Epithelial Transport*: Springer; 1996. p. 118-45.
152. Wills N, Clausen C. Transport-dependent alterations of membrane properties of mammalian colon measured using impedance analysis. *The Journal of membrane biology*. 1987;95(1):21-35.
153. Clausen C, Lewis SA, Diamond JM. Impedance analysis of a tight epithelium using a distributed resistance model. *Biophysical journal*. 1979;26(2):291-317.
154. Boron W, Boulpaep E. *Medical Physiology*. New York: Saunders; 2008. 1352 p.
155. Silverthorn DU, Johnson BR, Ober WC, Ober CE, Impagliazzo A, Silverthorn AC. *Human Physiology: An Integrated Approach*: Pearson Education, Incorporated; 2019.
156. Feher JJ. *Quantitative human physiology: an introduction*: Academic press; 2017.
157. Keener J, Sneyd J. *Mathematical Physiology: I: Cellular Physiology*: Springer New York; 2014.
158. Layton AT, Edwards A. *Mathematical Modeling in Renal Physiology*: Springer Berlin Heidelberg; 2014.
159. Wostyn P, Van Dam D, Audenaert K, Killer H, Paul D, Groot V. A new glaucoma hypothesis: A role of glymphatic system dysfunction. *Fluids and barriers of the CNS*. 2015;12:16.
160. Norman RE, Flanagan JG, Sigal IA, Rausch SM, Tertinegg I, Ethier CR. Finite element modeling of the human sclera: influence on optic nerve head biomechanics and connections with glaucoma. *Experimental eye research*. 2011;93(1):4-12.
161. Ethier C, Norman R, Sigal I, Rausch S, Tertinegg I, Eilaghi A, et al. Finite Element Modeling of the Human Sclera: Influence on ONH Biomechanics and Connections With Glaucoma. *Investigative Ophthalmology & Visual Science*. 2009;50(13):4889-.
162. Gardiner BS, Smith DW, Coote M, Crowston JG. Computational modeling of fluid flow and intra-ocular pressure following glaucoma surgery. *PLoS One*. 2010;5(10):e13178.
163. Stefan C, Cojocaru I, Pop A. [Glaucoma and ocular surface]. *Oftalmologia*. 2011;55(1):30-3.
164. Wang N. *Intraocular and Intracranial Pressure Gradient in Glaucoma*: Springer Singapore; 2019.
165. Jin B-J, Smith AJ, Verkman AS. Spatial model of convective solute transport in brain extracellular space does not support a “glymphatic” mechanism. *Journal of General Physiology*. 2016;148(6):489-501.
166. Kaur J, Davoodi-Bojd E, Fahmy LM, Zhang L, Ding G, Hu J, et al. Magnetic Resonance Imaging and Modeling of the Glymphatic System. *Diagnostics*. 2020;10(6):344.
167. Zhu Y, Xu S, Eisenberg RS, Huang H. A Tridomain Model for Potassium Clearance in Optic Nerve. *arXiv:201203303*. 2020.
168. Stephenson JL. 23 - The Mathematical Theory of Renal Function. In: Brown JHU, Gann DS, editors. *Engineering Principles in Physiology*: Academic Press; 1973. p. 283-320.
169. Kuo IY, Ehrlich BE. Ion Channels in Renal Disease. *Chemical Reviews*. 2012;112(12):6353-72.
170. Thomas SR, Layton AT, Layton HE, Moore LC. *Kidney Modeling: Status and Perspectives*. *Proceedings of the IEEE*. 2006;94(4):740-52.
171. Layton AT, Layton HE. Countercurrent multiplication may not explain the axial osmolality gradient in the outer medulla of the rat kidney. *American Journal of Physiology-Renal Physiology*. 2011;301(5):F1047-F56.
172. Eisenberg RS, Rae JL. Current-voltage relationships in the crystalline lens. *J Physiol*. 1976;262(2):285-300.
173. Eisenberg RS, Engel E. The spatial variation of membrane potential near a small source of current in a spherical cell. *J Gen Physiol*. 1970;55(6):736-57.
174. Engel E, Barcion V, Eisenberg RS. The interpretation of current-voltage relations recorded from a spherical cell with a single microelectrode. *Biophys J*. 1972;12(4):384-403.
175. Kevorkian J, Cole JD. *Multiple Scale and Singular Perturbation Methods*. New York: Springer-Verlag; 1996. pp. 1-632. p.

176. Eisenberg RS, Barcion V, Mathias RT. Electrical properties of spherical syncytia. *Biophys J*. 1979;25(1):151-80.
177. Peskoff A. Electric potential in cylindrical syncytia and muscle fibers. *Bulletin of mathematical biology*. 1979;41(2):183-92.
178. Forrester JV, Dick AD, McMenamin PG, Roberts F, Pearlman E. *The Eye: Basic Sciences in Practice*: Elsevier; 2021.
179. Donaldson P, Kistler J, Mathias RT. Molecular solutions to mammalian lens transparency. *News Physiol Sci*. 2001;16:118-23.
180. Mathias RT, Kistler J, Donaldson P. The lens circulation. *Journal of Membrane Biology*. 2007;216(1):1-16.
181. Rae JL. The potential difference of the frog lens. *Exp Eye Res*. 1973;15(4):485-94.
182. Mathias RT, Rae JL. The lens: local transport and global transparency. *Exp Eye Res*. 2004;78(3):689-98.
183. Donaldson PJ, Musil LS, Mathias RT. Point: A critical appraisal of the lens circulation model--an experimental paradigm for understanding the maintenance of lens transparency? *Invest Ophthalmol Vis Sci*. 2010;51(5):2303-6.
184. Donaldson PJ, Grey AC, Maceo Heilman B, Lim JC, Vaghefi E. The physiological optics of the lens. *Prog Retin Eye Res*. 2017;56:e1-e24.
185. Mathias RT, Rae JL. Transport properties of the lens. *The American journal of physiology*. 1985;249(3 Pt 1):C181-90.
186. Mathias RT, Riquelme G, Rae JL. Cell to cell communication and pH in the frog lens. *J Gen Physiol*. 1991;98(6):1085-103.
187. Mathias RT, Rae JL, Baldo GJ. Physiological properties of the normal lens. *Physiological reviews*. 1997;77(1):21-50.
188. Gong X, Baldo GJ, Kumar NM, Gilula NB, Mathias RT. Gap junctional coupling in lenses lacking alpha3 connexin. *Proceedings of the National Academy of Sciences of the United States of America*. 1998;95(26):15303-8.
189. Kushmerick C, Varadaraj K, Mathias RT. Effects of lens major intrinsic protein on glycerol permeability and metabolism. *The Journal of membrane biology*. 1998;161(1):9-19.
190. Baldo GJ, Gong X, Martinez-Wittinghan FJ, Kumar NM, Gilula NB, Mathias RT. Gap junctional coupling in lenses from alpha(8) connexin knockout mice. *J Gen Physiol*. 2001;118(5):447-56.
191. Gao J, Sun X, Martinez-Wittinghan FJ, Gong X, White TW, Mathias RT. Connections between connexins, calcium, and cataracts in the lens. *J Gen Physiol*. 2004;124(4):289-300.
192. Martinez-Wittinghan FJ, Sellitto C, White TW, Mathias RT, Paul D, Goodenough DA. Lens gap junctional coupling is modulated by connexin identity and the locus of gene expression. *Invest Ophthalmol Vis Sci*. 2004;45(10):3629-37.
193. McNulty R, Wang H, Mathias RT, Ortwerth BJ, Truscott RJ, Bassnett S. Regulation of tissue oxygen levels in the mammalian lens. *J Physiol*. 2004;559(Pt 3):883-98.
194. Varadaraj K, Kumari S, Shiels A, Mathias RT. Regulation of aquaporin water permeability in the lens. *Invest Ophthalmol Vis Sci*. 2005;46(4):1393-402.
195. Wang H, Gao J, Sun X, Martinez-Wittinghan FJ, Li L, Varadaraj K, et al. The effects of GPX-1 knockout on membrane transport and intracellular homeostasis in the lens. *The Journal of membrane biology*. 2009;227(1):25-37.
196. Gao J, Sun X, Moore LC, White TW, Brink PR, Mathias RT. Lens intracellular hydrostatic pressure is generated by the circulation of sodium and modulated by gap junction coupling. *J Gen Physiol*. 2011;137(6):507-20.
197. Gao J, Wang H, Sun X, Varadaraj K, Li L, White TW, et al. The effects of age on lens transport. *Invest Ophthalmol Vis Sci*. 2013;54(12):7174-87.
198. Hall JE, Mathias RT. The aquaporin zero puzzle. *Biophys J*. 2014;107(1):10-5.
199. Sindhu Kumari S, Gupta N, Shiels A, FitzGerald PG, Menon AG, Mathias RT, et al. Role of Aquaporin 0 in lens biomechanics. *Biochemical and biophysical research communications*. 2015;462(4):339-45.
200. Gao J, Minogue PJ, Beyer EC, Mathias RT, Berthoud VM. Disruption of the lens circulation causes calcium accumulation and precipitates in connexin mutant mice. *Am J Physiol Cell Physiol*. 2018;314(4):C492-C503.
201. Berthoud VM, Gao J, Minogue PJ, Jara O, Mathias RT, Beyer EC. Connexin Mutants Compromise the Lens Circulation and Cause Cataracts through Biom mineralization. *Int J Mol Sci*. 2020;21(16).

202. Delamere NA, Shahidullah M, Mathias RT, Gao J, Sun X, Sellitto C, et al. Signaling Between TRPV1/TRPV4 and Intracellular Hydrostatic Pressure in the Mouse Lens. *Invest Ophthalmol Vis Sci*. 2020;61(6):58.
203. Weinstein A, Stephenson J. Coupled water transport in standing gradient models of the lateral intercellular space. *Biophysical journal*. 1981;35(1):167-91.
204. Vereshchaga Y, Arnold N, Baumgartner W. Physiological relevance of epithelial geometry: New insights into the standing gradient model and the role of LI cadherin. *PLOS ONE*. 2018;13:e0208791.
205. O'Brien S. Lin & Segel's Standing Gradient Problem Revisited: A Lesson in Mathematical Modeling and Asymptotics. *SIAM Review*. 2011;53:775-96.
206. McLaughlin S, Mathias RT. Electro-osmosis and the reabsorption of fluid in renal proximal tubules. *J Gen Physiol*. 1985;85(5):699-728.
207. Mathias RT. Epithelial water transport in a balanced gradient system. *Biophys J*. 1985;47(6):823-36.
208. Mathias RT, Wang H. Local osmosis and isotonic transport. *The Journal of membrane biology*. 2005;208(1):39-53.
209. Ebihara L, Mathias RT. Linear impedance studies of voltage-dependent conductances in tissue cultured chick heart cells. *Biophys J*. 1985;48(3):449-60.
210. Mathias RT. Steady-state voltages, ion fluxes, and volume regulation in syncytial tissues. 1985;48(3):435-48.
211. Kelvin L. On the theory of the electric telegraph. *Proceedings of the Royal Society (London)*. 1855;7:382-99.
212. Kelvin L. On the theory of the electric telegraph. *Philosophical Magazine*. 1856;11:146-60.
213. Gordon JS. *A Thread Across the Ocean: The Heroic Story of the Transatlantic Cable*: Paw Prints; 2008.
214. Hodgkin AL, Rushton WAH. The electrical constants of a crustacean nerve fiber. *Proc Roy Soc (London) Ser B*. 1946;133:444-79.
215. Davis LD, Jr., de No RL. Contribution to the Mathematical Theory of the electrotonus. *Studies from the Rockefeller Institute for Medical Research*. 1947;131:442-96.
216. Jack JJB, Noble D, Tsien RW. *Electric Current Flow in Excitable Cells*. New York: Oxford, Clarendon Press.; 1975.
217. Weinberg AM. Nerve Conduction with Distributed Capacitance. *Journal of Applied Physics*. 1939;10(2):128-34.
218. Weinberg A. On the formal theory of nerve conduction. *The Bulletin of mathematical biophysics*. 1940;2(3):127-33.
219. Weinberg AM. Weber's theory of the Kernleiter. *The Bulletin of mathematical biophysics*. 1941;3(2):39-55.
220. Weinberg AM. Green's functions in biological potential problems. *The Bulletin of mathematical biophysics*. 1942;4(3):107-15.
221. Weinstein AM. A mathematical model of the rat proximal tubule. *American Journal of Physiology-Renal Physiology*. 1986;250(5):F860-F73.
222. Weinstein AM. Analysis of volume regulation in an epithelial cell model. *Bull Math Biol*. 1992;54(4):537-61.
223. Weinstein AM. Mathematical models of tubular transport. *Annual review of physiology*. 1994;56:691-709.
224. Weinstein AM. A mathematical model of rat proximal tubule and loop of Henle. *American journal of physiology*. 2015;308(10):F1076-97.
225. Weinstein AM. A mathematical model of the rat kidney: K(+)-induced natriuresis. *American journal of physiology*. 2017;312(6):F925-F50.
226. Starling EH. On the Absorption of Fluids from the Connective Tissue Spaces. *The Journal of Physiology*. 1896;19(4):312-26.
227. Michel CC, Woodcock TE, Curry FRE. Understanding and extending the Starling principle. *Acta Anaesthesiologica Scandinavica*. 2020;64(8):1032-7.
228. Nikolaev P. A new method to obtain the Carnahan–Starling equation and its generalization. *Moscow University Physics Bulletin*. 2017;72(1):23-8.
229. Dohrn R, Prausnitz JM. A simple perturbation term for the Carnahan-Starling equation of state. *Fluid phase equilibria*. 1990;61(1-2):53-69.
230. Hansen-Goos H, Roth R. A new generalization of the Carnahan-Starling equation of state to additive mixtures of hard spheres. *The Journal of chemical physics*. 2006;124(15):154506.

231. Erstad BL. The revised Starling equation: the debate of albumin versus crystalloids continues. *Annals of Pharmacotherapy*. 2020;54(9):921-7.
232. Song Y, Mason E, Stratt RM. Why does the Carnahan-Starling equation work so well? *The journal of physical chemistry*. 1989;93(19):6916-9.
233. Civetta JM. A new look at the Starling equation. *Critical Care Medicine*. 1979;7(3):84-91.
234. Rowlinson JS. *The Perfect Gas*. New York: Macmillan; 1963. 136 p.
235. Eisenberg RS. Computing the field in proteins and channels. *Journal of Membrane Biology*. 1996;150:1–25. Preprint available on physics arXiv as document 1009.2857.
236. Eisenberg B. Self-organized model of selectivity. Institute of Mathematics and its Applications. 2009; IMA University of Minnesota (on-line): <http://www.ima.umn.edu/2008-9/W12.8-12.08/abstracts.html> and also <http://arxiv.org/0906.5173>.
237. Eisenberg B. Ion Channels, Natural Nanovalves. In: Savinell RF, Ota K-i, Kreysa G, editors. *Encyclopedia of Applied Electrochemistry (in the press)* Also available at <http://arxiv.org/abs/12061253> as arXiv:12061253v1. New York: Springer; 2013.
238. Eisenberg B. Electrostatic effects in living cells. *Physics Today*. 2013;66(7):10-1.
239. Eisenberg B. Shouldn't we make biochemistry an exact science? *ASBMB Today*. 2014;13(9, October):36-8, Available on arXiv as <https://arxiv.org/abs/1409.0243>.
240. Eisenberg B. Rate Constant Models cannot Describe Movement of Charged Atoms or Molecules. *Biophysical Journal*. 2015;108(2):577a.
241. Eisenberg R. PNP what is in a name July 25-1 2019. pdf 10.31224/osf.io/2739d. engrXiv August 3. 2019.
242. Eisenberg RS. Kirchhoff's Law can be Exact. arXiv preprint available at <https://arxiv.org/abs/190513574>. 2019.
243. Eisenberg RS. Updating Maxwell with Electrons, Charge, and More Realistic Polarization. arXiv preprint available at <https://arxiv.org/abs/190409695>. 2019.
244. Eisenberg RS. Electrodynamics Correlates Knock-on and Knock-off: Current is Spatially Uniform in Ion Channels. Preprint on arXiv at <https://arxiv.org/abs/200209012>. 2020.
245. Eisenberg RS. Core Maxwell Equations are Exact, Universal, and Scary, for that reason. Slide Show: DOI: 10.13140/RG.2.2.24122.31687. 2021.
246. Eisenberg R. A Necessary Addition to Kirchhoff's Current Law of Circuits, Version 2. Engineering Archive EngArXiv. 2022; <https://doi.org/10.31224/2234>.
247. Shockley W. *Electrons and Holes in Semiconductors to applications in transistor electronics*. New York: van Nostrand; 1950. 558 p.
248. Van Roosbroeck W. Theory of flow of electrons and holes in germanium and other semiconductors. *Bell System Technical Journal*. 1950;29:560-607.
249. Blotekjaer K. Transport equations for electrons in two-valley semiconductors. *Electron Devices, IEEE Transactions on*. 1970;17(1):38-47.
250. Streetman BG. *Solid State Electronic Devices*. 4th ed. Englewood Cliffs, NJ: Prentice Hall; 1972. 462 p.
251. Sze SM. *Physics of Semiconductor Devices*. New York: John Wiley & Sons; 1981. 838. p.
252. Selberherr S. *Analysis and Simulation of Semiconductor Devices*. New York: Springer-Verlag; 1984. pp. 1-293. p.
253. Ferry DK. *Semiconductor Transport*. New York: Taylor and Francis; 2000. 384 p.
254. Hess K. *Advanced Theory of Semiconductor Devices*. New York: IEEE Press; 2000. 350 p.
255. Vasileska D, Goodnick SM, Klimeck G. *Computational Electronics: Semiclassical and Quantum Device Modeling and Simulation*. New York: CRC Press; 2010. 764 p.
256. Eisenberg B. The value of Einstein's mistakes. *Letter to the Editor: "Einstein should be allowed his mistakes ..."* *Physics Today*. 2006;59(4):12.
257. Mott NF. The theory of crystal rectifiers. *Proc Roy Soc A*. 1939;171:27-38.
258. Grimley TB, Mott NF. I. General and theoretical. The contact between a solid and a liquid electrolyte. *Discussions of the Faraday Society*. 1947;1:3.
259. Mott NF, Gurney RW. *Electronic processes in ionic crystals*. 1948.
260. Goldman DE. Potential, impedance and rectification in membranes. *J Gen Physiol*. 1943;27:37–60.
261. Hodgkin AL, Katz B. The effect of sodium ions on the electrical activity of the giant axon of the squid. *J Physiol*. 1949; 108:37–77.
262. Hille B. *Ion Channels of Excitable Membranes*. 3rd ed. Sunderland: Sinauer Associates Inc.; 2001. 1-814. p.
263. Eisenberg B. *Proteins, Channels, and Crowded Ions*. *Biophysical chemistry*. 2003;100:507 - 17.

264. Eisenberg B. Crowded Charges in Ion Channels. In: Rice SA, editor. *Advances in Chemical Physics*. New York: John Wiley & Sons, Inc.; 2011. p. 77-223 also on the arXiv at <http://arxiv.org/abs/1009.786v1>.
265. Boda D, Nonner W, Henderson D, Eisenberg B, Gillespie D. Volume Exclusion in Calcium Selective Channels. *Biophys J*. 2008;94(9):3486-96.
266. Boda D, Nonner W, Valisko M, Henderson D, Eisenberg B, Gillespie D. Steric selectivity in Na channels arising from protein polarization and mobile side chains. *Biophys J*. 2007;93(6):1960-80.
267. Boda D, Valisko M, Henderson D, Eisenberg B, Gillespie D, Nonner W. Ionic selectivity in L-type calcium channels by electrostatics and hard-core repulsion. *Journal of General Physiology*. 2009;133(5):497-509.
268. Chen R-C, Li C-L, Chen J-H, Eisenberg B, Liu J-L. Differential Capacitance of Electric Double Layers: A Poisson-Bikerman Formula arxiv:2012.13141. Available at <https://arxiv.org/pdf/201213141.pdf>. 2020.
269. Jimenez-Morales D, Liang J, Eisenberg B. Ionizable side chains at catalytic active sites of enzymes. *European Biophysics Journal*. 2012;41(5):449-60.
270. Edsall JT, Wyman J. *Biophysical Chemistry: Thermodynamics, Electrostatics, and the Biological Significance of the Properties of Matter*: Elsevier Science; 2014.
271. Tanford C. *Physical Chemistry of Macromolecules*. New York: Wiley; 1961. 710 p.
272. Kubo R, Toda M, Hashitsume N. *Statistical Physics II: Nonequilibrium Statistical Mechanics Second Edition* ed. New York: Springer-Verlag; 1995. 295 p.
273. Eisenberg B. Vignette Applications of Physical Chemistry, a Biological Example. In: Berry S, Rice SA, Ross J, editors. *Physical Chemistry*: Oxford; 2005.
274. Hille E, Schwartz W. Potassium channels as multi-ion single-file pores. *J Gen Physiol*. 1978;72:409-42.
275. Chen D, Xu L, Tripathy A, Meissner G, Eisenberg R. Rate Constants in Channology. *Biophys J*. 1997;73(3):1349-54.
276. Eisenberg RS. Atomic Biology, Electrostatics and Ionic Channels. In: Elber R, editor. *New Developments and Theoretical Studies of Proteins*. 7. Philadelphia: World Scientific; 1996. p. 269-357. Published in the Physics ArXiv as arXiv:0807.715.
277. Barcion V, Chen D, Eisenberg RS, Ratner M. Barrier crossing with concentration boundary conditions in biological channels and chemical reactions. *J Chem Phys*. 1993;98:1193-211.
278. Eisenberg RS, Kłosek MM, Schuss Z. Diffusion as a chemical reaction: Stochastic trajectories between fixed concentrations. *J Chem Phys*. 1995;102:1767-80.
279. Schuss Z, Nadler B, Eisenberg RS. Derivation of Poisson and Nernst-Planck equations in a bath and channel from a molecular model. *Phys Rev E Stat Nonlin Soft Matter Phys*. 2001;64(3 Pt 2):036116.
280. Schuss Z, Nadler B, Singer A, Eisenberg R. A PDE formulation of non-equilibrium statistical mechanics for ionic permeation. *AIP Conference Proceedings*, 3-6 September 2002: Unsolved Problems Of Noise And Fluctuations, UPoN 2002, 3rd International Conference on Unsolved Problems of Noise and Fluctuations in Physics, Biology, and High Technology 2002;665.
281. Nadler B, Schuss Z, Singer A, Eisenberg B. Diffusion through protein channels: from molecular description to continuum equations. *Nanotechnology*. 2003;3:439.
282. Schuss Z, Nadler B, Singer A, Eisenberg RS, editors. *A PDE Formulation of Non-Equilibrium Statistical Mechanics for Ionic Permeation*. *AIP Conference Proceedings*, 3-6 September 2002: Unsolved Problems Of Noise And Fluctuations, UPoN 2002, 3rd International Conference on Unsolved Problems of Noise and Fluctuations in Physics, Biology, and High Technology 2003: AIP.
283. Nadler B, Schuss Z, Singer A, Eisenberg R. Ionic diffusion through confined geometries: from Langevin equations to partial differential equations *Journal of Physics: Condensed Matter*. 2004;16:S2153-S65.
284. Singer A, Schuss Z, Nadler B, Eisenberg RS. Models of boundary behavior of particles diffusing between two concentrations. In: Abbot D, Bezrukov SM, Der A, Sanchez A, editors. *Fluctuations and Noise in Biological, Biophysical, and Biomedical Systems II: series Vol 5467*. 5467. New York: SPIE Proc.; 2004. p. 345-58.
285. Cooper K, Jakobsson E, Wolynes P. The theory of ion transport through membrane channels. *Progress in Biophysics and Molecular Biology*. 1985;46:51-96.
286. Cooper KE, Gates PY, Eisenberg RS. Surmounting barriers in ionic channels. *Quarterly Review of Biophysics*. 1988;21: 331-64.
287. Cooper KE, Gates PY, Eisenberg RS. Diffusion theory and discrete rate constants in ion permeation. *J Membr Biol*. 1988;109:95-105.
288. Gates P, Cooper K, Rae J, Eisenberg R. Predictions of diffusion models for one-ion membrane channels. *Prog Biophys Mol Biol*. 1989;53(3):153-96.
289. Xu X, Liu C, Qian T. Hydrodynamic boundary conditions for one-component liquid-gas flows on non-isothermal solid substrates. *Communications in Mathematical Sciences*. 2012;10(4 (December 2012)):1027-53.

290. Eisenberg B. Multiple Scales in the Simulation of Ion Channels and Proteins. *The Journal of Physical Chemistry C*. 2010;114(48):20719-33.
291. Eisenberg B. Mass Action in Ionic Solutions. *Chemical Physics Letters*. 2011;511:1-6.
292. Eisenberg B. Life's Solutions are Not Ideal. Posted on arXiv.org with Paper ID arXiv:11050184v1. 2011.
293. Eisenberg B. A Leading Role for Mathematics in the Study of Ionic Solutions. *SIAM News*. 2012;45(9):11-2.
294. Eisenberg B. Ionic Interactions Are Everywhere. *Physiology*. 2013;28(1):28-38.
295. Eisenberg B. Interacting ions in Biophysics: Real is not ideal. . *Biophysical Journal*. 2013;104:1849-66.
296. Martell AE, Smith RM. *Critical Stability Constants: First Supplement*: Springer US; 2013.
297. Martell AE, Smith RM. *Critical Stability Constants: Second Supplement*: Springer US; 2013.
298. Martell AE, Motekaitis R. *Determination and Use of Stability Constants*. New York: VCH; 1988. 216 p.
299. Martell AE, Hancock RD. *Metal Complexes in Aqueous Solutions*: Springer US; 2013.
300. Martell AE, Hancock RD. *Metal Complexes in Aqueous Solutions*. New York: Plenum; 1996. 253 p.
301. Zhu Y, Xu S, Eisenberg RS, Huang H. A Bidomain Model for Lens Microcirculation *Biophysical Journal*. 2019;116(6):1171-84 Preprint available at <https://arxiv.org/abs/810.04162>.
302. Zhu Y. *Mathematical Modeling of Coupled Ion and Water Transport in Biological Tissues*. 2021.
303. Mathias RT. An analysis of the electrical properties of a skeletal muscle fiber containing a helicoidal T system. *Biophys J*. 1978;23(2):277-84.
304. Eisenberg RS, Mathias RT. Structural analysis of electrical properties of cells and tissues. *CRC Crit Rev Bioeng*. 1980;4(3):203-32.
305. Mathias RT. Analysis of Membrane Properties Using Extrinsic Noise. In: Eisenberg R, Frank M, Stevens C, editors. *Membranes, Channels, and Noise*: Springer US; 1984. p. 49-116.
306. Xu S, Eisenberg B, Song Z, Huang H. Osmosis through a Semi-permeable Membrane: a Consistent Approach to Interactions. arXiv preprint arXiv:180600646. 2018.
307. Chanson H. *Applied Hydrodynamics: An Introduction*: CRC Press; 2013.
308. Varadaraj K, Gao J, Mathias RT, Kumari S. C-Terminal End of Aquaporin 0 Regulates Lens Gap Junction Channel Function. *Invest Ophthalmol Vis Sci*. 2019;60(7):2525-31.
309. Berthoud VM, Gao J, Minogue PJ, Jara O, Mathias RT, Beyer EC. The Connexin50D47A Mutant Causes Cataracts by Calcium Precipitation. *Invest Ophthalmol Vis Sci*. 2019;60(6):2336-46.
310. Vaghefi E, Donaldson PJ. The lens internal microcirculation system delivers solutes to the lens core faster than would be predicted by passive diffusion. *American Journal of Physiology-Regulatory, Integrative and Comparative Physiology*. 2018.
311. Petrova RS, Webb KF, Vaghefi E, Walker K, Schey KL, Donaldson PJ. Dynamic functional contribution of the water channel AQP5 to the water permeability of peripheral lens fiber cells. *Am J Physiol Cell Physiol*. 2018;314(2):C191-C201.
312. Schey KL, Petrova RS, Gletten RB, Donaldson PJ. The Role of Aquaporins in Ocular Lens Homeostasis. *Int J Mol Sci*. 2017;18(12).
313. Kumari S, Gao J, Mathias RT, Sun X, Eswaramoorthy A, Browne N, et al. Aquaporin 0 Modulates Lens Gap Junctions in the Presence of Lens-Specific Beaded Filament Proteins. *Invest Ophthalmol Vis Sci*. 2017;58(14):6006-19.
314. Vaghefi E, Kim A, Donaldson PJ. Active Maintenance of the Gradient of Refractive Index Is Required to Sustain the Optical Properties of the Lens. *Invest Ophthalmol Vis Sci*. 2015;56(12):7195-208.
315. Young JZ. Fused neurons and synaptic contacts in the giant nerve fibres of cephalopods. *Philosophical Transactions of the Royal Society of London Series B, Biological sciences*. 1939;229(564):465-503.
316. Young J, Keynes R. The Functioning of the Giant Nerve Fibres of the Squid. 1938-JZ and the discovery of squid giant nerve fibres. *The Journal of experimental biology*. 2005;208(Pt 2):179-80.
317. Young JZ. The functioning of the giant nerve fibres of the squid. *Journal of Experimental Biology*. 1938;15(2):170-85.
318. Hodgkin AL. *Chance and Design*. New York: Cambridge University Press; 1992. 401 p.
319. Huxley AF. From overshoot to voltage clamp. *Trends in Neurosciences* 2002;25 (11):553-8.
320. Huxley A. Hodgkin Obituary. *The Independent (newspaper)*. 1999;January 4, 1999 <http://www.independent.co.uk/arts-entertainment/obituaries-professor-sir-alan-hodgkin-1044924.html>.
321. Huxley A. Sir Alan Lloyd Hodgkin, O. M., K. B. E. 5 February 1914-20 December 1998. *Biographical Memoirs of Fellows of the Royal Society*. 2000;46:221-41.
322. Hodgkin AL. Evidence for electrical transmission in nerve: Part II. *J Physiol*. 1937;90(2):211-32.
323. Hodgkin AL. Evidence for electrical transmission in nerve: Part I. *J Physiol*. 1937;90(2):183-210.
324. Hill AV. *Chemical Wave Transmission in Nerve*: Cambridge University Press; 1932. 74 p.

325. Huxley AF. Kenneth Stewart Cole 1900-1984. A biographical memoir by Sir Andrew Huxley. Washington DC: National Academies Press; 1996.
326. Rubinstein I, Zaltzman B. Convective diffusive mixing in concentration polarization: from Taylor dispersion to surface convection. *Journal of Fluid Mechanics*. 2013;728:239-78.
327. Abu-Rjal R, Chinarian V, Bazant MZ, Rubinstein I, Zaltzman B. Effect of concentration polarization on permselectivity. *Phys Rev E Stat Nonlin Soft Matter Phys*. 2014;89(1):012302.
328. Tanaka Y. 6 - Concentration Polarization. *Ion Exchange Membranes (Second Edition)*. Amsterdam: Elsevier; 2015. p. 101-21.
329. Wang Y, Liu C, Eisenberg B. On variational principles for polarization in electromechanical systems. *arXiv preprint arXiv:210811512*. 2021.
330. Frankenhaeuser B, Hodgkin AL. The after-effects of impulses in the giant nerve fibres of *Loligo*. *Journal of Physiology (London)*. 1956;131:341-76.
331. Taylor RE, Bezanilla F, Rojas E. Diffusion models for the squid axon Schwann cell layer. *Biophys J*. 1980;29(1):95-117.
332. Salzberg BM, Bezanilla F. An optical determination of the series resistance in *Loligo*. *J Gen Physiol*. 1983;82(6):807-17.
333. Kuffler SW, Nicholls JG, Orkand RK. Physiological properties of glial cells in the central nervous system of amphibia. *Journal of neurophysiology*. 1966;29(4):768-87.
334. Orkand RK, Nicholls JG, Kuffler SW. Effect of nerve impulses on the membrane potential of glial cells in the central nervous system of amphibia. *Journal of neurophysiology*. 1966;29(4):788-806.
335. Kuffler SW, Nicholls JG. The physiology of neuroglial cells. *Ergeb Physiol*. 1966;57:1-90.
336. Kuffler SW. Neuroglial cells: physiological properties and a potassium mediated effect of neuronal activity on the glial membrane potential. *Proceedings of the Royal Society of London Series B, Containing papers of a Biological character*. 1967;168(1010):1-21.
337. Cohen MW, Gerschenfeld HM, Kuffler SW. Ionic environment of neurones and glial cells in the brain of an amphibian. *J Physiol*. 1968;197(2):363-80.
338. Zhu Y, Xu S, Eisenberg RS, Huang H. Optic nerve microcirculation: Fluid flow and electrodiffusion. *Physics of Fluids*. 2021;33(4):041906.
339. Zhu Y, Xu S, Eisenberg RS, Huang H. Membranes in Optic Nerve Models. *arXiv preprint arXiv:210514411*. 2021.
340. Gabbiani F, Cox SJ. *Mathematics for Neuroscientists*. New York: Academic Press; 2010.
341. Huxley AF. The quantitative analysis of excitation and conduction in nerve. 1963. 242-60 p.
342. Gao J, Sun X, Yatsula V, Wymore R, Mathias R. Isoform-specific function and distribution of Na/K pumps in the frog lens epithelium. *The Journal of membrane biology*. 2000;178(2):89-101.
343. Cheng X, Mwaura BW, Chang Stauffer SR, Bezanilla M. A Fully Functional ROP Fluorescent Fusion Protein Reveals Roles for This GTPase in Subcellular and Tissue-Level Patterning. *Plant Cell*. 2020;32(11):3436-51.
344. Moss J, Williams A. Opening the floodgates to the brain. *Science*. 2020;367(6483):1195.
345. Ray L, Iliff JJ, Heys JJ. Analysis of convective and diffusive transport in the brain interstitium. *Fluids and Barriers of the CNS*. 2019;16(1):6.
346. Iliff JJ, Wang M, Liao Y, Plogg BA, Peng W, Gundersen GA, et al. A paravascular pathway facilitates CSF flow through the brain parenchyma and the clearance of interstitial solutes, including amyloid  $\beta$ . *Science translational medicine*. 2012;4(147):147ra11-ra11.
347. Witthoft A, Filosa JA, Karniadakis GE. Potassium buffering in the neurovascular unit: models and sensitivity analysis. *Biophysical journal*. 2013;105(9):2046-54.
348. Killer H, Jaggi G, Flammer J, Miller NR, Huber A, Mironov A. Cerebrospinal fluid dynamics between the intracranial and the subarachnoid space of the optic nerve. Is it always bidirectional? *Brain*. 2007;130(2):514-20.
349. Kofuji P, Newman EA. Potassium homeostasis in glia. *Encyclopedia of Neuroscience: Elsevier Ltd*; 2009. p. 867-72.
350. Rouach N, Duc KD, Sibille J, Holcman D. Dynamics of ion fluxes between neurons, astrocytes and the extracellular space during neurotransmission. *Opera Medica et Physiologica*. 2018;4(1).
351. Murakami S, Kurachi Y. Mechanisms of astrocytic K<sup>+</sup> clearance and swelling under high extracellular K<sup>+</sup> concentrations. *The Journal of Physiological Sciences*. 2016;66(2):127-42.
352. Lewcock JW, Schlepckow K, Di Paolo G, Tahirovic S, Monroe KM, Haass C. Emerging Microglia Biology Defines Novel Therapeutic Approaches for Alzheimer's Disease. *Neuron*. 2020.
353. Pajevic S. Exploring the Dynamics of Brain Extracellular Space. *Biophys J*. 2019;117(10):1781-2.

354. O'Connell R, Mori Y. Effects of Glia in a Triphasic Continuum Model of Cortical Spreading Depression. *Bull Math Biol.* 2016;78(10):1943-67.
355. Filippidis AS, Zarogiannis SG, Ioannou M, Gourgoulianis K, Molyvdas P-A, Hatzoglou C. Permeability of the arachnoid and pia mater. The role of ion channels in the leptomeningeal physiology. *Child's Nervous System.* 2012;28(4):533-40.
356. Qiao Q, Zhang W, Vencent C, Ren Q. Electric Stimulation of Optic Nerve Fiber: A Simulation Study. *Advances in Cognitive Neurodynamics ICCN 2007: Springer; 2008.* p. 609-15.
357. Wei Y, Ullah G, Schiff SJ. Unification of Neuronal Spikes, Seizures, and Spreading Depression. *The Journal of Neuroscience.* 2014;34(35):11733-43.
358. Ullah G, Wei Y, Dahlem MA, Wechselberger M, Schiff SJ. The Role of Cell Volume in the Dynamics of Seizure, Spreading Depression, and Anoxic Depolarization. *PLoS computational biology.* 2015;11(8):e1004414.
359. Ayata C, Lauritzen M. Spreading Depression, Spreading Depolarizations, and the Cerebral Vasculature2015 2015-07-01 00:00:00. 953-93 p.
360. Nye BL, Thadani VM. Migraine and Epilepsy: Review of the Literature. *Headache: The Journal of Head and Face Pain.* 2015;55(3):359-80.
361. Somjen G. Ions in the Brain: Normal Function, Seizures, and Stroke. New York: Oxford; 2004. 504 p.
362. Miura RM, Huang H, Wylie JJ. Cortical spreading depression: An enigma. *The European Physical Journal.* 2007;147:287-302.
363. Filippidis A, Zarogiannis S, Ioannou M, Gourgoulianis K, Molyvdas PA, Hatzoglou C. Transmembrane resistance and histology of isolated sheep leptomeninges. *Neurol Res.* 2010;32.
364. Verikouki CH, Hatzoglou CH, Gourgoulianis KI, Molyvdas PA, Kallitsaris A, Messinis IE. Rapid effect of progesterone on transepithelial resistance of human fetal membranes: evidence for non-genomic action. *Clin Exp Pharmacol Physiol.* 2008;35.
365. Adams EA, Choi HM, Cheung CY, Brace RA. Comparison of amniotic and intramembranous unidirectional permeabilities in late-gestation sheep. *Am J Obstet Gynecol.* 2005;193.
366. Vogiatzidis K, Hatzoglou C, Zarogiannis S, Matafia G, Gourgoulianis K, Molyvdas PA. mu-opioid influence on transmesothelial resistance of isolated sheep pleura and parietal pericardium. *Eur J Pharmacol.* 2006;530.
367. Zarogiannis S, Vogiatzidis K, Hatzoglou C, Liakopoulos V, Potamianos S, Eleftheriadis T, et al. mu-opioid stimulation of isolated parietal sheep peritoneum decreases peritoneal permeability in vitro. *Adv Perit Dial.* 2007;23.
368. Zarogiannis S, Liakopoulos V, Hatzoglou C, Kourti P, Vogiatzidis K, Potamianos S, et al. Effect of sodium-potassium pump inhibition by ouabain on the permeability of isolated visceral sheep peritoneum. *Adv Perit Dial.* 2007;23.
369. Zarogiannis S, Kourti P, Hatzoglou C, Liakopoulos V, Poultsidi A, Gourgoulianis K, et al. Influence of the sodium transport inhibition by amiloride on the transmesothelial resistance of isolated visceral sheep peritoneum. *Adv Perit Dial.* 2005;21.
370. Stefanidis I, Liakopoulos V, Kourti P, Zarogiannis S, Poultsidi A, Mertems PR, et al. Amiloride-sensitive sodium channels on the parietal human peritoneum: evidence by Ussing-type chamber experiments. *ASAIO J.* 2007;53.
371. Simon M. Peritoneal mesothelium in vitro: an electrophysiologic study. *Perit Dial Int.* 1996;16.
372. Li FK, To CH, Leung JK, Chan TM, Lai KN. Electrophysiology and glucose transport of human peritoneal mesothelial cells: implications for peritoneal dialysis. *Perit Dial Int.* 2001;21.
373. Zarogiannis S, Deligiorgi T, Stefanidis I, Liakopoulos V, Gourgoulianis K, Molyvdas PA, et al. Dexamethasone decreases the transmesothelial electrical resistance of the parietal and visceral pleura. *J Physiol Sci.* 2009;59.
374. Zarogiannis S, Hatzoglou C, Stefanidis I, Liakopoulos V, Gourgoulianis K, Molyvdas PA. Adrenergic influence on the permeability of sheep diaphragmatic parietal pleura. *Respiration.* 2007;74.
375. Zarogiannis S, Hatzoglou C, Stefanidis I, Ioannou M, Paraskeva E, Gourgoulianis K, et al. Comparison of the electrophysiological properties of the sheep isolated costal and diaphragmatic parietal pleura. *Clin Exp Pharmacol Physiol.* 2007;34.
376. Sarkos S, Hatzoglou C, Dahabre J, Gourgoulianis KI, Molyvdas PA. Effect of amiloride in human and sheep parietal pleura. *Respir Physiol Neurobiol.* 2002;132.
377. Payne DK, Kinasewitz GT, Gonzalez E. Comparative permeability of canine visceral and parietal pleura. *J Appl Physiol.* 1988;65.
378. Hatzoglou CH, Gourgoulianis KI, Molyvdas PA. Effects of SNP, ouabain, and amiloride on electrical potential profile of isolated sheep pleura. *J Appl Physiol.* 2001;90.

379. Weller RO. Microscopic morphology and histology of the human meninges. *Morphol.* 2005;89.
380. McLone DG. The subarachnoid space: a review. *Childs Brain.* 1980;6.
381. Jayatilaka AD. An electron microscopic study of sheep arachnoid granulations. *J Anat.* 1965;99.
382. Barshes N, Demopoulos A, Engelhard HH. Anatomy and physiology of the leptomeninges and CSF space. *Cancer Treat Res.* 2005;125.
383. Alcolado R, Weller RO, Parrish EP, Garrod D. The cranial arachnoid and pia mater in man: anatomical and ultrastructural observations. *Neuropathol Appl Neurobiol.* 1988;14.
384. Redzic ZB, Segal MB. The structure of the choroid plexus and the physiology of the choroid plexus epithelium. *Adv Drug Deliv Rev.* 2004;56.
385. Johanson CE, Duncan JA, Klinge PM, Brinker T, Stopa EG, Silverberg GD. Multiplicity of cerebrospinal fluid functions: new challenges in health and disease. *Cereb Fluid Res.* 2008;5.
386. Amin MS, Reza E, Wang H, Leenen FH. Sodium transport in the choroid plexus and salt-sensitive hypertension. *Hypertension.* 2009;54.
387. Pollay M, Hisey B, Reynolds E, Tomkins P, Stevens FA, Smith R. Choroid plexus Na<sup>+</sup>/K<sup>+</sup>-activated adenosine triphosphatase and cerebrospinal fluid formation. *Neurosurgery.* 1985;17.
388. Filippidis AS, Kalani MY, ReKate HL. Hydrocephalus and aquaporins: lessons learned from the bench. *Child Nerv Syst ChNS Off J Int Soc Pediatr Neurosurg.* 2011;27.
389. Damkier HH, Brown PD, Praetorius J. Epithelial pathways in choroid plexus electrolyte transport. *Physiol Bethesda.* 2010;25.
390. Segal MB. Extracellular and cerebrospinal fluids. *J Inherit Metab Dis.* 1993;16.
391. Segal MB, Pollay M. The secretion of cerebrospinal fluid. *Exp Eye Res.* 1977;25.
392. McComb J. Recent research into the nature of cerebrospinal fluid formation and absorption. *J Neurosurg.* 1983;59.
393. Cserr HF. Physiology of the choroid plexus. *Physiological reviews.* 1971;51.
394. Mori Y. A multidomain model for ionic electrodiffusion and osmosis with an application to cortical spreading depression. *Physica D: Nonlinear Phenomena.* (0).
395. Mori Y, Jerome JW, Peskin CS. A three-dimensional model of cellular electrical activity. *Bulletin of the Institute of Mathematics, Academia Sinica (New Series).* 2007;2(2):367-90.
396. Mori Y, Fishman GI, Peskin CS. Ephaptic conduction in a cardiac strand model with 3D electrodiffusion. *Proceedings of the National Academy of Sciences of the United States of America.* 2008;105(17):6463-8.
397. Mori Y, Liu C, Eisenberg RS. A multidomain model for electrodiffusion and water flow. *Poster 511. Biophysical Journal.* 2010;98:96a.
398. Mori Y, Liu C, Eisenberg RS. A model of electrodiffusion and osmotic water flow and its energetic structure. *Physica D: Nonlinear Phenomena.* 2011;240(22):1835-52.
399. Mori Y. Mathematical properties of pump-leak models of cell volume control and electrolyte balance. *J Math Biol.* 2012;65(5):875-918.
400. Mori Y. A multidomain model for ionic electrodiffusion and osmosis with an application to cortical spreading depression. *Physica D: Nonlinear Phenomena.* 2015;308:94-108.
401. Stinchcombe AR, Mori Y, Peskin CS. Well-Posed Treatment of Space-Charge Layers in the Electroneutral Limit of Electrodiffusion. *Communications on Pure and Applied Mathematics.* 2015;n/a-n/a.
402. Wei N, Mori Y, Tolkacheva EG. The dual effect of ephaptic coupling on cardiac conduction with heterogeneous expression of connexin 43. *Journal of theoretical biology.* 2016;397:103-14.
403. Mori Y, Young Y-N. Electrohydrodynamics of Leaky Dielectrics as the Weak Electrolyte Limit of an Electrodiffusion Model. *arXiv preprint arXiv:170902321.* 2017.
404. Tuttle A, Riera Diaz J, Mori Y. A computational study on the role of glutamate and NMDA receptors on cortical spreading depression using a multidomain electrodiffusion model. *PLoS computational biology.* 2019;15(12):e1007455.
405. Ellingsrud AJ, Solbrå A, Einevoll GT, Halnes G, Rognes ME. Finite element simulation of ionic electrodiffusion in cellular geometries. *Frontiers in Neuroinformatics.* 2020;14:11.
406. Genocchi B, Cunha A, Jain S, Hyttinen J, Lenk K, Ellingsrud AJ, editors. Parametric exploration of cellular swelling in a computational model of cortical spreading depression. *2020 42nd Annual International Conference of the IEEE Engineering in Medicine & Biology Society (EMBC); 2020: IEEE.*
407. Daversin-Catty C, Richardson CN, Ellingsrud AJ, Rognes ME. Abstractions and automated algorithms for mixed domain finite element methods. *ACM Transactions on Mathematical Software (TOMS).* 2021;47(4):1-36.
408. Ellingsrud AJ, Boullé N, Farrell PE, Rognes ME. Accurate numerical simulation of electrodiffusion and water movement in brain tissue. *Mathematical Medicine and Biology: A Journal of the IMA.* 2021;38(4):516-51.

409. Ellingsrud AJ, Daversin-Catty C, Rognes ME. A cell-based model for ionic electrodiffusion in excitable tissue. *Modeling Excitable Tissue*: Springer, Cham; 2021. p. 14-27.
410. Ellingsrud AJ, Enger R, Dukefoss DB, Halnes G, Pettersen KH, Rognes ME. Validating a computational framework for ionic electrodiffusion with cortical spreading depression as a case study. *bioRxiv*. 2021.
411. Sacco R, Airoidi P, Mauri AG, Jerome j. Three-dimensional simulation of biological ion channels under mechanical, thermal, and fluid forces-available on arXiv at <http://arxiv.org/pdf/150907301pdf>.
412. Zhu Y, Xu S, Eisenberg RS, Huang H. A tridomain model for potassium clearance in optic nerve of *Necturus*. *Biophysical journal*. 2021;120(15):3008-27.
413. Hill A. Solute-solvent coupling in epithelia: a critical examination of the standing-gradient osmotic flow theory. *Proceedings of the Royal Society of London Series B Biological Sciences*. 1975;190(1098):99-114.
414. Diamond JM, Bossert WH. Standing-gradient osmotic flow: A mechanism for coupling of water and solute transport in epithelia. *The Journal of general physiology*. 1967;50(8):2061-83.
415. Curran PF, Macintosh JR. A model system for biological water transport. *Nature*. 1962;193:347-8.