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Less Technical Summary

Electricity is central to most of our lives. The main uses of electricity are in the transmission of power and information in wires that form circuits. This paper shows how to analyze current flow in circuits.

Most scientists think of electricity as a force produced by charges. They think of current flow as movement of those charges, much like movement of water, because that is what they were taught in school. But the bits (and bytes) of information in our computers last only 10^{-9} seconds. The current that forms those bits and bytes of information is not like the current of water. The current that forms those bits of information is a fluid that can flow from satellites in orbit through the vacuum of space to our phones and WiFi. Charges do not exist and do not flow in a vacuum. Water does not flow in a vacuum. Total current flows as a fluid through a vacuum.

If we actually try to keep track of charges as they flow, and try to compute their interactions, we have a problem. The number of charges in a circuit is much too large to compute, something like the number of stars in galaxies. The idea of total current resolves this problem, as Maxwell did in 1865: see footnotes on p.2 & 4 below. He included the displacement current $\varepsilon_0\ \partial\mathbf{E}/\partial t$ in his definition of total current, our eq.(4). Here, $\mathbf{E}(x,y,z;t)$ describes electric forces at any location and time t, and ε_0 is the electrical constant that never varies anywhere and anytime, as far as we can tell. Maxwell included the displacement current to explain how electricity and magnetism combine to produce a total current that flows as a perfectly incompressible fluid as it propagates light between sun and earth.

Scientists rarely use Maxwell's idea of total current. They describe current flow by just the flow of charges. They do not include displacement current. The mathematics they use to describe current flow is Kirchhoff's law for the movement of charge. Textbooks have lost sight of displacement current in their treatment of Kirchhoff's law, since roughly 1905. Displacement current must be included, however, according to mathematics. Without displacement current, Kirchhoff's law is not compatible with the Maxwell equations that describe electricity, magnetism, and light. Without displacement current, Kirchhoff's laws are an approximation that makes little sense for the 10^{-9} second signals travelling to, or used by our computers, although the approximation works fine for water in pipes. Without displacement current, it is difficult to understand how charges accumulate as they create the electric fields of Ohm's law.

Here, I show how easily displacement current can be included without approximation, if we follow Maxwell's practice and use the total current in Kirchoff's law. Fortunately, little change is needed in two main applications of electricity. They already include displacement current, without saying so. Classical mathematics uses a single dielectric constant that includes displacement current. Classical circuit theory uses stray capacitances that include displacement current. Thinking of total current has little effect on the important classical work in these fields because it already includes displacement current, albeit in disguise.

But the atomic scale is different. Scientists tend to think of atomic motions in a mechanical way as independent uncorrelated thermal noise of colliding hard spheres. But atoms are charged and so the Maxwell equations enforce strong correlations of charge movement that are not like mechanical forces. In fact, the Maxwell equations enforce the perfect conservation of total current that I call the Kirchhoff law of fields, our eq. (5). Conservation implies perfect correlation when total current is confined to circuits. Correlation is defined precisely by the coherence function of stochastic signal theory, our eq. (8). Atoms and charges do not move independently in circuits.

Kirchhoff correlations of atoms reduce the complexity of analysis tremendously. Instead of dealing with astronomical numbers of charges, we deal with the number of currents. Kirchhoff correlations provide a coarse graining that makes analysis easy, using widely available computer programs, when total current is confined to circuits. Kirchhoff coarse graining is unusual, however, because it is true on all the time scales of the Maxwell equations themselves. It reduces the number of variables and interactions involved enormously, but it does not reduce time resolution. Other forms of coarse graining usually reduce resolution significantly.

When circuit components are all in a row—in series—total currents are the same everywhere at every time. Kirchhoff correlation produces equality of total current in series circuit. The equality of total current has striking effects in the components of our circuits and the ion channels of our nerve membranes, on all scales of time and distance. The total current is equal everywhere ***no matter what the mechanism of charge movement.*** A spatial variable is not needed to describe total current when components are in a row, because then there is no spatial dependence of total current. The consequences of this dramatic simplification are yet unknown, for theory, computation, and understanding of atomic motions.

Abstract

Electricity flows in circuits that bring us power and information. The current flow in circuits is defined by the Maxwell equations that are as exact and universal as any in science. The Maxwell-Ampere law defines the source of the magnetic field as a current. In a vacuum, like that between stars, there are no charges to carry that current. In a vacuum, the source of the magnetic field is the displacement current, $\varepsilon_0 \, \partial\mathbf{E}/\partial t$. Inside matter, the source of the magnetic field is the flux of charge added to the displacement current. This total current obeys a version of Kirchhoff's current law that is implied by the mathematics of the Maxwell equations, and therefore is as universal and exact as they are. Kirchhoff's laws provide a useful coarse graining of the Maxwell equations that avoids calculating the Coulombic interactions of 10^{23} charges yet provide sufficient information to design the integrated circuits of our computers. Kirchhoff's laws are exact, as well as coarse grained because they are a mathematical consequence of the Maxwell equations, without assumption or further physical content. In a series circuit, the coupling in Kirchhoff's law makes the total current exactly equal everywhere at any time. The Maxwell equations provide just the forces needed to move atomic charges so the total currents in Kirchhoff's law are equal for any mechanism of charge movement. Those movements couple processes for any physical mechanism of charge movement. In biology, Kirchhoff coupling is an important part of membrane transport and enzyme function. For example, it helps the membrane enzymes cytochrome c oxidase and ATP-synthase produce ATP, the biological store of chemical energy.

Introduction

Electricity flows in circuits. Circuits bring us power and information. Kirchhoff's law is used to analyze circuits more than anything else. Kirchhoff's current law is used to deal with the 50 or 60 Hz currents in our power systems. Kirchhoff's law is usually justified by discussions of water flow and derivations are presented as low frequency, long time approximations.

Kirchhoff's law is also used, however, at high frequencies and short times. It is used to analyze bits of information in our computers of duration $\sim 10^{-9}$ seconds [1-12] on a time scale where electric current flow is not like the flow of water. Something as important to our technology as Kirchhoff's law should be derived and justified in a more reasonable way, in my view.

A derivation of an extended Kirchhoff's law valid under a wide range of conditions is easy with a small change in the definition of current [13-16] to include the displacement current $\varepsilon_0\, \partial\mathbf{E}/\partial t$ that might be called an aethereal current for historical reasons [17].

The discussion of this 'whole current' ([18] p. 176) or total current ($\mathbf{J}_{total}$ of our eq. (4) see [19][1]) disappeared from textbooks ([20, 21] and innumerable successors) along with the aether, perhaps because the term seemed so small when telegraph signals lasted say 10^{-1}seconds. Signals today last 10^{-9} sec and so the displacement term $\varepsilon_0\ \partial\mathbf{E}/\partial t$ has a practical significance, some 10^8 times what it did when it disappeared from textbooks.

$\varepsilon_0\, \partial\mathbf{E}/\partial t$ could not disappear, however, from the Maxwell equations themselves—specifically the Maxwell Ampere law—if light is to propagate in a vacuum. The displacement current $\varepsilon_0\, \partial\mathbf{E}/\partial t$ flows everywhere, even in the vacuum between stars, everywhere in the universe—whether an aetherial [22] or vacuum universe [23]—whether or not it is used in the derivation of Kirchhoff's law in textbooks today.

Despite this history, almost everyone today thinks of current as the flow $\mathbf{J}$ of charge (with mass), just as the currents in our plumbing and in the ocean are the flow of mass. But current of this sort—of charge with mass—is not enough to explain electrodynamics, particularly in the vacuum between stars. Current flow $\mathbf{J}$ of charge (with mass) is only part of the source term for the magnetic field, the right-hand side of the Maxwell Ampere law, eq. (2). Kirchhoff's law for current flow $\mathbf{J}$ of only this type is in fact incompatible with the Maxwell Ampere law, as we show later in this paper, near eq. (6).

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[1]Vol.1, Article §328, p. 377, eq. (3) defines the 'total current' and uses it, including the displacement current, in Ohm's law, as I advocate here. Maxwell also used the phrase displacement current defined as $\varepsilon_r \varepsilon_0\ \partial\mathbf{E}/\partial t$**.**

Theory and Methods

Electrodynamics is universal. Electrodynamics successfully describes current in matter. It also describes light in the vacuum of space. Light cannot propagate between stars without a magnetic field and an electric field. Charges do not exist in a vacuum and so cannot flow in space between stars.

The magnetic field in a vacuum is created by the right-hand side of the Maxwell-Ampere law. In a vacuum, the entire right-hand side of the Maxwell-Ampere law is the displacement term $\varepsilon_0\,\partial\mathbf{E}/\partial t$. The displacement current is the total current that flows in a vacuum to create electromagnetic radiation, including light.

$$\frac{1}{\mu_0}\,\mathbf{curl\ B} = \mathbf{J} + \varepsilon_0 \partial\mathbf{E}/\partial t; \qquad \mathbf{J} \text{ is zero in a vacuum} \tag{1}$$

$\mathbf{J}$ is the flux of charge with mass, however brief, small, or transient. $\mathbf{J}$ includes the polarization charge of dielectrics and charge movements driven by diffusion, convection and other forces [24], including chemical reactions [25, 26].

Ampere's law (as generalized in the Maxwell equations) implies a natural definition of total current $\mathbf{J}_{total}$ as the source of $\mathbf{curl\ B}$.

$$\frac{1}{\mu_0}\,\mathbf{curl\ B} = \mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t\,; \qquad \mathbf{J} \text{ is zero in a vacuum} \tag{2}$$

$\mathbf{J}_{total}$ is the source of $\mathbf{B}$ because $\mathbf{div\ B} = \mathbf{0}$. $\mathbf{J}_{total}$ is a "perfect[ly] incompressible fluid" in Kirchhoff's law eq. (5), as was well understood long ago [27], p. 107.

Total current flows in matter and vacuum. It contains polarization currents of ideal dielectrics and also the universal vacuum displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$. In a vacuum, $\mathbf{J}_{total}$ is the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$.

In an ideal dielectric $\mathbf{J}_{total}$ contains an additional component included in the dielectric current $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ where ε_r is the dielectric constant (more formally the relative permittivity). ε_r is a single positive real number in an ideal dielectric. $\varepsilon_r \geq 1$: the dielectric constant is never less than one. ε_r contains the displacement current as an independent component with the same value in matter and in vacuum because the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is a property of space—not matter—and is the same everywhere.

Physics textbooks (e.g., [18, 28]) show how the displacement current arises from special relativity as a property of space. As Einstein pointed out (in my paraphrase), electrodynamics and relativity are nearly the same thing. In quotation: "The special theory of relativity ... was simply a systematic development of the electrodynamics of Clerk Maxwell and Lorentz" (p.57 of [29]), to which I might add the name Poincare [17]. The displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is the same in matter and in a vacuum because it is a property of space and time and not of matter itself, in special relativity.

Charge has a special place in the universe according to relativity [18, 28, 29]. Charge on a particle (in coulombs, like the charge on an electron) is 'Lorentz invariant'. Unlike distance, time, and (relativistic) mass (that defines momentum, for example), the charge on a particle does not vary even when the particle moves close to the speed of light.

The Maxwell Ampere law implies the conservation of total current, as we shall now show using the mathematical definition of conservation. Note that total current $\mathbf{J}_{total}$ involves properties of space away from a particle as well as the location and motion of the particle. Calculations that only deal with particles—and do not deal with $\varepsilon_0\, \partial\mathbf{E}/\partial t$ away from particles—deal with only part of the total current.

Conservation laws are defined by the divergence operator of vector analysis. The mathematical way of saying ***SOMETHING*** is conserved is

$$\mathbf{div}\ \ (\textit{SOMETHING}) = 0$$

This mathematical statement of conservation was popularized by a chemist (J.W. Gibbs [30]) for a reason. It implies specific differential equations and boundary conditions, so the statement is beyond words. It is precise, computable, and can be checked by experiment.

Vector analysis deals with fields with divergence. It deals with fields with curl, as well. The divergence of the curl of anything is zero,

$$\mathbf{div\ curl}\ (\textit{ANYTHING}) = 0 \tag{3}$$

This identity is not fancy mathematics. It is easy to derive as shown in any textbook that includes vector calculus. The identity is taught in the first weeks of graduate physics, and in many undergraduate courses as well. A convincing physical explanation of eq. (3) is not known to me.

We then can derive Kirchhoff's law for fields using the Maxwell-Ampere law eq.(2) and the definition of $\mathbf{J}_{total}$. This definition (and this name) was used by Maxwell in his treatment of conduction in imperfect dielectrics[2].[19]

$$\mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial t \tag{4}$$

Kirchhoff's law for fields shows that $\mathbf{J}_{total}$ is a perfectly incompressible fluid as was well known long ago ([18], p. 176; [16] p. 161 eq 4.18).

$$\mathbf{div}\ \mathbf{J}_{total} = \mathbf{div\ curl\ B}\ = 0 \tag{5}$$

Eq. (5) can serve as an experimental check of the mathematical identity (3), if one does not trust the mathematics. Eq. (5) then can depend on physics, not mathematics using measurements of $\mathbf{J}_{total}$ and $\mathbf{B}$.

The displacement term $\varepsilon_0\, \partial\mathbf{E}/\partial t$ in equations (4) & (5) is not found in textbook presentations of Kirchhoff's law, probably for historical reasons. Displacement current has a

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[2] Vol.1, Article §328, p. 377, eq. (3) defines the 'total current' and uses it, including the displacement current, in Ohm's law, as I advocate here. Maxwell also used the phrase displacement current defined as $\varepsilon_r\varepsilon_0\, \partial\mathbf{E}/\partial t$. Maxwell used the word 'insulator' to describe what we today might call an imperfect dielectric. Maxwell only used displacement current when dealing with imperfect insulators, as far as I can tell. He seems to not include displacement current in his treatment of 'conductors'. The word 'conductor' described what we today might call a resistive solid. The distinction between imperfect dielectrics and resistive solids has blurred as measurements have increased in resolution in the time since Maxwell.

practical significance today—when signals last 10^{-9} sec—that it did not have a century ago, when telegraph signals lasted say 10^{-1}seconds. The duration of dots and dashes were limited by manual human constraints that do not limit the duration of the bits in our computers.

Any form of Kirchhoff's law must contain a term for the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$ if the laws are to apply in a vacuum and account for the propagation of light between stars.[28] The textbook form of Kirchhoff's law that includes only $\mathbf{J}$ Is incompatible with the Maxwell equations. It is not correct to use the textbook form of Kirchhoff's law that includes only $\mathbf{J}$ on short time scales where $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is large. $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is large in the semiconductor circuits of our computers or in the charge movements of simulations of molecular dynamics.

The steady state $\partial\mathbf{E}/\partial t = 0$ might seem a useful special case. Indeed, it is, in introductory courses on electricity and circuits. the steady-state $\partial\mathbf{E}/\partial t = 0$ provides a simplified gateway to Kirchhoff's law and electric circuit theory as natural to beginners as conservation of mass of water flow in a pipe and is widely taught for that reason.

But modern applications require more than the steady state, as does mathematics itself (eq. (6) and below), as does the propagation of light in a vacuum. The charges in simulations of molecular dynamics move in 10^{-15} seconds. The bits of information in our computers are not in steady state. Beginners cannot understand the applications of currents in computer circuits with the steady state approximation they were taught. They must move on to deal with the time dependence present in modern applications.

Even the currents in our power systems involve significant time dependence. The hundreds of volts in our power system force significant displacement currents (often 100 picoamps) to flow from power system to our computer chips. These currents—that couple power circuits to signal circuits—interfere with successful function if they are not shielded by proper grounds [4, 6, 7, 9, 11, 12, 31].

Steady-states are misleading. The steady state obscures the underlying physics by hiding the role of charge on the boundaries and in the initial condition. To be specific, the steady state $\varepsilon_0\,\partial\mathbf{E}/\partial t = 0$ leaves out charges that are the source of the electric field. It is seriously misleading for that reason and must be abandoned in all but elementary discussions.

Simplifications of this sort have been extensively studied in mathematics because—as useful as they are interesting—they are easily misused. Approximations that leave out derivatives are singular perturbations [32] because they leave out boundary conditions as well as the derivative terms. They cannot serve as uniform approximations because they do not describe behavior correctly soon after a system starts, or close to boundaries.

Singular approximations leave out the sources for the physics being studied when sources appear in boundary conditions. The charge that is a source of the electric field is not visible in steady state analysis, because the steady state analysis does not deal with initial conditions, and often boundary conditions, that contain charge. These issues are explored in the context of Kirchhoff's law in detailed worked examples in [14]. We return to this issue later, when we show that the continuity equation (9) is not a practical replacement for Kirchhoff's field equation: numerical issues prevents its easy use in systems where atomic detail is needed. The continuity equation is not incorrect. It is just inadequate to deal with the main uses of electricity which are in the circuits of our power grids and computers.

Summary. To be precise, if somewhat unkind: Kirchhoff's law for fields eq. (5) implies that the flux of charges $\mathbf{J}$ is **NOT** conserved and so is in conflict with the elementary understanding of Kirchhoff's law present in textbooks.

The flux $\mathbf{J}$ of charge is not conserved. Rather, it accumulates and changes the electric field (also see the continuity equation (9) that expands $\mathbf{E}$ in the right hand side of eq.(6)).

$$\mathbf{div}\,\mathbf{J} \;=\; -\,\varepsilon_0\,\mathbf{div}\frac{\partial \mathbf{E}}{\partial t} \tag{6}$$

The textbook form of Kirchhoff's sets the right-hand side of eq. (6) to zero and is incorrect, if the Maxwell equations are a correct description of electrodynamics.

The textbook form of Kirchhoff's law describes conservation of flux $\mathbf{J}$, not conservation of current $\mathbf{J}_{total}$.

$$\mathbf{div}\,\mathbf{J} = 0 \text{ is incorrect because it is in conflict with the Maxwell equations} \tag{7}$$

Kirchhoff's law of circuits arises from Kirchhoff's law of fields (5) if total current $\mathbf{J}_{total}$ is confined to branched networks and the components of the network are well behaved, as they are in the idealized circuits of engineering texts.

Kirchhoff's law of circuits cannot be justified in general because it depends on the particulars of each circuit or setup. It is easy to check experimentally, however, whether the Kirchhoff law of circuits is correct, i.e., an adequate approximation. The Kirchhoff circuit law is correct if (1) total currents $\mathbf{J}_{total}$ are equal in series circuits and (2) total currents that are measured (in general circuits) add up properly—i.e., are conserved.

Classical electrostatics and the classical theory of idealized circuits automatically include displacement current because both usually include a dielectric constant. Indeed, classical equations are often derived from equations for Maxwell's **D** field that include the polarization currents of dielectrics, as much as authors [33, 34] and textbooks complain about the use of the **D** field [28, 35, 36]. Classical results satisfy Kirchhoff's laws for fields eq. (5) automatically because the displacement currents that are described by the **D** field are $\varepsilon_r\varepsilon_0\ \partial\mathbf{E}/\partial t$. These displacement currents (of dielectrics, capacitors, and matter in general) correctly include the displacement current of the vacuum $\varepsilon_0\ \partial\mathbf{E}/\partial t$ as one of its components, because $\varepsilon_r \geq 1$.

Idealized circuits cannot be used for practical circuit design [4, 6, 7, 9, 11, 12, 31] until capacitors are added to describe displacement currents at more or less every node in the circuit. By convention, these are not shown in the diagrams of idealized circuits. Circuits cluttered by 'strays' would be ugly, and less useful than the idealized circuits we are used to. But the capacitors, and their displacement currents $\varepsilon_r\varepsilon_0\ \partial\mathbf{E}/\partial t$ must be included if circuits are to function as desired.

The idealized circuits need to be 'fleshed out' with what are jarringly[3] called 'stray capacitances' to include the displacement currents $\varepsilon_r\varepsilon_0\ \partial\mathbf{E}/\partial t$ with its component$\varepsilon_r\varepsilon_0\ \partial\mathbf{E}/\partial t$ as they must according to Maxwell's equations and Kirchhoff's law. Stray capacitances must be included if circuits [37] are to function in practice as they are described in theory.

Stray capacitances are needed, for sure, but they are not enough to design real circuits.

Real Circuits require more. Any description of real circuits must contain additional information beyond the stray capacitances and their displacement currents.[7] Kirchhoff's laws are necessary for the design of real circuits but they are not sufficient to describe or construct real components or circuits that actually work.[15]

Series Circuits are a simple example. Kirchhoff's laws apply in a particularly simple way to series circuits in which the same total current flows through all elements. Series circuits play a surprisingly large role in the systems that deliver electric power. They also describe components of integrated circuits and ion channels, even some ion transporters, in biological membranes.

The total currents $\mathbf{J}_{total}$ in a series circuit are equal everywhere. Spatial dependence does not occur in components in series as every engineer knows. In the language of signals, Kirchhoff's current law is a perfect low pass spatial filter, in series circuits. It can even create a signal with zero spatial dependence in a noisy environment. Kirchhoff's current law can create a signal with zero spatial dependence from thermal noise found along the length of a resistor.[38] This, even though the spatial variance of flux $\mathbf{J}$ is actually infinite in the classical representations of noise as a stochastic process [39, 40] in the number density of charges and $\mathbf{J}$.

Kirchhoff's law acts as a perfect low pass spatial filter for the electric field. It filters the infinite spatial variance $\mathbf{J}$ of (models of) thermal noise into the zero variance $\mathbf{J}_{total}$ measured in electrical measurements in space, i.e., in measurements of the electric field. The spatial variance in the field $\mathbf{J}_{total}$ is constant, namely zero[4], however silly it is to assume a constant electric field.[51]

Kirchhoff's laws show that total currents are equal in a series circuit for any mechanism of flow of charge. The law is silent about the properties or mechanism of $\mathbf{J}$. This fact is seen most vividly if one considers a resistor of salt water (in which current is carried by NaCl in water), in series with a metal film resistor (in which current is carried by electrons), and in series with a semiconductor resistor in which current is carried by the quasi particles holes and electrons, not to be confused with the electrons in the wires that connect the components. In those systems, electron current depends on salt concentration because Kirchhoff's law couples the flow of current in one component to that in another, even though the mechanisms of charge movement are quite different in the components. Extensive physical discussion is found near Fig.2 in [41].

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[3] These capacitances cannot stray far. They always include a universal component that is a property of space—not matter—and so cannot wander away.

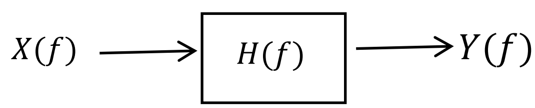
[4] Noise in real circuits of course almost always has passed through some sort of low pass filter to make the variance finite, however large. Circuits with infinite noise are likely to be only occasionally useful, and then as noise generators.

‘Everything’ interacts with everything else. ‘Everything’ is coupled to everything else because of Kirchhoff’s law.

Kirchhoff’s law shows that the current in all the resistors in this series system of resistors depends on the concentration of NaCl, even though NaCl does not exist in the semiconductor, metal film, or in the wires for that matter. Nothing is said (in Kirchhoff’s laws) about the mechanism of the flow of charge or the relative size of the flux of charge and the displacement current. The total currents are equal, but the components of the total current may not be equal at all. Both components may vary with location, in the quite dramatic way seen in a significant biological molecule, the voltage sensor [42, 43] of the nerve membrane [44]. The displacement current varies with location. The flux of charges varies with location. When summed from its components, the total current is found not to vary with location—Fig. 4 of [44]—despite the dramatic variation of its components.

Kirchhoff’s law makes a local theory of current flow impossible. The current anywhere depends on the current everywhere.

Stochastic signal theory shows this dependence in its precisely eloquent way. The correlation of currents is unity [45]:



The coherence function describing elements of an ideal circuit is unity.

$$C_{xy}(f) = \frac{|H(f)G_{xx}|^2}{G_{xx}(f)|H(f)|^2 G_{xx}} = 1 \qquad when \;\; H(f) = \frac{Y(f)}{X(f)} \tag{8}$$

The physical reason for this perfect correlation is revealing. Correlation is perfect because ***the Maxwell equations provide exactly the forces needed to move charges so that Kirchhoff’s law is precise and universal***, as are the Maxwell equations themselves, which are valid on all scales including those of atomic motion and time.

These correlations are so well known to engineers that it seems unnecessary to discuss the underlying physics, even in textbooks. Every engineer knows that currents are equal in a series circuit. But what seems obvious to engineers is less obvious to chemists, biochemists and biophysicists. They focus on the atomic details ([46], popularized by [47]) that control their systems. They rarely think of the electric fields that correlate atomic motions, and almost never discuss the role of Kirchhoff’s field equations in those motions. Electrodynamic sources of correlation are almost never discussed in the literature of chemistry, biochemistry, or biology and are rarely discussed in the literature of molecular dynamics.

We are used to thinking of interactions of atoms as if they are mechanical, when in fact the dominant forces are electrical (third paragraph of [28]), summarized by the Kirchhoff form of the Maxwell equations eq. (5). The ***Kirchhoff correlation exists because of mathematics, without approximation or additional physical content.*** The atomic motions seen in simulations are those needed to conserve total current according to equation (5).

The forces on atoms provided by the electric and magnetic fields of the Maxwell equations automatically provide the atomic motions seen in molecular dynamics that are needed to conserve total current in Kirchhoff's laws. To repeat, this is a consequence of mathematics, not physics. The conservation of total current is an unavoidable mathematical consequence of the Maxwell Ampere equation as shown by the derivation of eq. (5).

Coupling implied by Kirchhoff's law occurs in biological systems and electrical technology.

Coupling occurs in the biological systems where atoms control macroscopic function. It is important for physical scientists to be reminded that biological function is controlled by a handful of atoms and that fact is exploited in hundreds or thousands of laboratories every day, using site directed mutagenesis (for example) to design drugs, molecules, and binding sites on proteins.

Moving from biology to engineering, we see that Kirchhoff's law provides a productive way to understand circuits in general. Thinking of charges and their interactions in circuits is not productive and is not done in circuit design, probably for that reason. The number of pairwise interactions of charges is something like 10^{23} factorial, a number incomprehensibly too large to deal with, let alone compute.

Coarse Graining. The usual way to deal with overwhelmingly large calculations, arising from too high resolution, is to find a way to reduce that resolution by averaging or coarse graining the calculation.

Coarse graining is of course possible for calculations using Coulomb's law, but those procedures must produce results compatible with the Maxwell equations or they are not useful. The enormous strength of the electric field requires that coarse graining be accurate if it is to be useful (third paragraph of [28]; Appendix of [41]). Indeed, Kirchhoff's law of ***total*** current is a form of coarse graining, that unlike most forms of coarse graining, is exact, not an approximation. Because it is exact, Kirchhoff's law shows how simple analysis is adequate—indeed, exact—in a channel so narrow that ions cannot move past each other.[38] Here the coarse grained (total) current is independent of location whereas the knock on, knock off behavior of individual ions is too bewilderingly complex to be described in a single way or by an agreed upon mechanism [48-50].

In physics, Kirchhoff's law implies that the movement of charges in Brownian motion are correlated as we have discussed, eq. (8). The usual Langevin description—used nearly everywhere since it was introduced by Sutherland [51, 52], Einstein [51], and Langevin—is inadequate because it does not compute electrical forces from the charges [53, 54] of the Brownian particles [55] and so usually does not include fluctuations of electrical forces, with the notable exception of [56]. Electrical correlations have always been included in work in computational electronics [25-30] and these automatically satisfy Kirchhoff's current law.

The irony is that the Brown of Brownian motion originally studied 'colloidal' particles [57] that have long been thought to be highly charged. And Einstein studied sucrose[5], p. 55 of [58].

[5] Sucrose is now known to be a polar highly charged molecule—even though it has zero net charge—with a dielectric constant ~60 measured in ref. [60], nearly that of water, presumably arising from the asymmetrical local charge distribution of (many of) its highly charged hydroxyl **OH** groups.

Einstein said 'sugar'—at least in translation—but almost certainly meant sucrose, perhaps because he thought it was uncharged, without net charge, and nonpolar, without significant local charge densities. Einstein was certainly aware that moving charges produce moving fields [29, 59], and so fluctuating densities of charges, that are the essential feature of thermal and Brownian motion, must produce fluctuating electrical forces. The neglect of electrical fluctuations in Einstein's paper [51] must be understood in that light, both logically and psychologically (in my opinion). The neglect of fluctuations in electric fields in the later literature of Brownian motion of matter is harder to understand, or justify logically, including a few of my own papers.

In noise analysis, Kirchhoff's law implies that the total current in a series system is uniform in space. The spatial independence of noise in a series system implies that a spatial variable is not needed to describe the total current. The simplification seems profound because it reduces partial differential equations to ordinary differential equations, reducing the need to describe the details of knock on and knock off mechanisms in many cases [61-64] but this special property resulting from Kirchhoff's law has not been exploited in mathematical or numerical analysis to the best of my knowledge, with the exception of [65].

Of course, some properties of such systems depend on the movements and flux of charges as separate chemical species, beyond their electrical current. These movements are much harder to understand than the total current because they do not follow a conservation law without accumulation. Fortunately, most properties of the circuits of our technology depend only on total current. Circuits can be designed by Kirchhoff's circuit law for total current without considering charges at all.[1-12] In biology, circuit representation of signaling in the nervous system, skeletal and cardiac muscle work quite well [66] without separate consideration of fluxes.

In chemistry, Kirchhoff's laws imply that total current is conserved as electrons change orbitals while changing covalent bonds, even as they emit light because Kirchhoff's law is true on the time scale of the Maxwell equations involved in quantum processes.[67-71] Kirchhoff's law imply that reactants and products are coupled by the flow of total current, as well as conservation of chemical species. In the chemical networks of biochemistry, Kirchhoff's circuit law is an important constraint not always followed in classical kinetic analysis.[72] It is distinct from the conservation of chemical species, or the conservation of charge because it includes displacement current. Analysis confined to the steady state is misleading as previously discussed, near eq.(6).

<u>Kirchhoff coupling in channels and transporters</u>. In biology, Kirchhoff's law provides the coupling that creates the propagating nerve signal from the currents through the conductances of ion channels in membranes. Propagation of the action potential signal in nerve—and muscle and the heart—is electrical, as the 23 year old Alan Hodgkin showed in 1937 [73, 74]. Chemical coupling is not involved despite the views of the leader of English biophysics at the time, Nobel Laureate A.V. Hill [75]. Hodgkin and Huxley computed the propagating action potential from the properties of the conductances measured independently in different experiments in voltage clamp conditions [66, 76] in which control amplifiers prevent the formation of a signal. The fluxes accompanying the nerve signal were much harder to measure, particularly on the relevant time scale in axons (without inactivation, fast and slow) during voltage clamp experiments. [77-79]

In biology, Kirchhoff's law helps couple ions in transporters. Kirchhoff's law guarantees that the sum of membrane currents is zero [80] in a closed system, like a mitochondrion or vesicle or spherical (or short) cell. A plot of the flux of one species of ion *vs.* another will be a straight line in that case.

The coupling of ions in transporters occurs because of the Maxwell equations and the equivalent Kirchhoff's law for total current. The coupling need not depend on chemical interactions, any more than the coupling of sodium and potassium conductances in a nerve fiber depends on chemical interactions.

(It is an interesting historical fact that Hodgkin exploited Kirchhoff coupling in his first experiments using the voltage clamp (see Fig. 10 of [81] as he explained to me decades later) but he did not consider the coupling implied by Kirchhoff's law when he initiated membrane studies of active transport [82-86]. In my opinion, Hodgkin only considered Kirchhoff coupling in the propagating action potential because the cable equation automatically produced such coupling, as it did in the trans-Atlantic cable designed by Kelvin [87, 88]. All in Cambridge UK [89] in the 1930's were proud of Kelvin's leading the world in the 1850's in this revolutionary technology. The idea that electrodynamics was important within proteins was not known to Hodgkin, until Warshel introduced the idea, decades later [90, 91] and I brought Warshel's contribution [91] to Hodgkin's attention, in the context of the PNP treatment of open ionic channels [53, 92, 93].)

In biology, Kirchhoff's law helps generate ATP, the universal currency of chemical energy. Kirchhoff's law helps couple the flow of electrons, 'protons' (as workers call positively charged forms of water) and substrates within cytochrome c oxidase [94]. Kirchhoff coupling powers ATPsynthase using current flow from the oxidase, no matter what charge moves to carry the current.

Kirchhoff coupling. Kirchhoff's law couples electrons, substrates and protons, in many places in the respiratory chain, and photosynthetic pathways, and in the multitude of other transporters in biological membranes. It is universal in physical systems, so it is universal in biology and biophysics, as well as electronics.

Kirchhoff's law guarantees that flux measurements of such transporters will depend on the setup in which they are measured and the boundary conditions at the boundaries. Coupling is a central idea in the literature of transporters, oxidative phosphorylation, and photosynthesis, as a glance at the literature or textbooks shows. Coupling is defined quantitatively by the plot of the flux of one charge carrier vs another, e.g., [53, 82-86]. Coupling depends on the ratio of fluxes, and those depend on current flow in circuits defined by Kirchhoff's law. ***Kirchhoff coupling is unavoidable in transporters, as in all physical systems***, although of course there may also be chemical coupling.

In a voltage clamp bilayer setup, Kirchhoff coupling does not exist. The voltage clamp prevents it. In finite cells or organelles, like mitochondria, Kirchhoff coupling guarantees that every membrane current is coupled to every other, with the sum of membrane currents zero, as it was in the special case studied by Hodgkin (Fig. 10 of [81]). Flux ratios measured from the same transporters will differ in the two cases, the artificial voltage clamp and the natural organelle.

Discussion

Kirchhoff's law is responsible for many of the important properties of electricity. Most of the technology of electricity depends on circuits and Kirchhoff's law summarizes the physics of circuits. Kirchhoff's law is the mathematics of circuits actually used by engineers in practical designs [1-12] that deliver electrical power and information throughout our civilization.

Kirchhoff and Continuity Equations. Kirchhoff's law does not have a particularly prominent place in textbooks of electricity and magnetism despite its importance as the mathematics of the electrical circuits that dominate our technology, if not our daily lives. Kirchhoff's law has hardly any place in textbooks of electrochemistry or physical chemistry. It is natural to wonder then if there might be another way to describe circuits without using Kirchhoff's law explicitly. Do chemists (and mathematicians for that matter) avoid Kirchhoff's law because they have dealt with its role in a different way?

The continuity equation might appear to be an equivalent representation to Kirchhoff's law. The continuity equation links the buildup of charge density $\varepsilon_0\ \partial\rho/\partial t$ with flow of charges.

$$\mathbf{div\ J} = -\varepsilon_0\ \partial\rho/\partial t \tag{9}$$

Here ρ is called the density of free charge in systems with an ideal dielectric constant. The general meaning of ρ is vividly discussed on p. 500-507 of [35] with particular attention to the ambiguously defined polarization current of its dielectric component. Robinson [95] presents electrodynamics for polarization currents of more complexity (in time and frequency) than in most textbooks.

But this continuity equation is something of a mirage. Reality emerges from the mirage when one tries to use the continuity equation to design something, i.e., circuits.

Circuits contain macroscopic numbers of charge, say 10^{23} charges. Computations involving ρ made of that many charges are beyond practical description when the interactions of charges are described by Coulomb's law, as they are in simulations with atomic resolution. Such computations of circuits are nearly unspeakable, certainly impracticable, because the number of interactions far exceeds 10^{23} factorial required by Coulomb's law. Coulomb's law requires the computation of $\sim 10^{23}!$ Interactions for just two charges.

Physical scientists need to be reminded that handfuls of atoms control much of biology and chemistry and so atomic resolution computations are needed to understand that control. Modern electronics also depends on atomic details. Circuits of our computers involve structures of 3 to 5 nanometers and are more and more dependent on realistic computations at the atomic scale of say 0.1 nanometers.

Charge, Currents, and Circuits. The continuity equation shows what is an unfortunate emphasis on charges and fields—not currents and circuits— in the teaching and literature of electricity, in my view.

For historical and pedagogical reasons textbooks and teaching focus on charges and their motion. The previous discussion shows, however, that no consideration of charges can deal with circuits. Currents in circuits are constantly changing their nature, sometimes (and places) they are displacement currents $\varepsilon_0\,\partial\mathbf{E}/\partial t$; sometimes, material displacement currents $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$; sometimes ionic currents; often, electron currents.[61, 63] Charge analysis cannot deal with this. Kirchhoff's law can, if it deals with total current, as we have shown at length, following Maxwell.

Charge analysis is not wrong. It is simply inadequate. It cannot deal with the circuits that are the main use of electricity in our world.

Coarse Graining, revisited. The usual way to deal with overwhelmingly large calculations, arising from too high resolution, is to find a way to reduce that resolution by averaging or coarse graining the calculation.

Coarse graining is of course possible for calculations using the continuity equation and Coulomb's law, but those procedures must produce results compatible with the Maxwell equations or they are not useful. The enormous strength of the electric field requires that coarse graining be accurate if it is to be useful (third paragraph of [28]; Appendix of [41].

Kirchhoff's laws provide a useful coarse graining of the Maxwell equations that avoids calculating the Coulombic interactions of 10^{23} charges yet provide sufficient information to design the integrated circuits of our computers. Kirchhoff's laws are exact, as well as coarse grained because they are a mathematical consequence of the Maxwell equations, without assumption or further physical content. In a series circuit, the coupling in Kirchhoff's law makes the total current exactly equal everywhere at any time. The Maxwell equations provide just the forces needed to move atomic charges so the total currents in Kirchhoff's law are equal for any mechanism of charge movement in a series circuit. Those movements couple processes for any physical mechanism of charge movement. A striking example is the way Kirchhoff's law couples the conductance of chemically independent ion channels to make the propagating nerve signal, the action potential.

Conclusion

Kirchhoff's law for total current may be the optimal coarse graining of the electric field in circuits. Kirchhoff's law is exact as well as coarse grained, an unusual combination. Indeed, Kirchhoff's law may be so useful in designing the circuits of our computers precisely because it is exact, as well as coarse grained.

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