Electrodynamics of Circuits: Version 2

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Abstract

Circuits are found by the billions throughout our technology. Kirchhoff's laws are widely used to design circuits, but the words 'circuit' and 'Kirchhoff' are not found in the index of some widely used textbooks of electrodynamics. Electrodynamics and circuit theory could benefit if Kirchhoff's current law and circuit theory were taught as an integral part of electrodynamics. Circuits cannot be analyzed by Coulomb's or Gauss' law applied to charges. Circuits contain too many charges to allow interactions to be calculated by Coulomb's law, charge by charge. Circuits can be analyzed by the currents that flow in them, traditionally by Kirchhoff's current law applied to conduction current. Maxwell insisted that a generalization of Kirchhoff's current law is needed that includes displacement current $\varepsilon_r \varepsilon_0 \, \partial \mathbf{E}/\partial t$ as well as the conduction current. Circuit designers provide the displacement currents intuitively, using imagined circuit elements called 'parasitic capacitors'. Computer packages like LT-Spice could use Maxwell's generalization, thereby using mathematics instead of imagination to deal with displacement current. Using Maxwell's generalization would be a continuous reminder of the importance of displacement current in circuit theory and electrodynamics, uniting both disciplines. Using Maxwell's generalization would show respect for historical precedent while increasing the realism of circuit simulations, although not to perfection: details of layout and device properties would be needed as well to design computer circuits that work at computer speeds. This paper shows how circuit analysis can be taught and used as part of electrodynamics.

Electrodynamics is the science of moving electricity. Electricity moving in circuits has provided the signals and power of our technology since the invention of the telegraph. Telegraph circuits were built to 'complete the circuit' and analyzed with Kirchhoff's laws before electrodynamics was defined by the Maxwell equations.

Circuits are perhaps the most widely used application of electrodynamics because so many are found in our computers and phones. One might expect circuits and Kirchhoff's laws to occupy a prominent place in the teaching and textbooks of electrodynamics. But 'circuit' and 'Kirchhoff' do not appear in the index of the widely used text of Griffiths [1] and the otherwise inclusive treatment of Jackson [2], making it harder to teach those ideas to students of electrodynamics. This paper shows a simple way to derive Kirchoff's law and its Maxwell generalization from classical electrodynamics. This derivation can be taught and used as part of classical electrodynamics.

'Circuits' must be defined mathematically to proceed with precision. At first glance, it may seem that circuits can be defined by the movement of charges in the circuit using Coulomb's or Gauss' law. This is not true in practice.

Analysis of charge movement using Coulomb/Gauss does not provide a practical path to circuit analysis because of the large numbers of charges in even the smallest circuits. Circuits usually involve more than 10^{10} charges, often many more, as in Kelvin's original trans-Atlantic telegraph cable [3-5] that was thousands of kilometers long. Coulomb's law says charges interact in pairs even as they flow in circuits. Analyzing pairwise interactions in circuits would involve $(10^{10}!) \cdot (10^{10} - 2)!$ computations. The need for at least this many calculations prohibits circuit design based only on Coulombic interactions of charges.

Most circuit analysis avoids calculating interactions by studying the properties of currents, not charges. With this approach, circuits can be described by handfuls of currents instead of unwieldly numbers of interacting charges. Currents in circuits are described by the Maxwell Ampere law, not by the Coulomb equation or Gauss' law.

Coarse graining this way—using currents instead of charges—is exact because it uses extra physics not present in Gauss/Coulomb equations for charge interactions. The extra physics involves the magnetic **B** field and currents as described by the Maxwell-Ampere equation (1). The resulting current law (eq. (2)) is true and helpful whether the magnetic field is small or large. The current law eq. (2) is universal because it depends on the existence of magnetic and electric fields, not their size. The current law eq. (2) is universal because magnetic and electric fields are connected by the universal theory of relativity.

Electricity E and magnetism B were connected by considering the relative motion of an observer and a stream of moving charges that form an electric current because of that motion. The following sketch is meant to provide a pathway to the proper treatments in the chapters [1, 2] and discussions [6, 7] of electrodynamics texts. These expand and explain Einstein's original papers [8-10].

Consider observer #1 moving at the same speed as a stream of charges. The observer measures no current (relative to the motion of the observer) and so sees no magnetic field, only an electric field and electric force. But a resting observer #2 measures current relative to his/her motion. A magnetic field and magnetic force are observed but no electric field or electric force. What is an electric force according to observer #1 is a magnetic force according to observer #2.

Explicit detailed calculations of the example just described [1, 2] using the Lorentz transformation (of special relativity) establish the equality of forces measured by observer#1 and #2. In general, the principle of relativity requires that the forces observed are independent of the straight-line motion of the observers. Electricity **E** and magnetism **B** are intertwined as Einstein put it [9], p. 57: 'The special theory of relativity ... was simply a systematic development of the electrodynamics of Clerk Maxwell and Lorentz".

Circuits can be analyzed with the Maxwell-Ampere equation applied to a manageable number of currents.

Maxwell Ampere Equation $$\frac{1}{\mu_0}\,\mathbf{curl\ B} = \mathbf{J}_{total} = \hat{\mathbf{J}} + \varepsilon_r \varepsilon_0\, \partial \mathbf{E}/\partial t \qquad (1)$$

Here $\hat{\mathbf{J}}$ is the conduction current when the dielectric approximation is used.

Equation (1) is written in familiar form using the dielectric approximation to polarization despite its evident limitations p. 500-507 of [7] and [12], particularly when applied to liquids including the ionic solutions [13, 14] found throughout chemical and living systems.

<u>Polarization is the compressibility of material charge</u>. If more realistic descriptions of polarization are needed, the polarization current can be included in a revised definition of conduction current as shown in [15-17]. An auxiliary model is then used to describe the displacement current [13, 18, 19] produced by changes in the density of polarization. The treatment of polarization of charge density is similar to the treatment of compressibility (of mass density) in fluid mechanics and the Navier Stokes equations. Polarization describes the compressibility of the charge density of matter.

<u>Solenoidal currents</u>. The special properties of current $\mathbf{J}_{total}$ as a solenoidal field were discovered by Maxwell and exploited in his electrodynamics: the current $\mathbf{J}_{total}$ has no divergence (and, in just that sense, has no source!). Many of the special features of electrodynamics stem from this fact, particularly when the displacement current $\varepsilon_r \varepsilon_0\, \partial \mathbf{E}/\partial t$ is included, as Maxwell insisted on p. 232 of [20], discussed in more modern language in [16].

To see this, take the divergence of eq. (1) and use the fact that the divergence of the curl is always zero. The divergence is defined as the surface integral of flux across all the surface of a small volume element. 'Divergence (of a field) is zero' or 'a field is solenoidal' in a region means that the surface integral of flux surrounding that region is zero, and so there is no source of flux (for that field) in that region.

Maxwell Current Law $$\mathbf{div}\left(\hat{\mathbf{J}} + \varepsilon_r \varepsilon_0\, \partial \mathbf{E}/\partial t\right) = \mathbf{div}\ \mathbf{J}_{\boldsymbol{total}} = \mathbf{div\ curl\ B}\ = 0 \tag{2}$$

The **curl** function is defined by a circulation integral over a closed small volume. **Div curl** is defined by the flux integral of circulation. The symmetry of that integral forces **div curl** to be zero for any field smooth enough to satisfy the Maxwell equations.

<u>Total Current does not accumulate</u>. Because it is solenoidal and has no divergence, $\mathbf{J}_{total}$ cannot accumulate at all, at any time, for any interval of time, or under any circumstances that the Maxwell Ampere law is true. Unlike material flows, total current cannot accumulate at one time to be balanced by a depletion at a later time. Accumulation then depletion would require a time dependent term on the (extreme) right hand side of the Maxwell current law, eq. (2). The total current $\mathbf{J}_{total}$ cannot accumulate ever, at all, anywhere, for any period of time.

The universal properties of $\mathbf{J}_{total}$ depend on its displacement current term $\varepsilon_r \varepsilon_0\, \partial \mathbf{E}/\partial t$. Otherwise they (eq. (2)) would not be present in a vacuum. The universal properties of electricity seem to be what Maxwell had in mind when he said " … the time-variation of the electric displacement must be considered in estimating the total movement of electricity." as discussed in [16]. Quotation and supporting equations are in Vol. 2, Section 610, p. 232 of [20], Maxwell illustrates the general principle eq. (2) with examples and applications calculated in detail.

Series Circuits. If total current can be confined to an unbranched series circuit, it is equal everywhere at any time no matter what the microphysics involved. If the total current is zero in any place in a series circuit, it is zero everywhere and the circuit is said to be 'incomplete'. Current only flows in complete circuits.

Ref. [21] explains how the microphysics of some circuit components accomodates the general requirement of spatial equality in a series circuit. Ref. [23] explains some of the surprising consequences of spatial equality. Landauer [22] emphasizes the role of displacement current and criticizes hopping theories that ignore displacement current. One other consequence of spatial equality (in unbranched circuits) is the puzzling prediction that spatial thermal noise in total current is zero in series circuits or systems that are very narrow like ion channels. Hopping models of ion channels were introduced by Hodgkin and Keynes [24] who imagined that ions in a narrow channel interacted as if they were uncharged (billiard) balls. They ignored electric forces that Coulomb's law shows are very large indeed for contiguous spheres. They ignored electrodynamics and displacement current. Hopping models have been a central part of most visions of ions in channels following [25], despite their cavalier treatment of electrodynamics and Landauer's vocal objections to hopping models in general.

Circuits in general. If total current is confined to branched circuits, the classical Kirchhoff's law is the mathematical consequence of eq. (2), whether or not the dielectric approximation is made [15-17].

Kirchhoff Current Law $$\mathbf{div}\,\hat{\mathbf{J}} = \mathbf{div}\,\mathbf{curl}\,\mathbf{B} = \mathbf{0} \qquad \text{if } \hat{\mathbf{J}} \gg \varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t \tag{3}$$

or as written in circuit textbooks for the currents I_k at any node of a circuit:

$$\sum_k I_k = 0 \qquad \text{if } I_k \gg \varepsilon_r\varepsilon_0\,\partial V_k/\partial t \tag{4}$$

Complete the circuit. The current laws eq. (2)-(4) imply the "complete the circuit" principle, as we have seen. Electrical power and signals are distributed only by complete circuits. Incomplete circuits do not allow current or signals to move. This principle is central to the work of electricians to this day, as it has been to telegraphy for nearly two hundred years. Textbooks and courses that do not discuss Kirchhoff's laws or circuits are likely to omit this principle despite its importance for all electricians and a large fraction of the applications of electricity and electrodynamics.

Displacement current is not included in Kirchhoff's current laws eq. (3) & (4). Kirchhoff's laws describe conduction currents. In most modern applications, Kirchhoff's laws need to be extended to include displacement current $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ because in most signals of modern circuits $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t \geq \hat{\mathbf{J}}$. Most signals in computers have $\varepsilon_r\varepsilon_0\,\partial V_k/\partial t \gg I_k$. Displacement currents $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ are often included in circuits by introducing parasitic capacitances.

Parasitic capacitances are fictitious circuit elements used in circuit design to simulate the displacement current not included in Kirchhoff's law.

Parasitic capacitances are needed to make idealized circuits describe real circuits like those in our computers. Without parasitic capacitances, or their equivalent, the idealized circuits do not describe the real signals and circuits found in our computers.

Idealized circuits do not depend on the layout of components [11, 26-28] although real computer circuits do depend on layout. The parasitic capacitors are needed to add to idealized circuits the displacement currents $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ not written explicitly in eq. (3) & (4).

Parasitic capacitances play a large role in circuit design because "Parasitic capacitance is often the factor that ultimately limits the processor speed of a computer." [11] p. 313. Placing (and sizing) the fictitious elements is an art as much as a science. Experience and practice are needed to make circuits in actual layouts work the way idealized circuits suggest they should. The resulting considerable literature includes [16], p. 11; [27], Section 5.7; [11] p. 313; and [26, 28].

Voltages in computer circuits change in times much less than microseconds. Kirchhoff's law—without the supplemental displacement currents $\varepsilon_r \varepsilon_0 \, \partial \mathbf{E}/\partial t$ provided by parasitic capacitances—does not describe those signals. However, students are likely to be interested in signals actually found in modern technology. Students need to learn to design computer circuits.

Without parasitic capacitances, Kirchhoff's laws do not describe the temporal response of circuits at times like RC, where R characterizes the values of resistors in the circuit and C is roughly 10 pF, a decent beginning estimate of parasitic capacitance in many circuit layouts. R might be 100 kohms, implying $RC = 10^{-6}$ s. Circuits made of 100 kohm resistors will show prominent microsecond transients (in response to steps in current or voltage). The transients are absent in Kirchhoff treatments of ideal resistors and so students of electrodynamics observing such transients in real circuits will be confused, until they realize the importance of displacement currents in general, whether provided by fictitious capacitances or by Maxwell's current law.

Enclosures. Real circuits are usually enclosed in boxes of metal connected to ground. The box is a Faraday cage—an enclosure made of a conductor connected to ground potential The effects of enclosures surrounding real circuits and layouts appear mostly as 'capacitance to ground': what chemists call the self-energy, often estimated by the Born equation [29].

In experiments, a close-fitting Faraday cage dramatically lowers high frequency noise, without changing mean values.

It is possible that artificially increasing the 'capacitance to ground'/Born energy in simulations of molecular dynamics would also reduce high-frequency noise, without changing mean values. Reduction of noise could significantly increase the duration and utility of the calculations.

Maxwell's total current. The displacement current $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ of real circuits can be provided in another way without fictitious parasitic capacitances—as Maxwell advocated on p. 232 of [20]. The displacement current $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ can be included in the definition of total $\mathbf{J}_{total}$ (or true current, as Maxwell called it on that occasion) used in a current law. It is easy to implement Maxwell's approach in practical examples giving results identical to those using traditional parasitic capacitances [30, 31].

In general, total current can be used in a universal version of Kirchhof's current law, i.e., in Maxwell's current law, eq.(2). In this generalization, total current $\mathbf{J}_{total}$ of eq. (2) provides the displacement current not found in the in the classical versions of Kirchhoff's law, eq. (3) & (4). The total current $\mathbf{J}_{total}$ of eq. (2) replaces the conduction current found in the classical versions of Kirchhoff's law, eq. (3) & (4).

The $\mathbf{E}$ field is needed to compute the displacement current $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$ in Maxwell's current law eq.(2). Circuit approximations using Maxwell's total current estimate the displacement current by the one-dimensional circuit quantity $\varepsilon_r\varepsilon_0\,\partial V_k/\partial t$ where V_k is the electrical potential across component k in the circuit. More realistic analysis would use the circuit layout to compute $\mathbf{E}$ in three dimensions and then the displacement current $\varepsilon_r\varepsilon_0\,\partial V_k/\partial t$ in the circuit produced by $\varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t$. Nearby conductors and enclosures surrounding the layout would have an important role. Analysis with total current $\mathbf{J}_{total}$ will capture some—but not all—of the displacement current and nonideal behavior of real layouts [11, 26-28].

Total Current in LT-Spice. Using total current (instead of conduction current) in popular software packages like LT-Spice [32, 33] would introduce Maxwell's idea to a wide community. The importance of displacement current would then be apparent to circuit designers in general.

Descriptions of high-speed circuits in computer packages like LT-Spice ***must*** include displacement currents, one way or another if they are to realistically describe the properties of actual circuits in real layouts, as we have discussed. A mathematical treatment based on the Maxwell equations using total current seems preferable to an intuitive treatment requiring the artful choice and placement of fictitious capacitances, particularly because different investigators are likely to choose different locations and sizes of parasitic capacitances, using their different intuitions.

Using total current instead of conduction current in circuit simulations like LT-Spice would be historically appropriate and esthetically pleasing. Precise mathematics would replace fictitious capacitances that are subjective to some extent. Simulations with total current would increase realism and reduce the artful intuition needed to design computer circuits that perform as they as supposed to.

To be fully realistic, the Maxwell treatment needs to be supplemented by more detailed description of displacement currents (and other nonideal behaviors) as they are actually found in a specific circuit layout. Those details will certainly involve artful design. Nonetheless, a Maxwell analysis using total current $\mathbf{J}_{total}$ seems a good place to start, for students and simulation software, clearly better than a Kirchhoff analysis neglecting displacement currents altogether.

Conclusion. Circuit analysis using current laws is an important application of electrodynamics. The principle that current flows only in complete circuits has been an important part of electrical technology for nearly two hundred years. Both should be an integral part of electrodynamics.

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