Kirchhoff's Current Law and the Continuity Equation

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Abstract

Kirchhoff's current law says that current does not accumulate. The continuity equation says that current does accumulate when currents and voltages vary. The continuity equation is the more general result, derivable from conservation of charge or the Maxwell equations. Kirchhoff's current law is often a poor approximation when applied to the fast time varying signals of modern circuits. Engineers deal with this difficulty by supplementing circuits with parasitic (stray) capacitances that provide the displacement current ignored by Kirchhoff's law. Maxwell used a different unfamiliar definition of true (or total) current that avoids this difficulty, because it includes the displacement current. The engineering approach works well as our electronic technology demonstrates. Maxwell's approach may work better, after all he said " ... the time-variation of the electric displacement, must be considered in estimating the total movement of electricity."

Kirchhoff's current law has been used successfully to analyze signals for many years, including the very fast signals (10^{-9} s) in our computers.

The law is stated succinctly in Zangwill [1], p. 524, in general form in three dimensions

$$\mathbf{div\, j} = 0 \tag{1}$$

and in circuit form there and in most textbooks

$$\sum_k I_k = 0 \tag{2}$$

In words, current ***does not accumulate*** anywhere: eq. (1). Current does not accumulate at the nodes of a circuit: eq. (2). Current $\mathbf{j}$ or I_k here means the flux of charges with mass, e.g., the flux of electrons.

In contrast, the continuity equation of electrodynamics says that current ***does accumulate*** current accumulates as charge ρ for signals that vary with time.

$$\mathbf{div\, j} = -\frac{\partial \rho}{\partial t} \tag{3}$$

These equations (1) and (3) contradict each other, except when $\partial\rho/\partial t = 0$. Signals are in general time dependent — involving $\partial\rho/\partial t \neq 0$ — and so this contradiction undermines the use of Kirchhoff's law to analyze rapidly changing signals in circuits.

The continuity equation (3) is the more general result, a conservation law. The continuity equation is derived from conservation of charge as shown explicitly in the textbooks of Zangwill [1], p. 32, and Griffiths [2], p. 222, 356. The continuity equation (3) is also an inescapable corollary of the Maxwell equations (Appendix) and applies to quantum mechanical systems as well [3, 4].

Kirchhoff's Current Law is an Approximation. The conclusion is that Kirchhoff's current eq. (1) & (2) are not general laws. They are poor approximations when applied to rapidly changing signals in circuits.

The size of the error depends on the time dependence of signals but is unlikely to be small in circuits that operate in nanoseconds, as in our computers. The errors are immediately observable, for example, in a circuit made of one megohm resistors. Transients will be observed throughout the circuit on the (roughly) 1 microsecond time scale. Those transients are not predicted by eq. (1) or (2) when applied to a network of pure resistors, defined by Ohm's law applied to pure 1 megohm resistors. The microsecond time scale is much slower than the time scales used in the design of computer circuits. Most of those circuits are in fact designed with by eq. (2) as easily verified by reference to textbooks of circuit design.

This is a serious matter. Signals in circuits are perhaps the most important application of electrodynamics. They are certainly the most numerous. There are some seven billion computer powered mobile phones in use today, according to estimates by Google Search. Each contains billions of circuits. Hence the number of circuits analyzed with Kirchhoff's law is more than 7×10^{18}.

In my view, circuits should not be analyzed with a current law that is a poor approximation. In my view, the most numerous application of electrodynamics should use well defined approximations, avoiding arbitrary ambiguity and replacing art with science, wherever feasible.

Maxwell's redefinition of current. Maxwell himself proposed a definition of total or true current [5] that includes displacement current and so allows one to avoid these difficulties [6], implemented in [7]. Indeed, Maxwell insisted on that redefinition, using the words **" … that the time-variation of the electric displacement, must be considered in estimating the total movement of electricity."** Quotation and supporting equations are in Vol. 2, Section 610, p. 232 of his treatise [5].

The engineering approach, Engineers have dealt with this problem differently. They do not modify the definition of current, nor do they generalize Kirchhoff's current law into the Maxwell current law of the Appendix. Rather, they modify the model circuit (containing only idealize resistors and no capacitances, stray or otherwise) that is analyzed by Kirchhoff's law eq. (2). Engineers supplement circuit components of their model with a parallel capacitance called a stray or parasitic capacitance [8]. The parasitic capacitance does not exist as a separate physical capacitor.

The parasitic capacitances create an artificial circuit that is then analyzed by Kirchhoff's law eq. (2). The displacement current through the parasitic capacitance of the artificial circuit more or less provides the term missing in the model circuit. (The model circuit contains only idealized resistors and no capacitances, stray or otherwise. The artificial circuit contains the model circuit and supplements it with stray capacitances. The real circuit contains real resistors that are not described precisely by Ohm's law because the real components have displacement currents as universally requires by the Maxwell equations, see Appendix).

The artificial circuit can approximate the behavior of the real circuit with its displacement currents that arise from the unavoidable properties of the electric field. In this way, engineers reconcile [7] Kirchhoff's current laws eq. (1) and (2) with the continuity equation (3).

Much of our electronic technology is testimony to how well the engineering approach works. The engineering approach reaches even to microwave frequencies if handled with care [9, 10]. Kirchhoff's law eq. (2) is often an acceptable approximation to circuit behavior when stray capacitances are added to the circuit model to simulate displacement current in the real world circuit.

However, the engineering approach has problems

1) The engineering approach is a fix-up, as much an art as a science. Artful fixups are avoided as much as possible in mature sciences, because they are ambiguous. They are hard to teach to students or scientists who have not had time to learn the customary practices involved.

2) The values of the parasitic capacitances are not known from analysis. They are not unique. They are adjustable parameters that are best determined by fitting data from each circuit. Different scientists may fit data in different ways.

3) The engineering approach has difficulty dealing with the capacitances in the complex circuits of modern computers [11]. Indeed, a widely used encyclopedic guide to "The Art of Electronics" [8] devotes many pages to these issues: p.455-471 are used by these authors to "help illuminate this dark area of the electronic art". A search for the word "stray" or "parasitic" reveals many specific examples, where circuits do not work unless displacement currents of parasitic capacitances are dealt with artfully where "simplified models provide good circuit intuition, but [they] may often be inadequate." p. 908 of [8].

4) The capacitance to ground is often ignored in engineering applications despite its prominence in the chemical literature as the Born equation [12].

   In physics, engineering, and biophysics, most experiments require a large plate of metal connected to ground. The ground plate provides a capacitance to ground important for the function of high speed circuits [11]. The plate provides robust recording conditions and acts as a low pass filter to minimize noise, artifact, and pickup.

   The capacitance to ground is an important boundary condition that contributes to the energy of the system and so should be included in the simulations of molecular dynamics and transport Monte Carlo [13, 14]. Indeed, it is likely to decrease high frequency noise and artifact in simulations as it does in experiments.

5) Most mathematicians are reluctant to change the physical system being approximated. Rather, they prefer to deal with approximations explicitly as part of their numerical procedures.

Maxwell's redefinition of current provides an elegant and pleasing way to avoid these difficulties. On the other hand, his definition of true current is unfamiliar. It is also uncomfortable because its implications have not been worked out in the actual analysis of large numbers of practical circuits.

Conclusion. The treatment of Kirchhoff's current law in textbooks needs to be modified if it is to describe high speed signals. It is not satisfactory to derive the law for time independent systems and then use it to describe signals that vary rapidly in time. Engineering practice that allows the successful use or modification of Kirchhoff's current law should be clearly defined in textbooks relevant to modern applications. Maxwell's definition of true current should be applied to a wide range of actual circuits to see if it does as well in practice as it promises in principle.

Appendix

Derivation of Continuity Equation from Maxwell equations

We start with the Maxwell-Ampere equation.

Ampere-Maxwell Equation $$\frac{1}{\mu_0}\,\mathbf{curl\ B} = \mathbf{J}_{true} = \hat{\mathbf{J}} + \varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t \tag{4}$$

$\mathbf{J}_{true}$ is Maxwell's 'true current' [6] displayed prominently as his eq. **A**, p. 465, 480 of [5]. Perhaps Maxwell identified this equation as his first, his **A** equation because of the importance of displacement current in the propagation of light from the sun through the near vacuum of space. ε_0 is the electric constant. ε_r is the dielectric constant, more formally called the 'relative permittivity'. μ_0 is the magnetic constant. $\mathbf{J}_{true}$ is ***not*** zero in a vacuum. The treatment here uses the familiar dielectric approximation. A treatment without that approximation is straightforward [6, 15].

Take the divergence of $\mathbf{curl\ B}$ in eq. (4) and use the general mathematical result that divergence of the curl of a vector field is zero.

Maxwell Current Law $$\mathbf{div}\left(\hat{\mathbf{J}} + \varepsilon_r\varepsilon_0\,\partial\mathbf{E}/\partial t\right) = \mathbf{div}\,\mathbf{J}_{true} = \mathbf{div\ curl\ B} = \mathbf{0} \tag{5}$$

Equation (5) says in mathematics what Maxwell said in words. True current does not accumulate. Equation (5) is given a name because of its generality [6]. It is true whenever the Maxwell equations are an accurate approximation of electrodynamics.

If we introduce the charge density ρ using Gauss' law, the continuity equation (3) is derived.

References

1. Zangwill, A., *Modern Electrodynamics*. 2013, New York: Cambridge University Press. 977.
2. Griffiths, D.J., *Introduction to Electrodynamics, Third Edition*. 3rd ed. 2017: Cambridge University Press.
3. Ferry, D.K., X. Oriols, and J. Weinbub, *Quantum Transport in Semiconductor Devices*, in *Simulation using particles*. 2023, IOP Publishing.
4. Ferry, D.K., X. Oriols, and R. Eisenberg, *Displacement Current in Classical and Quantum Systems.* arXiv preprint arXiv:2408.13268, 2024.
5. Maxwell, J.C., *A Dynamical Theory of the Electromagnetic Field.* Philosophical Transactions of the Royal Society of London, 1865. **155**: p. 459-512.
6. Eisenberg, R.S., *Maxwell's True Current; see arXiv 2309.05667.* Computation, 2024. **12**(2): p. 22-46 see arXiv 2309.05667.
7. Eisenberg, B., et al., *What Current Flows Through a Resistor?* arXiv preprint arXiv:1805.04814, 2018.
8. Horowitz, P. and W. Hill, *The Art of Electronics*. Third Edition ed. 2015: Cambridge University Press. 1224.
9. Okoshi, T., *Planar circuits for microwaves and lightwaves*. Vol. 18. 2012: Springer Science & Business Media.
10. Mei, K.K. *From Kirchoff to Lorentz modifying-circuit theory for microwave and mm-wave structures*. in *Infrared and Millimeter Waves, 2000. Conference Digest. 2000 25th International Conference on*. 2000. IEEE.
11. Joffe, E.B. and K.-S. Lock, *Grounds for Grounding*. 2010, NY: Wiley-IEEE Press. 1088.
12. Laidler, K.J., J.H. Meiser, and B.C. Sanctuary, *Physical Chemistry*. Fourth ed. 2003: BrooksCole, Belmont CA. 1060.
13. Jacoboni, C. and P. Lugli, *The Monte Carlo Method for Semiconductor Device Simulation*. 1989, New York: Springer Verlag. pp. 1-356.
14. Vasileska, D., S.M. Goodnick, and G. Klimeck, *Computational Electronics: Semiclassical and Quantum Device Modeling and Simulation*. 2010, New York: CRC Press. 764.
15. Eisenberg, R.S., *Maxwell Equations Without a Polarization Field, Using a Paradigm from Biophysics.* Entropy, 2021. **23**(2): p. 172, also available on arXiv at https://arxiv.org/ftp/arxiv/papers/2009/2009.07088.pdf and 10.3390/e23020172.