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Communication

Kirchhoff's Current Law: A Derivation from Maxwell's Equations

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Abstract

Kirchhoff's current law was originally derived for systems such as telegraphs that switch in 0.1 s. It is used widely today to design circuits in computers that switch in ~0.1 nanoseconds, one billion times faster. Current behaves differently in one second and one-tenth of a nanosecond. A derivation of a current law from the fundamental equations of electrodynamics—the Maxwell equations—is needed. Here is a derivation in one line: $\mathbf{div\,curl\;B/\mu_0 = 0 = div\,(J + (\varepsilon_r - 1)\varepsilon_0\partial E/\partial t + \varepsilon_0\partial E/\partial t) = div\,J_{total}}$. Maxwell's 'true' current is defined as $\mathbf{J_{total}}$. The universal displacement current found everywhere is $\varepsilon_0\,\partial\mathbf{E}/\partial t$.The conduction current $\mathbf{J}$ is carried by any charge with mass, no matter how small, brief, or transient, driven by any source, e.g., diffusion. The second term $(\varepsilon_r - 1)\varepsilon_0\,\partial\mathbf{E}/\partial t$ is the usual approximation to the polarization currents of ideal dielectrics. The dielectric constant ε_r is a dimensionless real number. Real dielectrics can be very complicated. They require a complete theory of polarization to replace the $(\varepsilon_r - 1)\varepsilon_0\,\partial\mathbf{E}/\partial t$ term. The Maxwell current law $\mathrm{div}\,\mathbf{J_{total}} = 0$ defines the solenoidal field of total current that has zero divergence, typically characterized in two dimensions by streamlines that end where they begin, flowing in loops that form circuits. Note that the conduction current $\mathbf{J}$ is **not** solenoidal. Conduction current $\mathbf{J}$ accumulates significantly in many chemical and biological applications. Total current $\mathbf{J_{total}}$ does not accumulate in any time interval or in any circumstance where the Maxwell equations are valid. $\mathbf{J_{total}}$ does not accumulate during the transitions of electrons from orbital to orbital within a chemical reaction, for example. $\mathbf{J_{total}}$ should be included in chemical reaction kinetics. The classical Kirchhoff current law $\mathrm{div}\,\mathbf{J} = 0$ is an approximation used to analyze idealized topological circuits found in textbooks. The classical Kirchhoff current law is shown here by mathematics to be valid only when $\mathbf{J} \gg \varepsilon_0\,\partial\mathbf{E}/\partial t$, typically in the steady state. The Kirchhoff current law is often extended to much shorter times to help topological circuits approximate some of the displacement currents not found in the classical Kirchhoff current law. The original circuit is modified. Circuit elements—invented or redefined—are added to the topological circuit for that purpose.

Keywords: Kirchhoff law; maxwell equations; circuits; solenoidal

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1. Introduction

Kirchhoff's current law [1–6] was originally derived for systems such as telegraphs that switch in 0.1 s. See historical note in Appendix A. It is used widely today to design circuits in computers that switch in ~0.1 nanoseconds, one billion times faster.

Current behaves differently on the second and nanosecond time scale [7–10]. The Maxwell equations describe electrodynamics without significant measured error on all time scales [11,12]. A derivation of Kirchhoff's current law from the Maxwell equations is clearly needed. The derivation in this paper uses standard procedures of numerical analysis. It does not change the system being described by the current law. It provides error terms and validity conditions for the approximation itself.

2. Derivation

We start with the Maxwell Ampere law, which is one of the Maxwell equations. We define total current as Maxwell defined his 'true current' in Vol. 2, Section 610, p. 232 of his *A Treatise on Electricity and Magnetism* [13]. Variables are defined in the Details section below.

$$\mathbf{curl\ B} = \mu_0(\mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t) = \mu_0\mathbf{J_{total}} \tag{1}$$

where $\mathbf{J_{total}} = \mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t$

Next, take the divergence of both sides using an identity of vector calculus "div curl = 0" [14–16]. The identity can be simply derived from the usual differential definition of the vector operators in Cartesian coordinates.

$$\begin{aligned} \mathbf{div\ curl\ B} = \mathbf{0} &= \mathbf{div}\left(\mu_0(\mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t)\right) \\ &= \mathbf{div}(\mu_0\mathbf{J_{total}}) \end{aligned} \tag{2}$$

This establishes the three-dimensional version of the current law:

$$\textbf{Maxwell Current Law in Three Dimensions:}\ \mathbf{div\ J_{total}} = \mathbf{0} \tag{3}$$

3. Underlying Identity

The identity "div curl = 0" can be understood at many different levels of abstraction, ranging from elementary [15] to vector calculus [14], to the general Helmholtz decomposition of vector fields [11,14], to the exterior calculus and theory of differential forms [17]. Slides 39 and 40 of [18] present the Cartesian derivative derivation. Slide 40 also gives an integral derivation with an easily visualized representation.

We are motivated to use the variable $\mathbf{J_{total}} = \mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t$ because Maxwell gave it special significance. He called attention to the definition of total current $\mathbf{J_{total}}$ as 'One of the chief peculiarities of this treatise'. He called $\mathbf{J_{total}}$ 'the true current'.

4. True Current

Maxwell could hardly have chosen a stronger adjective than 'true' to describe $\mathbf{J_{total}}$. Maxwell was explicit about why he used the name true current. He said that "… estimating the total movement of electricity [requires] an equation of true currents" such as the Maxwell Ampere law Equation (1). See Vol. 2, Section 610, p. 232 of [13]. Maxwell gives a number of fully worked out examples that show how neglecting the displacement current in Equation (1) gives incorrect results.

The treatment in this paper, of course, depends on mathematics and not on Maxwell's opinion of what was true, however vigorously he said it. Maxwell's vigorous language is supported by the mathematics.

The mathematics provide a proof of Maxwell's opinion. It is easy to show—as in Equations (1)–(3) or in one line in the Abstract—that the Maxwell current law is a mathematical corollary of the Maxwell Ampere differential equation, made without physical or mathematical approximation or even argument.

5. Maxwell Current Law

The Maxwell current law embodies the same physics as the Maxwell equations of electrodynamics themselves. The Maxwell current law is as good a representation of electromagnetic phenomena as the Maxwell partial differential equations themselves. The Maxwell current law is true whenever and under any conditions that the Maxwell Ampere law is true. The Maxwell current law clearly applies on the time scale of gamma rays, 10^{-21} seconds and probably much faster.

The Maxwell current law applies at times much shorter than chemical reactions or thermal ('Brownian') motion. The Maxwell current law shows that total current does not accumulate even on these very rapid time scales. Total current flows out of a region as fast it flows in, without any delays **at all**. In a series of chemical reactions involving an electron changing orbitals, the total current does not accumulate. It involves no delay. Thus, in a series of chemical reactions, the total current $\mathbf{J}_{\mathbf{total}} = \mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t$ must be the same in each of the reactions. It must be exactly equal at all times and in all conditions.

6. Chemical Reactions

Total current is not usually considered in treatments of chemical reactions: terms involving the time derivative $\partial\mathbf{E} / \partial t$ are usually not included in the analysis of chemical reactions. They are likely to have significant effects on the rapid time scales involved in chemical reactions, including changes in electron orbitals. In my opinion, they need to be dealt with explicitly [19] if chemistry is to be as exact a science as electrodynamics.

Total current $\mathbf{J}_{\mathbf{total}}$ forms a divergence-free solenoidal field, as mathematicians call it, characteristic of flows of incompressible fluids [16]. Physical fluids compress before they become incompressible. They are not entirely incompressible. Physical fluids compress before they become divergence-free. Physical fluids can be nearly incompressible, but only after a time—nanoseconds to microseconds typically—determined by the speed of sound in the fluid.

Total current is different. It is exactly solenoidal. Total current is entirely incompressible. **Total current is divergence-free on all time scales in which the Maxwell equations are valid.** Those time scales extend over 33 orders of magnitude without measured error, and in all likelihood extend much further [20]. Few physical laws are accurate over such a large dynamic range.

7. Solenoidal Fields and Circuits

In two dimensions, streamlines of divergence-free solenoidal fields have special properties: the streamlines usually end where they start, forming looping circuits [21]. The circuits are generated and modified by physical constraints and boundary conditions that often supply energy. The physical constraints and boundary conditions arise in dipoles, not single charges, and perhaps also in the other coupled Maxwell equations and their boundary conditions. In three dimensions, flows are much more complex and streamlines are harder to define, as discussed at length in [11,22]. Current behaves differently on the second and nanosecond time scales in two and three dimensions, see [16].

Circuits are usually defined by simplified models that do not depend on the actual size and layout of the components of the circuit. I call these models—which are used in thousands of books and papers—topological circuits to distinguish them from representations that include the actual size of components and their nonideal properties.

The simplified topological circuits of engineering [4,23–25] are discrete versions of two-dimensional fields of total current $\mathbf{J}_{\mathbf{total}}$ in which all streamlines form loops. Real circuits [3–5,8,10,26–30] are built to approximate the two-dimensional topological circuits using the 'art of electronics' [9,25] with inspiration and hope [10]. Defining the relation of topological and real circuits is a tricky subject beyond the scope of this paper, see Appendix B. Loosely put, one can say that topological circuits show what would happen if total current could be confined entirely to a branching network of one-dimensional elements, each branch of which contained a series arrangement of circuit elements. It is hard to specify the qualitative let alone quantitative requirements that ensure this one-dimensional idealization and approximation. The topological circuit is most helpful if the branching network lies almost entirely in a two-dimensional plane, without much overlap.

The one-dimensional idealization allows the successful design of the circuits of our computers that function over an enormous range of time scales. Their success is likely to be related to the incompressible nature of total current over an even large range of time scales.

Details: ***There are no explicit adjustable parameters in this formulation of the Maxwell equations*** without dielectric constants. In this formulation, the Maxwell equations are not constitutive equations. In their classical formulations, the Maxwell equations are constitutive equations that depend in many ways on details of properties of materials [11][12].

$\mathbf{B}$ is the magnetic field. $\mathbf{E}$ is the electric field. The conduction current $\mathbf{J}$ is carried by any charge with mass, no matter how small, brief, or transient, driven by any source, e.g., diffusion [31]. The term $\mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t$ includes the usual approximation to the polarization currents of ideal dielectrics $(\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t$. ε_r is the dielectric constant, a dimensionless real number. The total current $\mathbf{J}_{\mathbf{total}} = \mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t + \varepsilon_0\, \partial\mathbf{E} / \partial t$ also includes the universal displacement current found everywhere $\varepsilon_0\, \partial\mathbf{E} / \partial t$. The magnetic constant is μ_0. The electric constant is ε_0.

The properties of matter are included only in the description of the conduction current and the terms $\mathbf{J} + (\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t$. They do not enter Equation (1) explicitly. There are no explicit adjustable parameters in this formulation of the Maxwell Ampere law. Real dielectrics can be very complicated because they involve all changes in charge density in response to applied electric fields [11,12]. Real dielectrics require a complete theory of polarization to replace the $(\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t$ term as shown in [31].

The formulation and approach used in Equation (1) is motivated by the treatment of compressibility (of mass) in fluid mechanics. Indeed, this approach is designed to illustrate explicitly the role of 'compressibility' of charge density [31]. A complete theory of polarization current—indeed, of the entire response of matter to changes in the electric field [32]—is needed to make Equation (1) a complete description of real situations. The complete theory must deal with anisotropy, frequency (time dependence), nonlinearity, and many other properties of real matter which are specific to the system and measurement of interest. In essence, the complete theory replaces the $(\varepsilon_r - 1)\varepsilon_0\, \partial\mathbf{E}/\partial t$ term in the Maxwell Ampere law Equation (1) and Maxwell current law Equation (3). Ref. [33] shows how a variational form of a complete theory can be used for this purpose.

Circuits confine current $\mathbf{J}$ to a network of one-dimensional components. Circuits are idealized topological representations used throughout engineering [4,23–25] to show the key features of current flow and electrical properties of the actual circuits of our computers and technology [3–5,8–10,25–30].

Topological circuits. Topological circuits have been used successfully to design real-world systems for some one hundred and seventy-five years. They are the 'bread and butter' of practical engineering. Appendix B describes some attempts to define these circuits more precisely. It can be difficult to define a priori in abstract language what mathematical features of current fields allow definitions of circuits, but it is clear that an enormous set of current fields do allow such definitions. The set includes essentially all the circuits of our signal, digital, and power technologies including the $> 10^{18}$ circuits and memory circuits used in the smartphones on our planet.

The connections of circuit components are shown in topological circuits used throughout electrical and electronic technology. Topological circuits do not depend on the size, dimensions, or details of the layout of the actual circuit: compare the idealized circuits of Horowitz and Hill in the original editions of their book [25] with the more realistic circuits in the updated "X-factor" edition of their text [9], as extended and verified by [7].

The current in each branch of a topological circuit is one-dimensional. It is the integral of the three-dimensional $\mathbf{J_{total}}(r, \theta, z)$ over the relevant cross-sectional area of that branch. The total current $\mathbf{J_{total}}$ is the same everywhere along a branch—i.e., everywhere in every component and wire in a series circuit—even though the microphysics of current flow in each component of the series circuit is very different. This counterintuitive behavior is illustrated and explained in physical terms in and near Figure 2 of [32].

The result is a generalized Kirchhoff current law that might be called the

Maxwell Current Law for Circuits:
The sum of total currents flowing into a node is zero. (4)

The Maxwell circuit law does not address the issue of when and whether circuits are an appropriate idealization or approximation to a system (see Appendix B). The existence of a wide class of such circuits is shown by their use throughout our computers and digital technology enumerated above.

The classical Kirchhoff current law is the Maxwell current law but it misses a term. The classical Kirchhoff current law does not include a time-dependent displacement current term involving $\partial \mathbf{E} / \partial \mathrm{t}$. As a matter of mathematics, we see that **the classical Kirchhoff current law approximates the Maxwell current law only when displacement current can be neglected; for example, at long times in systems that reach a steady state.** The contrast between Kirchhoff and Maxwell current laws is striking. There are no approximations or physical discussions involved in the derivation of Equation (3).

Kirchhoff Current Law for Circuits:
The sum of all one-dimensional currents $\mathbf{J}$ flowing into a node is zero (5)

is valid when $\mathbf{J} \gg \varepsilon_r \varepsilon_0\, \partial \mathbf{E} / \partial \mathrm{t}$ if we use the idealized ε_r approximation to dielectrics.

In three dimensions we have the

Three-dimensional Kirchhoff Current Law: $\mathbf{div\, J} = 0 \quad$ for $\mathbf{J} \gg \varepsilon_r \varepsilon_0\, \partial \mathbf{E} / \partial \mathrm{t}$ (6)

if we use the idealized ε_r approximation to dielectrics.

Kirchhoff's laws are not true in general but the three-dimensional Maxwell law (3) is true in general. The three-dimensional Maxwell law for total current is true whenever the Maxwell equations themselves are true. The three-dimensional Maxwell law Equation (3) can be used to generalize and rework the Kirchhoff laws to apply under wider conditions [1]. The Maxwell circuit law Equation (4) can be used when current is confined to a branching network of one-dimensional conductors that lie mostly in one plane.

The Maxwell total current $\mathbf{J_{total}}$ can be used in place of the conduction current $\mathbf{J}$ to analyze idealized topological circuits. The Maxwell total current $\mathbf{J_{total}}$ can be used where the classical Kirchhoff's current law uses the conduction current $\mathbf{J}$. Generalizing the Kirchhoff current law this way makes it **compatible with electrodynamics under all conditions at any time and any location on any scale.** This approach can be implemented in standard circuit software packages by small modifications of their code that deals with shunt capacitance [34–36], for example by using the Cpar parameter found within the component attribute editor of LTSpice. The paper "What Current Flows Through a Resistor?" [37] shows in detail one way to implement this modification.

The classical Kirchhoff current law is only true when $\mathbf{J} \gg \varepsilon_0\, \partial\mathbf{E} / \partial t$. It is usually a long-time (low-frequency) approximation. Like other long-time approximations, it fails to describe **even the qualitative properties** of currents outside the region of validity of the approximation. It does not predict transients at all (because it does not have time as a variable) and so does not describe one of the most prominent properties of real circuits. These short times are found in modern circuits almost everywhere.

8. Failures of the Classical Kirchhoff Law

The Kirchhoff current law fails qualitatively at short times because the displacement current is much larger than the conduction current at short times: $\varepsilon_0\, \partial\mathbf{E} / \partial t \gg \mathbf{J}$ at short times**.** This failure is an issue of mathematics, not physical science or tradition. This mathematical issue is not corrected by fixups that change the definition of circuit elements [7–10] in the topological circuits, however useful that change may be in applications [38–40].

In mathematics or numerical analysis, approximations are not considered quantitatively useful—and in that sense valid—when they are used beyond their region of validity. They are not considered valid when they miss some of the most prominent features of the system being described, such as transients. In mathematics, one does not change the object being approximated. In mathematics, one changes the approximation. Some workers identify the fixups that change circuits—and bring idealized circuits closer to reality—as "black magic", presumably for that reason [10].

The analysis of this paper allows quantitative estimation of errors in idealized circuits. If the branch of a circuit is a resistance R, the Kirchhoff approximation is accurate for times much longer than the RC time constant where C is the ideal stray parasitic capacitance, conservatively estimated as 10 pF (picofarads) [4,5] but often much larger.

The Kirchhoff approximation is seriously in error in many applications. Engineers examining circuits constructed only of physical resistors, roughly 10 kohms to 100 k ohms in value, will see transients 'everywhere' when they make measurements on the nanosecond time scales of modern circuits. Specifically, on the 0.1 nanosecond time scale used in computer circuits, RC might be 100 nsec to 1 microsecond in a well-implemented CMOS pull down/pull up circuit with R = 10 or 100 kohms. **The Kirchhoff current law is qualitatively in error in typical computer circuits because the time scale of the classical Kirchhoff current law is at least <u>four orders of magnitude</u> slower than the RC time constant of push down/pull up resistors.** There are billions of pull down/pull up resistors in our computer memories. The details of their performance are important determinants of the performance of our computers and smartphones.

9. Art of Electronics

The "art of electronics" [7–10,25] modifies idealized topological circuits so that they better approximate the properties of real circuits. The modification invents circuit elements and adds them into the original idealized circuit. Adding in the invented circuit elements supplements the idealized topology by providing some—but not all—of the displacement currents ignored in the classical Kirchhoff current law. The displacement currents include the universal displacement current $\varepsilon_0\, \partial\mathbf{E} / \partial t$ needed according to the Maxwell Ampere Equation (1). The invented circuit elements also include nonideal effects arising from the layout of components in the real circuit [7–10]. The invented supplemental elements are artfully placed so the Kirchhoff approximation can approximate reality.

The location and value of these invented supplemental components is subjective. **The supplemental elements do not have definite numerical values that will be the same in every implementation or numerical calculation.** Different people will choose different values and locations. The fixups are indeed 'black magic' [10] and do not satisfy the objectivity expected in a mature settled science particularly one so important for so much of the technology we use every day.

10. Supplementary Elements

The supplementary elements are not included in the classical idealized topological circuit diagrams because the invented elements and fixups depend on the size and details of the layout of circuits as discussed in detail in references [7–10]. The supplementary elements represent nonideal properties such as the inductance and coupling capacitance of real circuit components. These details vary from one real circuit to another even though they implement the same topological circuit. Compare the idealized early editions of Horowitz and Hill [25] with the "X-factor" used to describe these details in their more recent and realistic edition [9].

Nonetheless, well-placed supplemental elements added to topological circuits clarify the behavior seen in real circuits. Without them, the short time behavior of Kirchhoff's law in topological circuits is incompatible with the transients seen in nearly every experiment. Without them, the short time behavior of Kirchhoff's law is incompatible with the fundamental equations of electrodynamics, for example, the Maxwell Ampere law Equation (1). With the fixups, Kirchhoff approximations can be useful even at microwave frequencies [38–40].

11. Invented Elements

The invented elements of the fixups, however, have another difficulty. They hide the universal displacement current $\varepsilon_0 \, \partial \mathbf{E} / \partial t$ that is **always required** by the Maxwell Ampere differential equation. **The necessity of the displacement current is easy to overlook** when hidden in the invented circuit elements. An understanding of the universal nature of the displacement current $\varepsilon_0 \, \partial \mathbf{E} / \partial t$ is necessary to understand electrodynamics, as Maxwell demonstrated, long ago, when he introduced displacement current into Ampere's law. Without understanding the universal displacement current, one cannot understand how light propagates in a vacuum devoid of charge [41]. One cannot understand how one can see stars are astronomers see galaxies thousands of light years from earth.

12. Conclusions

The wide classical use of Kirchhoff's current law should not hide the following realities:

(1) The classical approach is often used far outside the realm of its validity, requiring 'black magic' [10] for justification. Most scientists would agree that black magic should be minimized in engineering and science, however important it is in understanding human behavior.

(2) Designs using supplemental invented or modified circuit elements do not provide unique numerical results because the placement and value of the supplemental elements is subjective. The placement and value of the invented elements is more art than science. Science and engineering should be objective wherever possible, most would agree.

(3) Maxwell current makes it possible to be more objective when dealing with topological or real circuits. It allows topological networks to be consistent with the laws of electrodynamics. Topological networks designed without the Maxwell current law are in fact inconsistent with the universal laws of electrodynamics. This is a disquieting situation given the enormous importance in our daily lives of circuits designed by topological networks. The designs using Maxwell current are derived by

mathematics from the equations of electrodynamics and so are unique, objective, and can be part of a mature settled science.

Invented subjective supplementary elements can still be used in an ad hoc way to supplement a Maxwell circuit design when that is useful in applications, even extending to microwave frequencies [38–40]. In this way, Maxwell circuit designs can deal with the widely varied (and rapidly evolving) nonideal properties of real circuit components and layouts over a very wide time scale in the remarkably successful tradition of the art of electronics [7–10,25].

(4) When Maxwell's current is used as just described, supplementary invented circuit elements depend **only** on the nonideal properties and layout of a real circuit. They are no longer surrogates for the universal displacement current $\varepsilon_0\, \partial \mathbf{E} / \partial t$ that always flows according to the Maxwell Ampere law Equation (1). The Maxwell circuit design clarifies the traditional art of electronics in this way as well as making it compatible with the Maxwell equations themselves and the experimentally observed behavior of real circuits, where transients are 'everywhere'. Maxwell circuit design brings some light to the dark arts and black magic used to design the circuits of our computers [7–10,25]. It restores transients—which are present in the real world—to the otherwise entirely DC analysis of traditional Kirchhoff current laws.

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Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Historical References

The reviewers were kind enough to mention important historical references to Kirchhoff's current law that should be included here [42–44].

Appendix B. Circuit Representations

This paper does not address the issue of the general validity of topological circuits. It seeks to present current laws to analyze circuits already known to be useful. That includes a significant number of circuits. It includes nearly all the circuits of our computers, of our digital technology, and of our power and signal distribution systems, including the more than 10^{18} circuits and memory circuits in the smartphones on our planet.

The referees kindly mentioned the following references on the general validity of circuits that should be included here [45–50].

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