Truly Incompressible: Maxwell's Total Current

DOI: 10.13140/RG.2.2.27505.80487

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May 24, 2025

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Abstract

Incompressible flows have special properties extensively studied in fluid mechanics using the theory of solenoidal or divergence-free fields. Fluids become compressible at short enough times. Here we point out that the total current defined by the Maxwell Ampere Law of electrodynamics forms a solenoidal divergence-free field on all measurable time scales ranging over more than 33 orders of magnitude.

Incompressible flows have special properties extensively studied in fluid mechanics [1-3] using the theory of solenoidal or divergence-free fields. Incompressible flows have no divergence. The flows form 'solenoidal fields' that do not have sources or sinks, in the ordinary monopolar meaning of those words. The circuits are generated by dipoles and the boundary conditions describing physical systems. In two dimensions, incompressible flow is usually along streamlines that can be precisely defined. The streamlines loop back on themselves, ending where they start, forming circuits. Three dimensional solenoidal fields are not so simple, Ch.5 & 6 of [4].

Incompressible fluids and their flows approximate the properties of fluids like water, which indeed do not change much in volume when pressure is changed.

Of course, it takes some time before incompressibility shows itself. Incompressible fluids are compressible on time scales set by the speed of sound. The time scale of compressibility can reach from nanoseconds to microseconds in typical cases. Incompressibility on all time scales is impossible when dealing with the mechanics of real fluids. The flow of total current is different, eq. (5).

This article identifies a flow that is truly incompressible, without measurable time dependence. It is the flow of total current defined by

$$\mathbf{J}_{\mathbf{total}} = \mathbf{J} + \varepsilon_0\, \partial\mathbf{E} \,/\, \partial \mathrm{t} \tag{1}$$

$\mathbf{J}$ is the flow of any charge with mass, however small, however brief and transient the flow, whatever its source, including, for example, diffusion. It includes the polarization charge of dielectric materials. $\mathbf{E}$ is the electric field. ε_0 is the electric constant.

Total current was identified by Maxwell himself as one of the main features of his theory of electrodynamics [5], Vol. 2 Section 610 p. 232: "One of the chief peculiarities of this treatise". Indeed, Maxwell called it "the true current" [6] that was required to estimate the flow of electricity.

In classical electrodynamics, the flow of total (i.e., true) current is described by the Maxwell Ampere Law [4, 7] where $\mathbf{B}$ is the magnetic field and μ_0 is the magnetic constant.

$$\mathbf{curl\ B} = \mu_0(\mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial \mathrm{t}) = \mu_0 \mathbf{J}_{\mathbf{total}} \tag{2}$$

The Maxwell Ampere law identifies the total current $\mathbf{J}_{\mathbf{total}}$ itself as the vorticity (vortex field) of $\mathbf{B}/\mu_0$ if we use the language of fluid mechanics [2].

The classical magnetic field is specified by its curl eq. (2) and its divergence, along with the experimental fact that magnetic charge (i.e., monopoles) do not exist. That is to say

$$\mathbf{div\ B} = 0 \tag{3}$$

The Maxwell Ampere law identifies the total current $\mathbf{J}_{\mathbf{total}}$ itself as the vorticity (vortex field) of $\mathbf{B}/\mu_0$ if we use the language of fluid mechanics [2].

These field equations are incomplete because they do not deal explicitly with physical boundary conditions (imposed by experiments or engineering devices) or the possibilities of intrinsic permanent magnetic ('spin') dipoles $\boldsymbol{\mu}$ mentioned in the closing paragraphs of this paper. See p. 384 of Griffiths [4] writing about spin "the world ***does*** include particles with intrinsic spin", but Griffiths makes clear that classical electrodynamics itself does not deal with spin dipoles: "they are ***not*** classical electrodynamics" (original has *italics*).

Taking the divergence shows that $\mathbf{J}_{\mathbf{total}}$ forms a solenoidal divergence-free field because $\mathbf{div\ curl} = \mathbf{0}$ is a well-known identity [2, 8] true for any function that satisfies the Maxwell equations.

$$\mathbf{div\ curl\ B} = 0 = \mathbf{div}\,\mu_0(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}\,/\,\partial t) = \mathbf{div}\,\mu_0\mathbf{J}_{\mathbf{total}} \tag{4}$$

<u>**Maxwell Current Law**</u>

$$\mathbf{div\ J}_{\mathbf{total}} = \mathbf{0} \tag{5}$$

The identity $\mathbf{div\ curl} = \mathbf{0}$ can be easily verified by writing the vector operators in Cartesian form. The identity has many interpretations at various levels of abstraction from calculus [8], to vector calculus [2] and its Helmholtz decomposition [4], to exterior calculus and differential forms [9]**.**

Note that no properties of matter are involved in the formulation of $\mathbf{J}_{\mathbf{total}}$ and no adjustable parameters are present [10, 11]. Only the electrical constant, magnetic constant and/or speed of light are involved. Those constants seem not to vary at all in experiments.

The Maxwell Ampere law is an accurate description of electrodynamics on the shortest time scales and smallest distance scales that have been measured. It is accurate on the time scale of gamma rays, some 10^{-20} seconds ***and also*** on the time scale of interstellar communication that is more than a million (light) years, 3×10^{13} seconds. It may fail at the enormous field strengths specified by the Schwinger limit, but experimental evidence of that failure has not yet been found as far as I know.

I conclude that $\mathbf{J}_{\mathbf{total}}$ is a perfectly incompressible fluid on all scales that matter. The divergence-free field of $\mathbf{J}_{\mathbf{total}}$ is as perfectly incompressible on all time scales as the Maxwell equations are accurate. There are few physical laws accurate over a time scale of 33 orders of magnitude!

<u>**Incompressibility**</u> of this order has remarkable implications because it means that $\mathbf{J}_{\mathbf{total}}$ ***does not accumulate at all, ever,*** even during extremely fast phenomena like thermal motion and electron rearrangements of chemical reactions that occur in ~10^{-17}seconds. Electrons changing orbital satisfy the Maxwell current law throughout their transition. Atoms in thermal motion satisfy the Maxwell current law. Chemical reactants satisfy the Maxwell current law in every sub-reaction. Chemical reactions in series have the same total current everywhere. ***Total current does not accumulate in any time interval no matter how brief, in a series of chemical reactions.*** Displacement current $\varepsilon_0\,\partial\mathbf{E}\,/\,\partial t$ and total current are usually not included in analyses of chemical reactions.

<u>**Circuits**</u>**.** Many of the most-used applications of electricity and electrodynamics involve electrical circuits [12]. Circuits deliver the power that enables our modern economy. Circuits provide the signals and information of our computers and smart phones.

The properties of $\mathbf{J}_{\mathbf{total}}$ involved in circuits can be analyzed with quite simple mathematics when $\mathbf{J}_{\mathbf{total}}$ is confined to networks of one-dimensional conductors [13-15]. In that case, the Kirchhoff's current law is widely used as an approximation of the exact Maxwell current law eq. (5). The classical Kirchhoff law of circuit analysis is accurate when displacement current $\varepsilon_0\,\partial\mathbf{E}\,/\,\partial t$ is small compared to conduction current $\mathbf{J}$. The art of electronics [16] has developed fix-ups [17-19] to deal successfully but approximately with the more general case. So further analysis is not really needed from a practical scientific point of view. But science is done by scientists and scientists are curious.

<u>What flows?</u> a speculation. Curious scientists ask "What flows according to the law $\mathbf{div}\,\mathbf{J_{total}} = \mathbf{0}$? Is there an aether that flows according to the law $\mathbf{div}\,\mathbf{J_{total}} = \mathbf{0}$? ". Such an aether need not be the aether of 19th century physics [20]; the aether could include the displacement current $\varepsilon_0\,\partial\mathbf{E}\,/\,\partial t$ and that might be enough to make that aether consistent with special relativity, Lorentz transformations, and the Michelson Morley experiment.

Flows of quasi-particles play a crucial role in the theory and practice of electronics [21-23]. The quasi-particles holes and 'electrons' are properties of the band structure of semiconductors that capture the correlations underlying the successful design of transistors. In this tradition, one can ask whether a quasi-particle for $\mathbf{J_{total}}$ can be usefully defined. It might be called a 'Maxion' to honor Maxwell's understanding of true current.

The flows $\mathbf{J_{total}}$ of a Maxion must include both the ethereal displacement current $\varepsilon_0\,\partial\mathbf{E}\,/\,\partial t$ and the flow $\mathbf{J}$ of charges with mass. Other properties of the Maxion are not preordained and can be specified for convenience.

It is important to remember that real electrons are not just charges. They have a permanent magnetic dipole $\boldsymbol{\mu}$, a vector called spin with magnitude $|\boldsymbol{\mu}| = \mu = \pm\frac{1}{2}$, as well as their permanent negative charge. Together the charge and spin form a real electron that might be called a 'Magtron' (see p. 384 of [4]). A Maxion is a Magtron along with its displacement current. The classical physics of Magtrons and Maxions seems worth investigating to isolate the effects of classical electrodynamics on spintronic devices and such chemical issues as structure of orbitals and the Pauli exclusion principle. Note that the Pauli exclusion principle and thus spin are involved in the properties of most chemical reactions.

A Maxion with spin forms a charged electromagnetic point-vortex, in the language of fluid dynamics [2, 24], a magnetic point-dipole with the field $\mathbf{B}_{\boldsymbol{\delta}} = \frac{\mu_0\mu}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$ in spherical coordinates that adds a source and source term $\boldsymbol{\mu}\,\mathbf{x}\,\boldsymbol{\nabla\delta_{3D}}$ to the $\mathbf{B}$ field found in eq. (2) & (3). $\boldsymbol{\nabla\delta_{3D}}$ is the gradient of the three dimensional Dirac delta function(al), see p. 48 of [4].

The magnetic point-dipole acts as a physical source that helps create the magnetic field $\mathbf{B}$ of real electrons with spin.

The magnetic point-dipole can be viewed either as a boundary condition or dipolar source term $\boldsymbol{\mu}\,\mathbf{x}\,\boldsymbol{\nabla\delta_{3D}}$ that supplements the classical Maxwell partial differential equations (3) & (2) $\mathbf{div}\,\mathbf{B} = \mathbf{0}$ and $\mathbf{curl}\,\mathbf{B} = \mu_0\mathbf{J_{total}}$. The mathematics of such dipoles needs further investigation. It "is ***not*** classical electrodynamics" (see p. 384 of [4] *italics* in the original). The point-dipole $\boldsymbol{\mu}\,\mathbf{x}\,\boldsymbol{\nabla\delta_{3D}}$ is singular and so it does ***not*** automatically satisfy the identities of vector calculus. The near singular behavior of the spin actually present in an electron means that vector identities like $\mathbf{div\,curl}\ (\boldsymbol{\mu} \times \boldsymbol{\nabla}\boldsymbol{\delta}_{3D}) = \mathbf{0}$ ***cannot*** be taken for granted. They may or may not apply to point-dipoles and layers of dipoles. The identities for dipole sources must be studied by the limiting processes of functional analysis that define $\boldsymbol{\nabla}\boldsymbol{\delta}_{3D}$.

An extension of classical electrodynamics to analyze an electron with spin and displacement current—a Magtron/Maxion—would certainly be interesting. Would it yield a Rydberg formula, Pauli exclusion principle and the periodic table of chemistry? What would a wave equation of a Magtron/Maxion tell us of spintronics?

The effect of the magnetic point-dipole $\boldsymbol{\mu} \times \boldsymbol{\nabla}\boldsymbol{\delta}_{3D}$ on the total current $\mathbf{J_{total}}$ will emerge from analysis. From the physical point of view of classical electrodynamics, $\mathbf{J_{total}}$ satisfies the Maxwell current law everywhere: in the plasma and near-vacuum of electron tubes (valves in UK usage); in delocalized orbitals that carry current in wires; in band structures of semiconductors. It is difficult to see how a single definition of a quasi-particle for total current could deal with the diversity of microphysics in all these systems, even if the magnetic point-dipole of the electron and displacement current is included. But mathematical analysis is needed.

Acknowledgement.

It is a joy to thank Ardyth Eisenberg for her contributions to this paper and to so much else in my life.

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