Kirchhoff's Current Law: A Derivation from Maxwell's Equations

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Abstract

Kirchhoff's current law was originally derived for systems that switch in 0.1 second. It is used widely today to design circuits in computers that switch in ~0.1 nanoseconds, one billion times faster. A derivation from the fundamental equations of electrodynamics—the Maxwell equations—is needed: in one line, $\mathbf{div\,curl\ B}/\mu_0 = \mathbf{0} = \mathbf{div}\,(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial\mathrm{t}) = \mathbf{div\,J_{total}}$. Here $\mathbf{J}$ is the current carried by any charge with mass, no matter how small, brief, or transient, driven by any source, e.g., diffusion. $\mathbf{J}$ includes the usual approximation to the polarization currents of ideal dielectrics. The Maxwell current law $\mathbf{div\,J_{total}} = 0$ defines the solenoidal field of total current that has zero divergence, typically characterized in two dimensions by streamlines that end where they begin forming loops or circuits. The Kirchhoff current law $\mathbf{div\,J} = \mathbf{0}$ is an approximation only valid when $\mathbf{J} \gg \varepsilon_0\,\partial\mathbf{E}/\partial\mathrm{t}$.

Kirchhoff's current law [1-5] was originally derived for systems that switch in 0.1 second [6]. It is used widely today to design circuits in computers that switch in ~0.1 nanoseconds, one billion times faster. A derivation from the fundamental equations of electrodynamics—the Maxwell equations—is clearly needed.

We start with the Maxwell Ampere Law which is one of the Maxwell equations that describe electrodynamics without significant error [7, 8]. And we define total current as Maxwell did.

$$\mathbf{curl\ B} = \mu_0(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t) = \mu_0\mathbf{J_{total}} \qquad \mathbf{J_{total}} = \mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t \tag{1}$$

Take the divergence of both sides using an identity of vector calculus [9-11] that is part of the general Helmholtz decomposition of vector fields [7, 9]

$$\mathbf{div\ curl\ B} = \mathbf{0} = \mathbf{div}\,(\mu_0(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t)) = \mathbf{div}(\mu_0\mathbf{J_{total}}) \tag{2}$$

This establishes the three-dimensional version a current law

Maxwell Current Law in Three Dimensions: $\mathbf{div\ J_{total}} = \mathbf{0}$ **(3)**

We are motivated to use the variable $\mathbf{J_{total}} = \mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t$ because Maxwell gave it special significance as 'One of the chief peculiarities of this treatise' [12]. He called it the true current. He could hardly have chosen a stronger adjective than 'true'. Maxwell was explicit about why he used the name true current. He said that " … estimating the total movement of electricity [requires] an equation of true currents" like the Maxwell Ampere Law eq. (1). See Vol. 2, Section 610, p. 232 of his "A Treatise on Electricity and Magnetism" [13].

The treatment here, of course, depends on mathematics not Maxwell's opinion of what was true. It is easy to show (as does eq. (2)) that the Maxwell Current Law is a mathematical corollary of the Maxwell Ampere differential equation. The Maxwell Current Law embodies the same physics as the Maxwell equations of electrodynamics themselves. It is as good a representation of electromagnetic phenomena as they are. The Maxwell current law is true whenever and under any conditions that the Maxwell Ampere law is true. The Maxwell current law applies on the time scale of gamma rays, 10^{-21}seconds. The Maxwell current law applies at times much shorter than chemical reactions or thermal ('Brownian') motion. Total current does not accumulate even on these very rapid time scales. Total current flows out of a region as fast it flows in, without any delays at all. Thus, in a series of chemical reactions involving an electron changing orbitals, the total current involves no delay. Total current is not usually considered in treatments of chemical reactions: terms involving the time derivative $\partial\mathbf{E}/\partial t$ are usually not included in the analysis of chemical reactions.

Total current $\mathbf{J_{total}}$ forms a solenoidal field, as mathematicians call it, characteristic of flows of incompressible fluids [11]. The streamlines of such fields have special properties: in two dimensions, the streamlines often end where they start, forming looping circuits in systems modified by physical constraints and boundary conditions, which arise in dipoles, not single charges and in the other coupled Maxwell equations. In three dimensions flows are much more complex and streamlines are hard to define [7]. The simplified topological circuits of engineering are discrete versions of two-dimensional fields of total current $\mathbf{J_{total}}$ in which all streamlines form loops.

<u>**Details**</u>: ***There are no explicit adjustable parameters in this formulation of the Maxwell equations*** [14, 15] without dielectric constants [16]. $\mathbf{B}$ is the magnetic field. $\mathbf{E}$ is the electric field. $\mathbf{J}$ is the conduction current of any charge with mass, however brief and transient the current, even if driven by forces not in the Maxwell equations, like diffusion [17]. In this formulation, $\mathbf{J}$ includes the movement of polarization charge of ideal dielectrics. The properties of matter are included only in the description of $\mathbf{J}$. They do not enter the equation (2) itself. There are no explicit adjustable parameters in this formulation. μ_0 is the magnetic constant. ε_0 is the electric constant.

<u>**Circuits**</u> confine current $\mathbf{J}$ to a network of one-dimensional components. Circuits are idealized topological representations used throughout engineering to show the key features of current flow and electrical properties of the actual circuits of our computers and technology. They show the connections of circuit components. Topological circuits do not depend on the size or dimensions or details of the layout of the actual circuit. The current in each branch of a circuit is the integral of $\mathbf{J_{total}}(r, \theta, z)$ over the cross-sectional area of the components of that branch. The total current is the same everywhere along a branch. The total current is the same everywhere in a series circuit even though the microphysics of current flow in each component of the series circuit is usually different as illustrated and explained in Fig. 2 of [18].

The result is a generalization of Kirchhoff's current law that might be called the

Maxwell Current Law for Circuits: The sum of all the total currents flowing into a node is zero. (4)

The classical Kirchhoff current law is the Maxwell current law without the time dependent term involving $\partial\mathbf{E} / \partial t$. ***The classical Kirchhoff current law approximates the Maxwell current law only at long times in systems that reach a steady state.*** (Remember that the total current $\mathbf{J_{total}}$ in an idealized topological circuit is the sum of conduction $\mathbf{J}$ and the universal displacement current $\varepsilon_0\, \partial\mathbf{E} / \partial t$. In real circuits the additional polarization currents of dielectrics are often crudely approximated [16] by adding a term $(\varepsilon_0 - 1)\varepsilon_0\, \partial\mathbf{E} / \partial t$

Kirchhoff Current Law for Circuits: The sum of all conduction currents $\mathbf{J}$ flowing into a node is zero (5)

Three-dimensional Kirchhoff Current Law: $$\mathbf{div}\,\mathbf{J} = 0 \tag{6}$$

Kirchhoff's law (6) is not true in general. The Maxwell law (3) for total current is true in general. The Maxwell law eq. (3) can be used as described in ref. [1] . The Maxwell total current can be used where Kirchhoff's current law uses the conduction current $\mathbf{J}$ to analyze idealized topological circuits. Generalizing the Kirchhoff current law this way makes it ***compatible with electrodynamics under all conditions at any time and any location on any scale.*** This approach can be implemented in standard circuit software packages by small modifications of their treatment of shunt capacitance [19-21]. In real circuits the additional polarization currents of dielectrics are often crudely approximated [16] by adding a term $(\varepsilon_0 - 1)\varepsilon_0\, \partial\mathbf{E} \,/\, \partial t$.

The classical Kirchhoff current law is only true when $\mathbf{J} \gg \varepsilon_0\, \partial\mathbf{E} \,/\, \partial t$. It is a long time (low frequency) approximation. Like other long-time approximations, it fails to describe ***even the qualitative properties*** of currents outside the region of validity of the approximation. The Kirchhoff current law fails qualitatively at short times because $\varepsilon_0\, \partial\mathbf{E} \,/\, \partial t \gg \mathbf{J}$ at short times**.** This is an issue of mathematics not physical science or tradition.

If the branch of a circuit is a resistance R, the Kirchhoff approximation is accurate for times much longer than RC time constant where C is the ideal stray parasitic capacitance, conservatively estimated as 10 pF [4, 5] but often much larger.

The Kirchhoff approximation is seriously in error on the 0.1 nanosecond time scale used in computer circuits, where RC might be 100 nsec to 1 microsecond in well implemented CMOS pull down/pull up circuit with R = 10 or 100 kohms. The Kirchhoff current law is qualitatively in error in typical computer circuits because their time scale is at least four orders of magnitude faster than the RC time constant of push down/pull up resistors.

The “the art of electronics” [22] modifies the idealized circuit so it better approximates the properties of real circuits. The modification invents circuit elements and adds them into the original idealized circuit. Adding in the invented circuit elements supplements the idealized topology with some of the properties of the real circuit. The invented circuit elements include nonideal effects arising from the layout of components in the real circuit. They also include the universal displacement current $\varepsilon_0\, \partial\mathbf{E} \,/\, \partial t$ needed according to the Maxwell Ampere equation (1). The invented supplemental elements are artfully placed so the Kirchhoff approximation can approximate reality.

The location and value of these invented supplemental components is subjective. The supplemental elements do not have definite numerical values that will be the same in every implementation or numerical calculation. Different people will choose different values and locations. The supplementary elements are not included in the classical idealized topological circuit diagrams because most of them depend on the details of the layout of circuits. The supplementary elements represent nonideal properties like the inductance and coupling capacitance of real circuit components. These details vary from one real circuit to another even though they implement the same topological circuit.

Nonetheless, well placed supplemental elements clarify the behavior seen in real circuits. Without them, the short time behavior of Kirchhoff's law is incompatible with experimental results and the fundamental equations of electrodynamics. They significantly extend the usefulness of the Kirchhoff approximation.

The invented elements hide the universal displacement current $\varepsilon_0 \, \partial \mathbf{E} \, / \, \partial t$ ***always required*** by the Maxwell Ampere differential equation. ***The necessity of the displacement current is easy to overlook.***

Conclusion. The wide use of Kirchhoff's current law in classical approaches should not hide the following realities:

(1) the classical approach is often used far outside the realm of its validity.

(2) Designs using supplemental circuit elements do not provide unique numerical results because the placement and value of the supplemental elements is subjective. It is more art than science.

(3) Maxwell current provides the general and unique designs appropriate for general topological networks, consistent with the laws of electrodynamics. The designs using Maxwell current are mathematics as well as science.

(4) Further invented elements can supplement a Maxwell circuit design as is useful in applications. In this way, Maxwell circuit designs can deal with nonideal properties of real circuit components and layouts in the tradition of the art of electronics.

(4) Supplementary circuit elements depend only on the nonideal properties and layout of a real circuit when Maxwell's current is used. They are no longer surrogates for the displacement current that always flows according to the Maxwell Ampere law eq.(1). The Maxwell circuit design clarifies the traditional art of electronics in this way as well as making it compatible with the Maxwell equations themselves.

Supplemental Note: The derivation in this paper depends on the vector calculus operators div and curl and their properties, particularly the identity $\mathbf{div}\,\mathbf{curl} = \mathbf{0}.$ The identity can be easily verified from substitution in the Cartesian definitions of the differential operators given in textbooks and interchange of the orders of differentiation. The identity can be understood at many different levels of abstraction, ranging from elementary [10] to vector calculus [9] to the exterior calculus and differential forms [23].

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References

1. Eisenberg, R. S. 2019. Kirchhoff's Law can be Exact. arXiv preprint available at https://arxiv.org/abs/1905.13574.
2. Eisenberg, R. 2023. Circuits, Currents, Kirchhoff, and Maxwell. Qeios Qeios ID: L9QQSH.2.
3. Balanis, C. A. 2012. Advanced engineering electromagnetics. John Wiley & Sons.
4. Ulaby, F. T., and M. M. Maharbiz. 2010. Circuits. NTS press.
5. Ulaby, F. T., and U. Ravaioli. 2015. Fundamentals of Applied Electromagnetics. Pearson.
6. Eisenberg, R., X. Oriols, and D. K. Ferry. 2022. Kirchhoff’s Current Law with Displacement Current. arXiv: 2207.08277.
7. Griffiths, D. J. 2024. Introduction to electrodynamics. Fifth Edition. Cambridge University Press.
8. Zangwill, A. 2013. Modern Electrodynamics. Cambridge University Press, New York.
9. Arfken, G. B., H. J. Weber, and F. E. Harris. 2013. Mathematical Methods for Physicists: A Comprehensive Guide. Elsevier Science.
10. Schey, H. M. 2005. Div, grad, curl, and all that: an informal text on vector calculus. WW Norton New York.
11. Chorin, A. J., J. E. Marsden, and J. E. Marsden. 1990. A mathematical introduction to fluid mechanics. Springer.
12. Eisenberg, R. S. 2024. Maxwell’s True Current; see arXiv 2309.05667. Computation 12(2):22-46 see arXiv 2309.05667.
13. Maxwell, J. C. 1865. A Treatise on Electricity and Magnetism (reprinted 1954). Dover Publications, New York.
14. Eisenberg, and R. S. 2025. Current Laws and the Maxwell Equations. DOI: 10.13140/RG.2.2.34171.63524, (posted).
15. Eisenberg, B. 2016. Maxwell Matters. Preprint on arXiv https://arxiv.org/pdf/1607.06691.
16. Eisenberg, R. S. 2019. Dielectric Dilemma. preprint available at https://arxiv.org/abs/1901.10805.
17. Eisenberg, B., C. Liu, and Y. Wang. 2022. On Variational Principles for Polarization Responses in Electromechanical Systems. Communications in Mathematical Sciences 20(6):1541-1550.
18. Eisenberg, R. S. 2016. Mass Action and Conservation of Current. Hungarian Journal of Industry and Chemistry Posted on arXiv.org with paper ID arXiv:1502.07251 44(1):1-28.
19. Antognetti, P., and G. Massobrio. 1993. Semiconductor device modeling with SPICE. McGraw-Hill, Inc.
20. Brocard, G. 2013. The LTspice IV simulator: manual, methods and applications. Würth Elektronik.

21. Analog_Devices. 2025. LTspice IV Getting Started Guide. (posted).

22. Horowitz, P., and W. Hill. 2015. The Art of Electronics. Cambridge University Press.

23. Zhdanov, M. S. 2020. Maxwell's equations and numerical electromagnetic modeling in the context of the theory of differential forms. Active geophysical monitoring. Elsevier, pp. 245-267.