From Maxwell to Circuits

Discussion, Slides, References
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Thanks to Tom Wong for the Invitation!

Presented as “Maxwell’s Equations, Exact, Universal and Scary”
Electrical and Computer Engineering Illinois Institute of Technology
February 14, 2025

Abstract

When the Maxwell equations are written without a dielectric constant, they are universal and exact, for fields less than the Schwinger limit (1.3×10^{18}v/m), from inside atoms to between stars. Dielectric and polarization phenomena need then to be described by stress strain relations for charge, that show how charge redistributes when the electric field is changed, in each system of interest.

Conservation of total current (including the ethereal displacement current) is then independent of any property of matter and as exact as the Maxwell equations themselves

Total current has zero divergence and is a solenoidal field, without sources or sink terms. Current flows in loops in this solenoidal field. The loops form circuits that are isolated in our electrical and electronic devices. Isolated loops follow Kirchhoff's laws, generalized to include displacement current, **on all time scales.**

Spatial dependence of total current disappears in unbranched series systems: conservation of total current becomes **exact equality at all times**. If total current is zero anywhere, it is zero everywhere. So current only flows in complete circuits. Hopping phenomena of total current disappear. Maxwell's Core Equations become a perfect (spatial) low pass filter. The infinite spatial variation of a Brownian model of thermal noise becomes the zero spatial variation of total current.

An Exact and Universal theory of Electrodynamics is a scary challenge to scientists like me, trained to be skeptical of sweeping claims to perfection.

Essence of Electrodynamics

is

Maxwell's Core Equations

for the

Flows and Forces of Charge and Current
in matter and space

Nearly Exact and Universal from Stars
to inside atoms

Ferry, Oriols, Eisenberg
Displacement Current in Classical and Quantum Systems.
Computation (2025) 13, 45
DOI 10.3390/computation13020045

Maxwell's Core Equations

$$\textbf{div } \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \textbf{div } \mathbf{B} = 0$$

CIRCUIT equation, as we shall see

$$\textbf{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \textbf{curl } \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Maxwell Equations have error* $< 10^{-8}$ for $|\mathbf{E}| \ll$ Schwinger limit* $= 1.32 \times 10^{8}$ volts/Angstrom

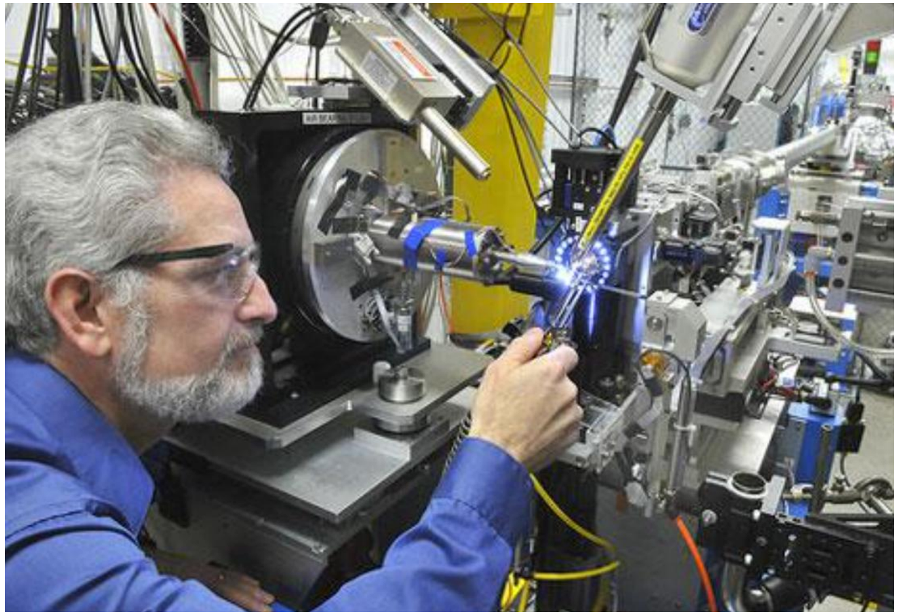
$\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field
$\mathbf{J}$ is flux of charge with mass, including brief dielectric transients of **P** and **D** fields
ρ is the the charge density of all types including brief dielectric transients of the **P** and **D** fields
ε_0 is the electrical constant, the permittivity of a vacuum
μ_0 is the magnetic constant the permeability of a vacuum
Velocity of light $c = (\varepsilon_0 \mu_0)^{-0.5}$ (!!)

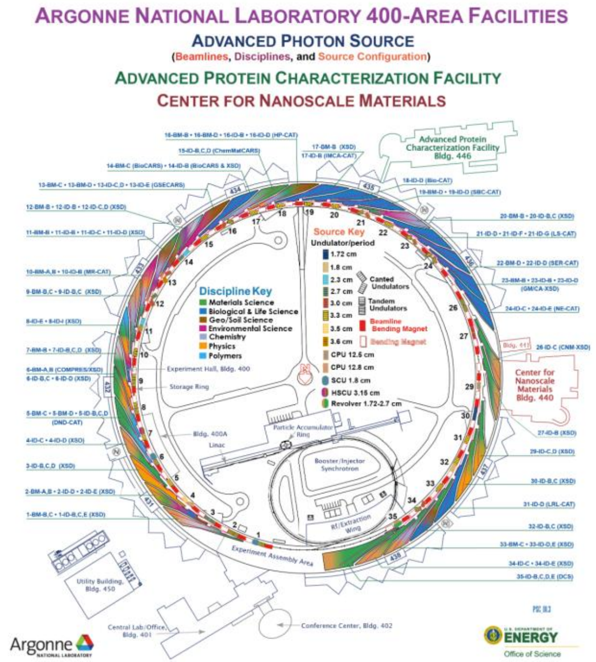
*from measurements of QED fine structure constant α

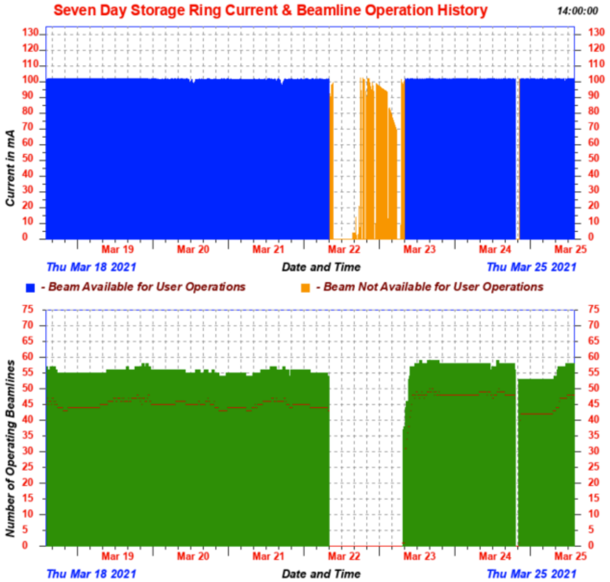
Advanced Photon Source Argonne National Laboratory

Error in Theory $< 10^{-10}$

Beam $\sim 10^{10}$ eV
Beam length 10^{3} m
Tolerance $< 10^{-7}$ m
Beam Current 100 mA
Beam Power 10^{9} watts







Source Internet

Don't recognize Core equations?

Polarization is too complex to define by one dielectric constant ε_r

Polarization is made part of J and ρ

Move the physics of dielectrics into $\boldsymbol{J}$ and ρ

$$\mathbf{div\,E} = \frac{\rho}{\varepsilon_0} \qquad \mathbf{div\,D} = \mathbf{div}\,\overbrace{\boldsymbol{\varepsilon_r}\varepsilon_0\mathbf{E}} = \rho_f$$

$$\frac{1}{\mu_0}\mathbf{curl\,B} = \mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad \frac{1}{\mu_0}\mathbf{curl\,B} = \tilde{\mathbf{J}} + \boldsymbol{\varepsilon_r}\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Polarization is part of J and ρ in the core equations

When nothing is known about polarization, when constitutive model is not known experimentally, it is customary and appropriate to include the dielectric constant ε_r as a single real positive constant $\varepsilon_r \geq 1$

Maxwell's Core Equations

$\mathbf{div\,E} = \frac{\rho}{\varepsilon_0}$ $\mathbf{div\,B} = 0$

$\mathbf{curl\,E} = -\,\partial\mathbf{B}/\partial t$ $\mathbf{curl\,B} = \mu_0(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t)$

Polarization is too diverse to define a dielectric constant ε_r

Polarization is part of J and ρ

in the core equations

Constitutive equations and models are needed to define J and ρ

> **When nothing is known about polarization,**
> **it is customary and appropriate to approximate the**
> **dielectric constant ε_r as a single real positive constant $\varepsilon_r \geq 1$**

Documentation

Polarization is too complex/diverse to define by one dielectric constant ε_r

Electronic Devices, Solid State: many references in

Ferry, Oriols, Eisenberg. 2025. Displacement Current in Classical and Quantum Systems. Computation 13, 45 DOI 10.3390/computation13020045

Ionic Solutions (thus life and most of chemistry): many references in

General: Barsoukov, E., and J. Ross Macdonald. 2018. *Impedance spectroscopy: theory, experiment, and applications (John Wiley & Sons).*
Optical Properties: Parsegian, V. Adrian. 2006. *Van der Waals Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists (Cambridge University Press: New York).*
Eisenberg, Robert S. 2019. '*Dielectric* Dilemma', *preprint* https://arxiv.org/abs/1901.10805.

Polarization $\varepsilon_0\, \partial \mathrm{E}/\partial t$ is Present in the Vacuum of Empty Space

Experimental Fact

Polarization $\varepsilon_0\, \partial \mathrm{E}/\partial t$ is Present Everywhere

Einstein Special Relativity

Einstein, Albert. 1905. 'On the electrodynamics of moving bodies', *Annalen der Physik, 17: 50.*

Explicitly calculated and eloquently explained in chapters in

***Griffiths**, D.J. 2017. Introduction to Electrodynamics,*

***Jackson**, J.D. 1999. Classical Electrodynamics*

Polarization ε_0 of Empty Space

Light travels through the Vacuum of Space

ethereal current $\varepsilon_0 \partial \mathrm{E}/\partial t$ flows in vacuum of space,

once thought to be filled with an 'aether'

Maxwell, 1865. Treatise on Electricity and Magnetism

Jeans, 1908. The Mathematical Theory of Electricity and Magnetism.

Whittaker, 1951. A History of the Theories of Aether & Electricity.

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$

Wave Equation

Corollary of
Maxwell Equations

$c = 1/\sqrt{\varepsilon_0 \mu_0} = \textit{velocity of light}$ **Experimental Fact**

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

Total Current is NOT the Flow of Charge

How do we know that?

Light Exists in Vacuum

Magnetic Fields Exist in Vacuum

$\varepsilon_0\, \partial\mathbf{E}/\partial t$ is the current that creates Electromagnetic Waves

LIGHT

$$\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = 0$$

Derivation of Wave Equations is in every textbook, starting with curl curl $\mathbf{E}$

$$\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = 0$$

Total Current is NOT the Flow of Charge
How do we know that?

Charge and Flow of Charge are ZERO in a vacuum

$\varepsilon_0\, \partial\mathbf{E}/\partial t$ is the current that creates Electromagnetic Waves

LIGHT

$\mu_0\varepsilon_0 \frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = 0$ **Derivation of Wave Equations is in every textbook, starting with $\mathbf{curl\ curl\ E}$** $\mu_0\varepsilon_0 \frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = 0$

$\boldsymbol{c = 1/\sqrt{\varepsilon_0\mu_0}}$ = ***velocity of light***

Experimental Fact

Electromagnetic Field Equations Exist Everywhere

because of the relativistic properties of space

Einstein, Albert. 1905. 'On the electrodynamics of moving bodies', *Annalen der Physik, 17: 50.*

1934. *Essays in science, originally published as Mein Weltbild 1933*

Explicitly Calculated and Eloquently Explained *in chapters in*

***Griffiths**, D.J. 2017. Introduction to Electrodynamics,*

***Jackson**, J.D. 1999. Classical Electrodynamics*

In Vacuum

$$\mathbf{div\ E} = 0 = \cancel{\frac{\rho}{\varepsilon_0}} \qquad \mathbf{div\ B} = 0$$

c is velocity of light

$$\mathbf{curl\ E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{curl\ B} = \cancel{\mu_0 \mathbf{J}} + \underbrace{c^{-2}\,\partial \mathbf{E}/\partial t}_{Ethereal\ Current}$$

Relativistic Property of Space NOT matter

Ethereal Current = Displacement Current

We use mostly Gauss Law or Coulomb's Law

$$\mathbf{div\ E} = \frac{\rho}{\varepsilon_0}$$

Source of Electric Field is charge

and the Maxwell Ampere Law

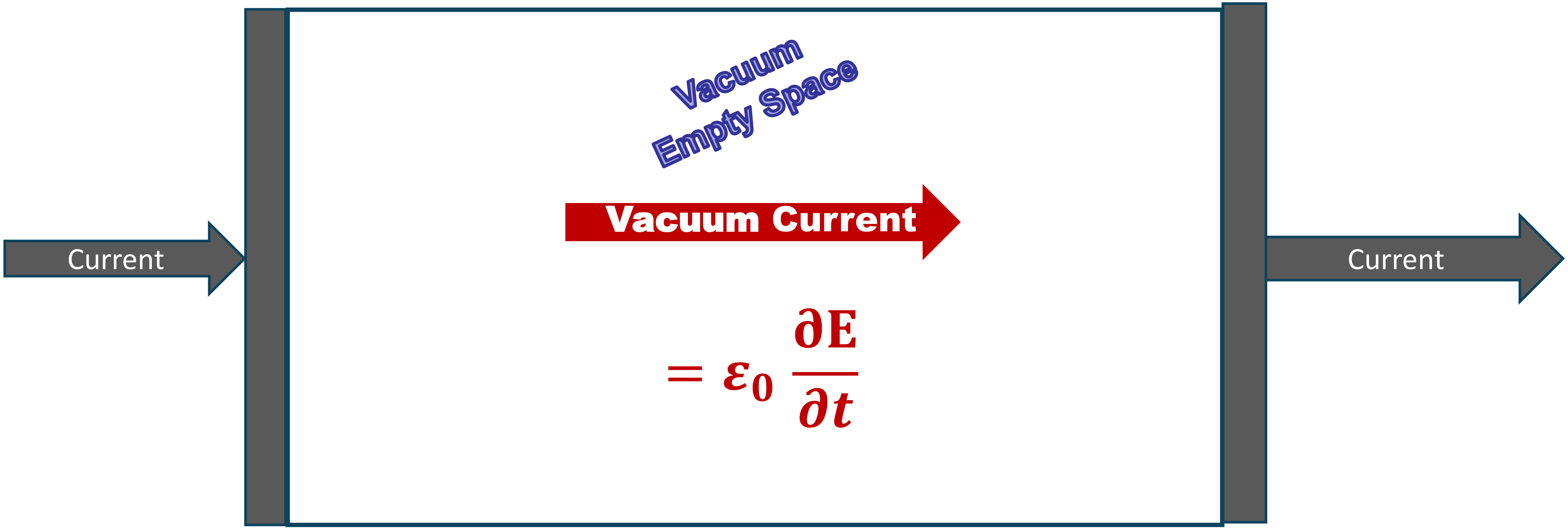
$$\mathbf{curl\ B} = \mu_0 \mathbf{J} + \underbrace{c^{-2}\,\partial\mathbf{E}/\partial t}_{Ethereal\ Current}$$

Source of CURL of Magnetic Field

Ethereal Current = Displacement Current

Note that $\mathbf{div\ B} = \mathbf{0}$

Well known Example of a Capacitor Circuit



Vacuum current = Ethereal current = Displacement Current
All are names for the same thing $\varepsilon_0\, \partial E/\partial t$

Circuits are the Most Used Application of Electrodynamics

1) **Signals in computers (2025) and in telegraphs (~1840)**
2) **Power delivery to computers (2025) and industry (~1890)**

A typical smartphone will contain > 10^9 memory circuits
Google says there are > 10^{10} smartphones

>> 10^{19} Circuits in the world

Circuits Implement the Devices of Engineering

Engineering is About Devices

From Telegraphs to Telephones to Integrated Circuits

Device Converts an Input to an Ouput

By a simple 'law'

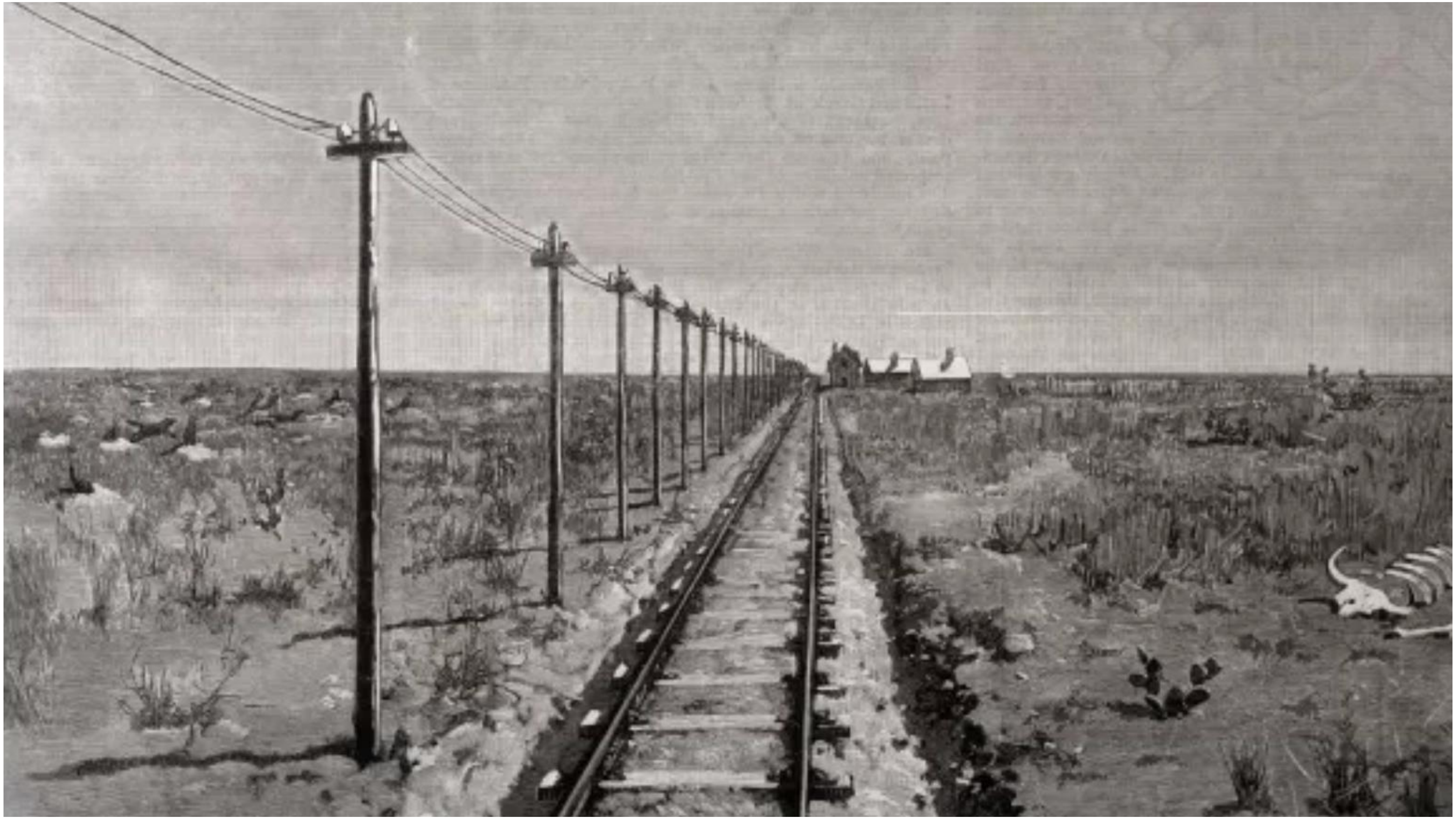
Devices are Useful because they are robust and transferrable

Devices and Circuits do what they are supposed to do

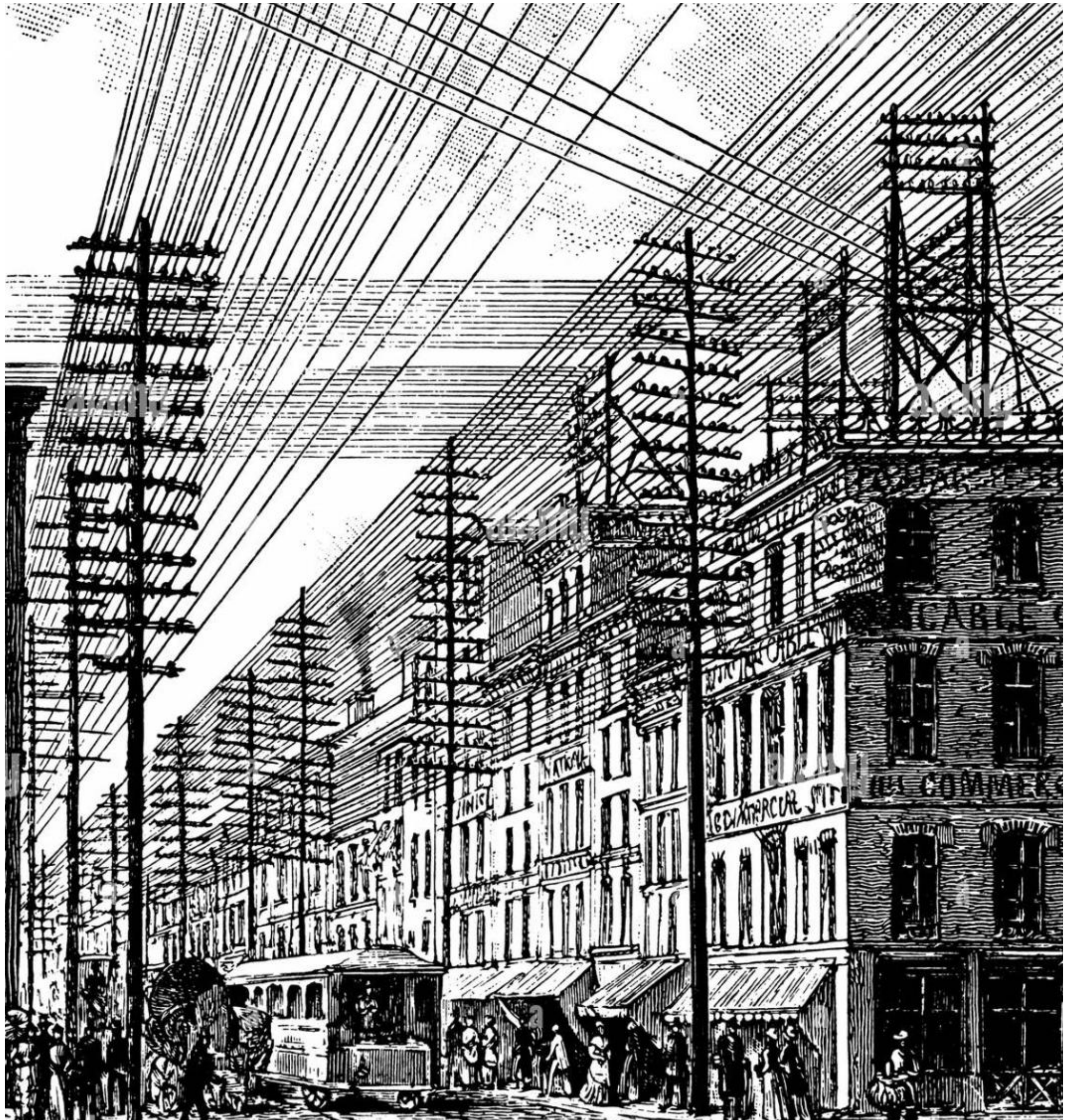
In chemistry and biochemistry, models are often NOT transferrable.

s

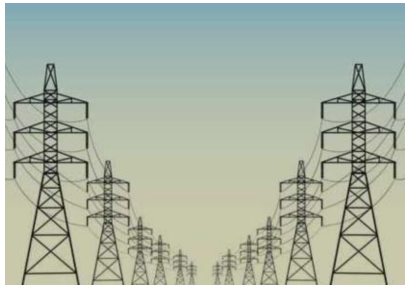
Telegraph Circuits in American West around 1850

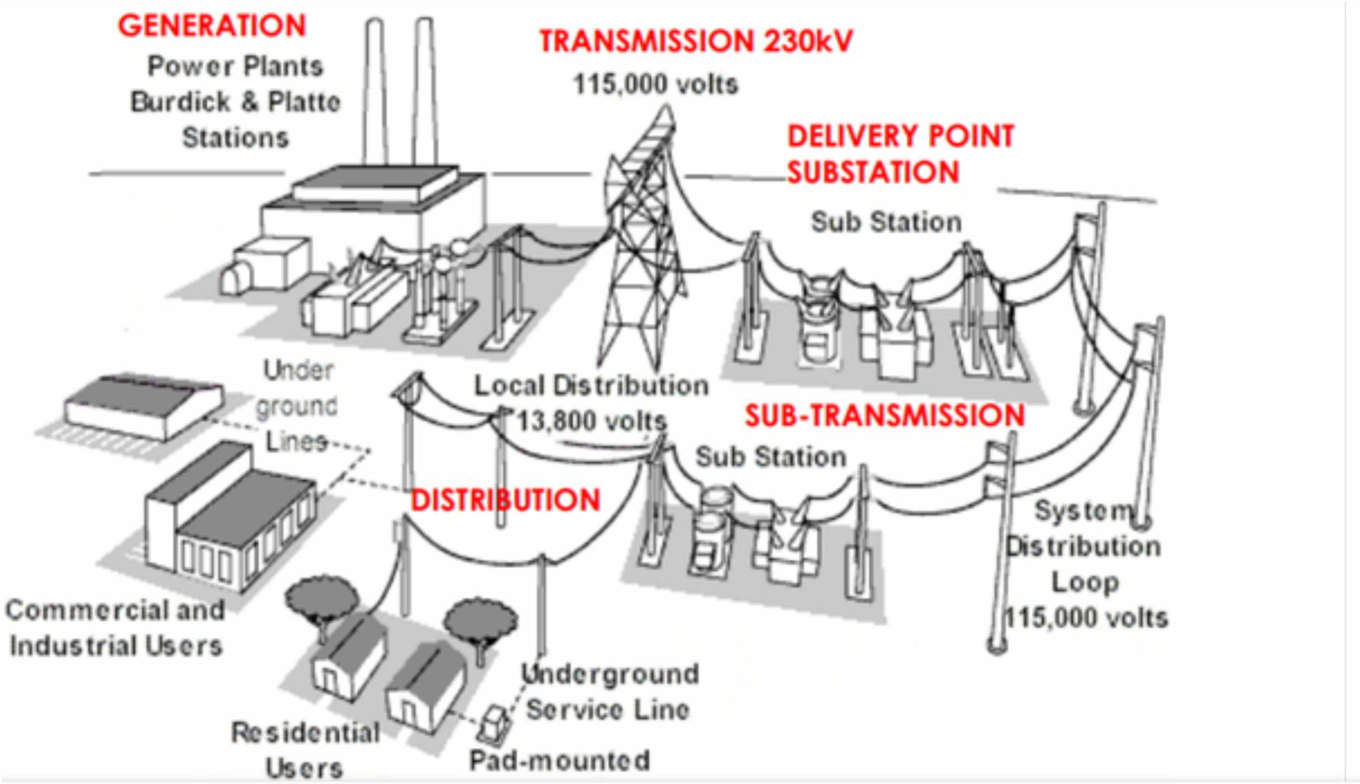


Telegraph and Telephone Wires Philadelphia 1890



Circuits Power Everything



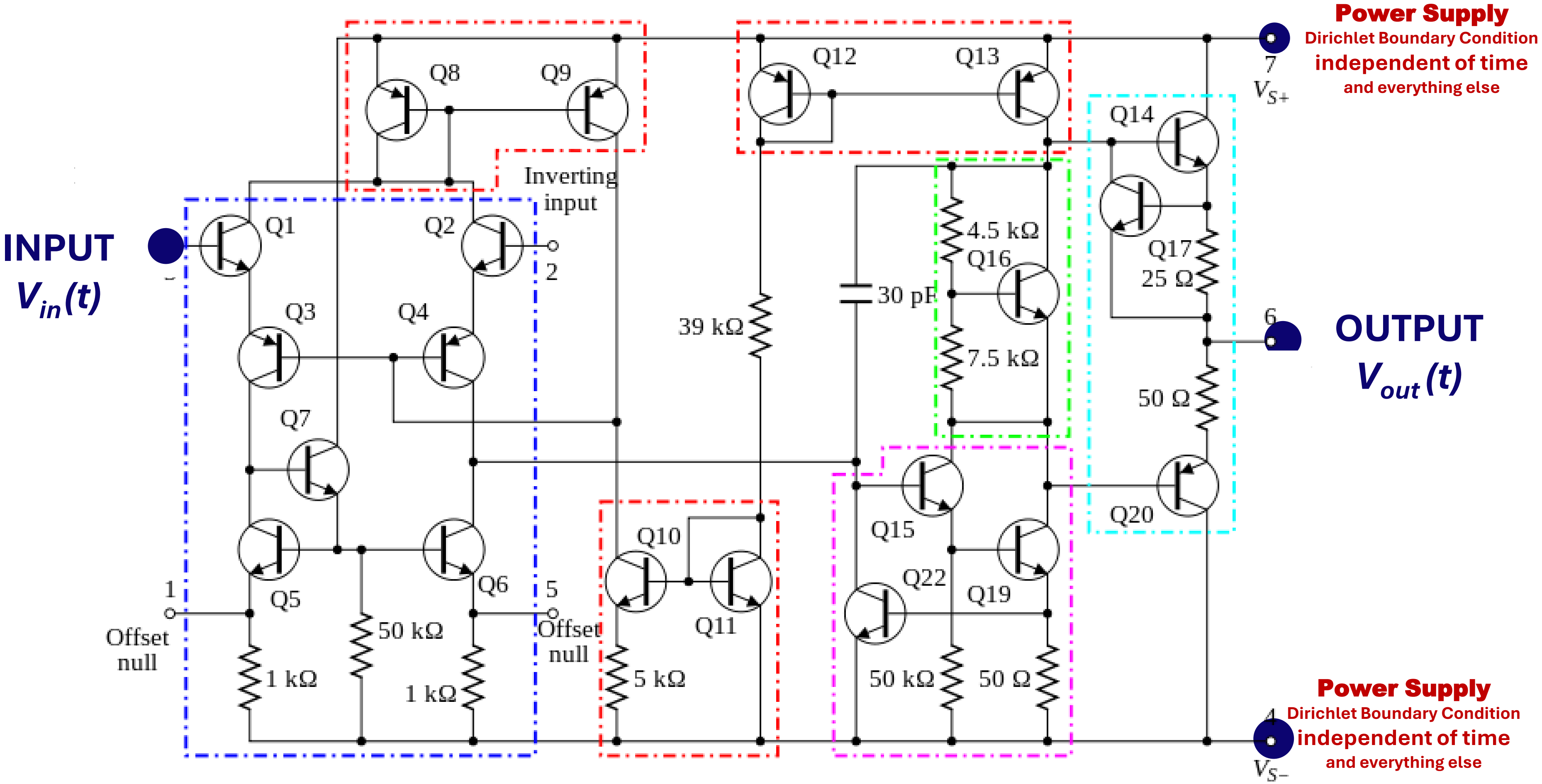


Circuits Power our Homes and Offices

Device converts Input to Output **by a simple 'law'**

Circuits create Devices

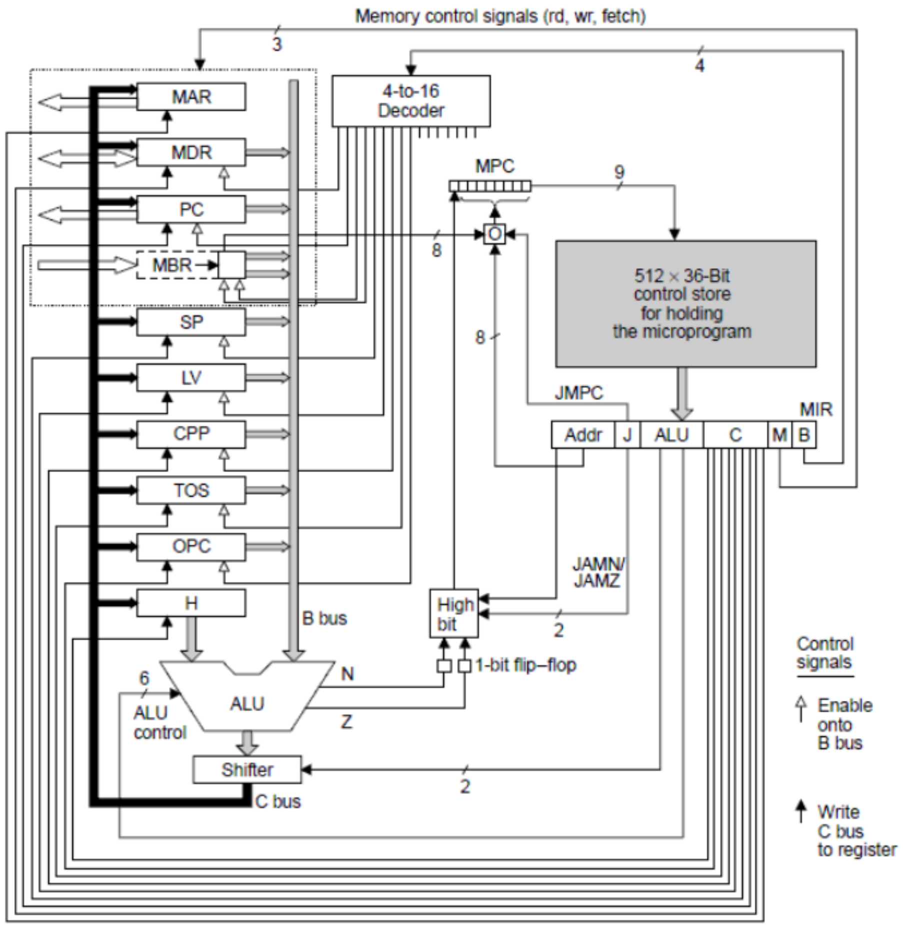
Circuit Diagram of common 741 op-amp: Twenty transistors needed to make linear robust device



Dotted lines outline: current mirrors (red); differential amplifiers (blue);
class A gain stage (magenta); voltage level shifter (green); output stage (cyan).

Arithmetic Logic Unit Circuit

'The Brain' of Computers

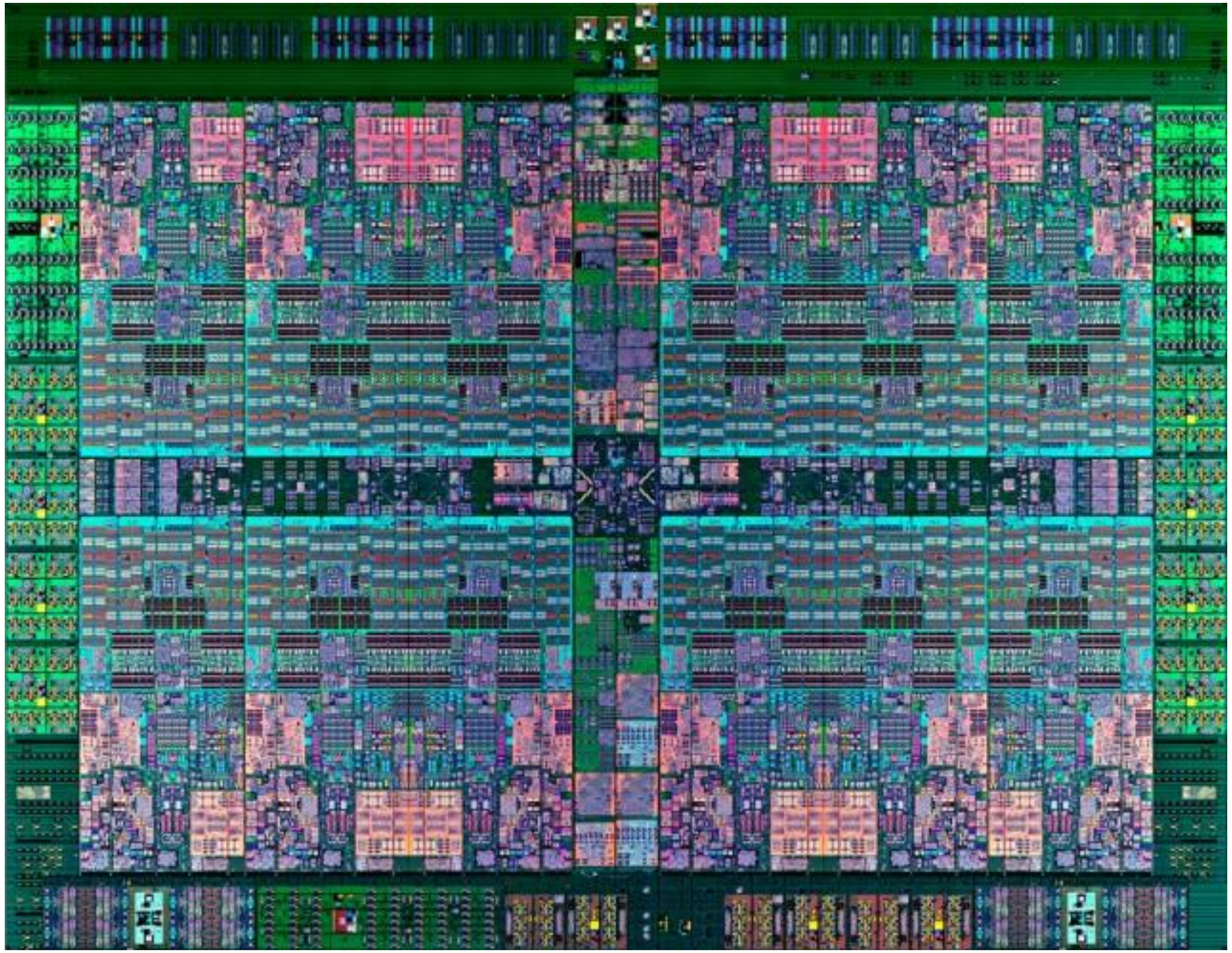


Integrated Circuit

Technology as of ~2014

IBM Power8

Too small to see!

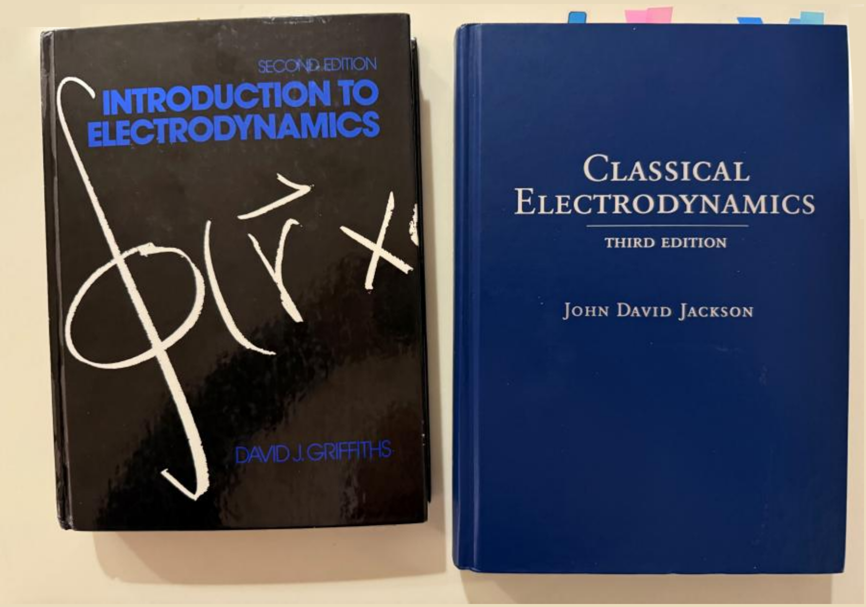


How derive properties of circuits/devices from the Maxwell Equations?

<u>Circuits are not mentioned</u> in the indexes of widely used textbooks of electrodynamics

Griffiths, D.J. 2017. *Introduction to Electrodynamics, Fourth Edition (Cambridge University Press).*

Jackson, J.D. 1999. *Classical Electrodynamics, Third Edition (Wiley: New York).*



Standard Methods to analyze circuits (for more than a century) use Kirchhoff's Laws

Circuits are taught in high school physics and university engineering to millions of students every year

Every electrician knows that current only flows in a complete circuit.
Incomplete circuits block current flow. Why?

Circuits and Kirchhoff's Laws are not mentioned in the indexes of widely used textbooks of electrodynamics

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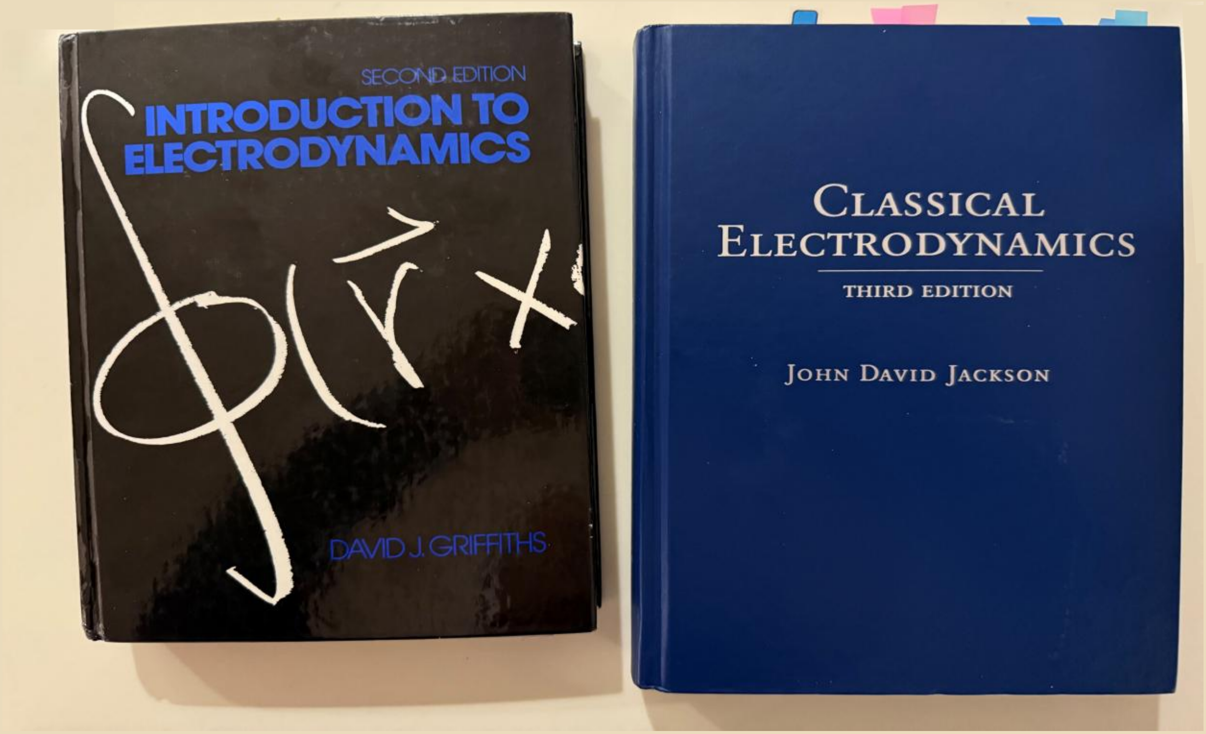


How derive properties of circuits from the Maxwell Equations?

Circuits are taught in high school physics and university engineering to millions of students every year

Every electrician knows that current only flows in a complete circuit.

Incomplete circuits block current flow.

Why?

i.e., derive from Maxwell Equations

CHALLENGE TO AUDIENCE

Cannot answer from Gauss Law or Coulomb's Law

Charges in Circuits Cannot be Computed.

Too Many Charges!!!

Charges interact by Gauss' Law (i.e., Coulomb's equation)

Number of pairwise interactions is something like $(10^{10}$ factorial$)^2$ **This is a fantastically large number**

Has anyone even tried?

Smallest Circuits Involve Flows of Nanoamps of Current for Seconds in Many Locations in Nanometer Structures

Nanoamp = 6.2 x 10^9 charges per second

Cannot actually use Gauss' Law/Coulomb Equation

Need to Know ALL charges at all times!!

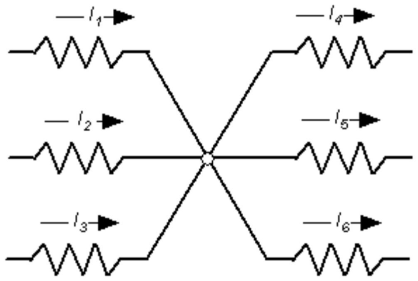
Hopeless to derive circuit laws because all ions interact with each other!

Kirchhoff's Current Law
Brings hope

It is NOT necessary to know all the charges!

Kirchhoff's Current law is (almost) enough

when properly generalized



Coarse Graining is the Answer to the Large Numbers of Charges

But Coarse Graining of Such Large Numbers is Awkward and Inaccurate
(and usually depends on parameters that can vary)

Maxwell Equations Provide an Exact Coarse Graining because they involve the extra physics of relativity and magnetism

in the form of the Maxwell Ampere Law

$$\mathbf{curl\ B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Corollaries of Maxwell Equations

Conservation of Current

Linking Current and Electric Field

Maxwell Ampere Equation $$\mathbf{curl\ B} = \mu_0(\mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial t)$$

Math Identity $$\mathbf{div\ curl\ B = 0} = \mu_0\mathbf{div}(\underbrace{\mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial t}_{\text{Total Current} = \mathbf{J}_{\mathbf{total}}})$$

$$\boxed{\mathbf{div\ J_{total} = 0}}$$

$\mathbf{J}$ = Flux of All Charges with mass, however small or transient

Div Curl is identically zero

for any function that is sufficiently smooth to satisfy the Maxwell equations

1) From derivative definition of div *yellow* and curl *blue*

by substitution and cancellation $for\ vector\ field\ \boldsymbol{F}\ with\ Components\ (F_x; F_y; F_z)$

$$\textbf{Div curl } \boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Schwarz/Clairaut Theorem: $\frac{\partial}{\partial x}\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F_z}{\partial x}; etc.$ ⇨ $\textbf{Div curl } \boldsymbol{F} = \mathbf{0}$

2) From integral definition of div and curl in a tiny volume element

Curl is a circulation integral with terms of equal magnitude but opposite sign

for any function that is sufficiently smooth to satisfy the Maxwell equations.

Divergence and flux have only one value and sign on the surfaces of the tiny element

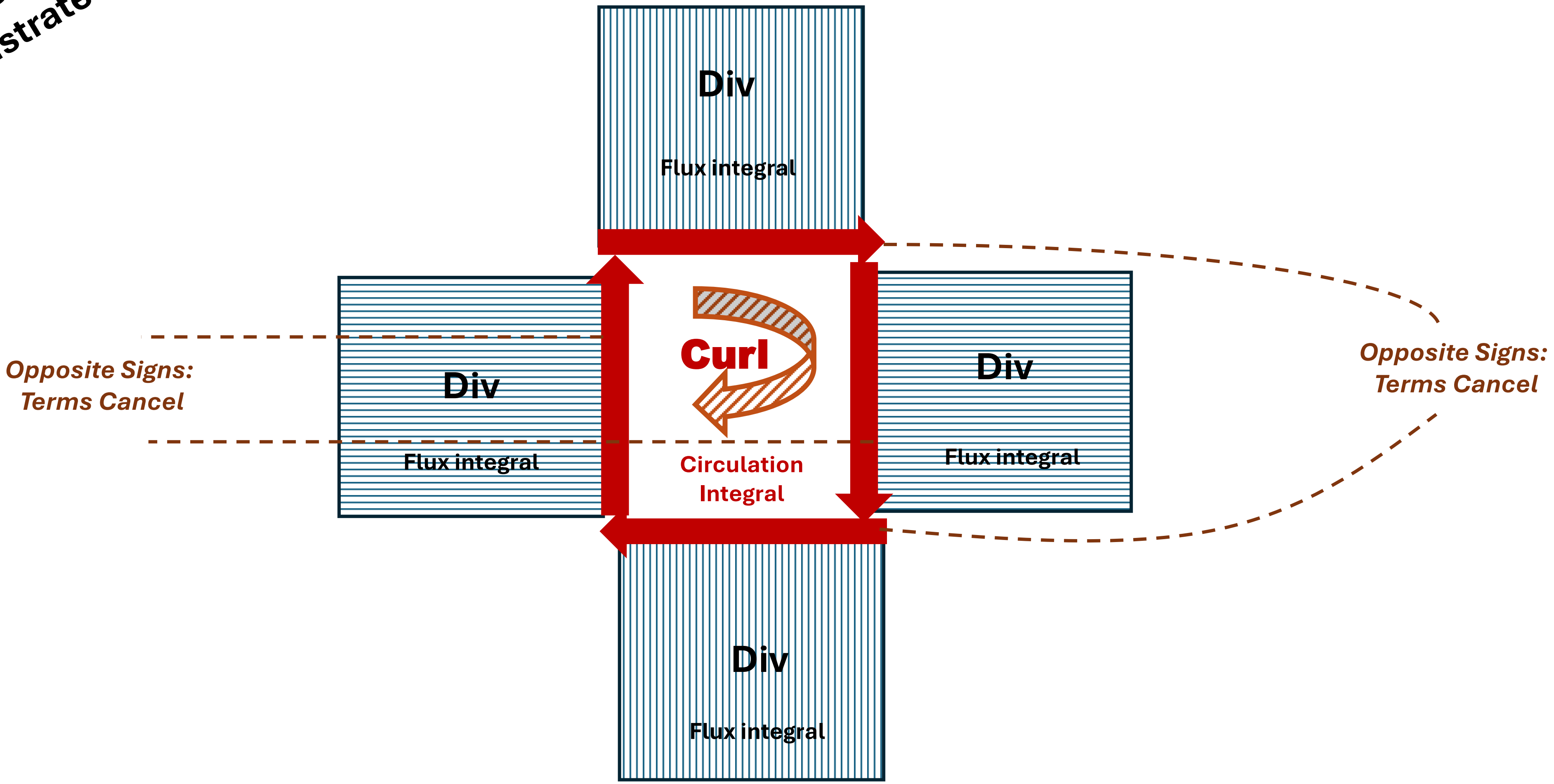
for any function that is sufficiently smooth to satisfy the Maxwell equations

Substitution shows that the terms of the integral cancel.

Div Curl = 0

always

Integral Derivation, illustrated



Div Curl = 0

Math identity

Corollaries of Maxwell Equations

Exact Coarse Graining

$$\mathrm{div}\ \mathbf{J}_{\mathbf{total}} = 0$$

This is a **Generalization of Kirchhoff's Current Law**
that Includes displacement current $\varepsilon_0\ \partial\mathbf{E}/\partial t$

Kirchhoff's Current Law for Circuits
All the Current that Flows into a Node Flows Out

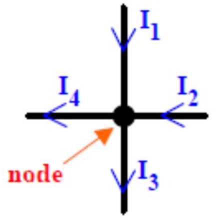
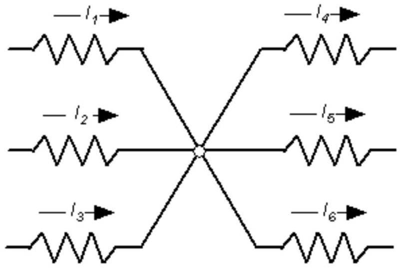


Fig. 2.1 A node with four connected branches



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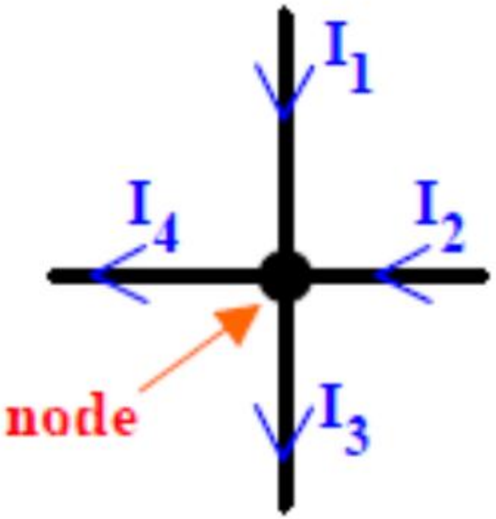

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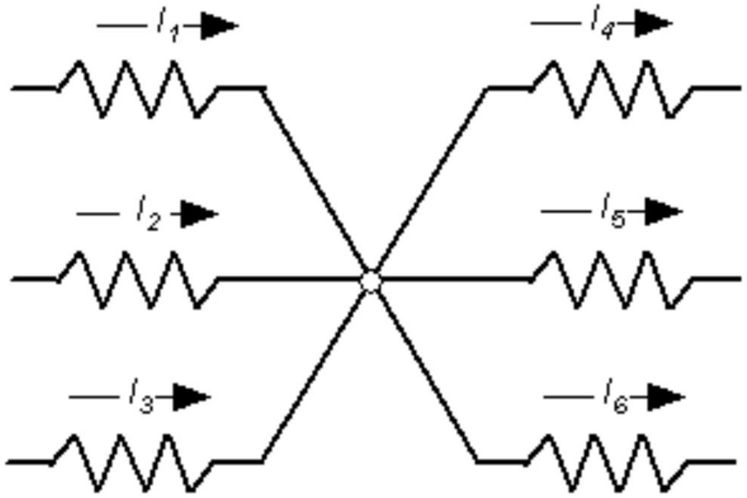


Corollaries of Maxwell Equations

Exact Coarse Graining

$$\mathbf{div}\ \mathbf{J}_{\mathbf{total}} = \mathbf{0}$$

This is a **Generalization of Kirchhoff's Current Law**
that Includes displacement current $\varepsilon_0\ \partial\mathbf{E}/\partial t$

It is a mathematical consequence of the
Maxwell Ampere Law of Magnetism
that I call Maxwell's Current Law
see 'Maxwell's True Current' Computation (2024) 12(2): 22

It is Scary Because it is Universal

Kirchhoff's Law Has been used for nearly two centuries to analyze Complex Circuits Successfully

MANY fewer currents are needed than charges!!!

With Current Law,

Large circuits are

Easily solved

with software available to every engineer like LTspice

Brocard, Gilles. 2013. *The LTspice IV simulator: manual, methods and applications (Würth Elektronik).*

Now we return to the challenge

Question:

Why does Current flow in complete circuits?

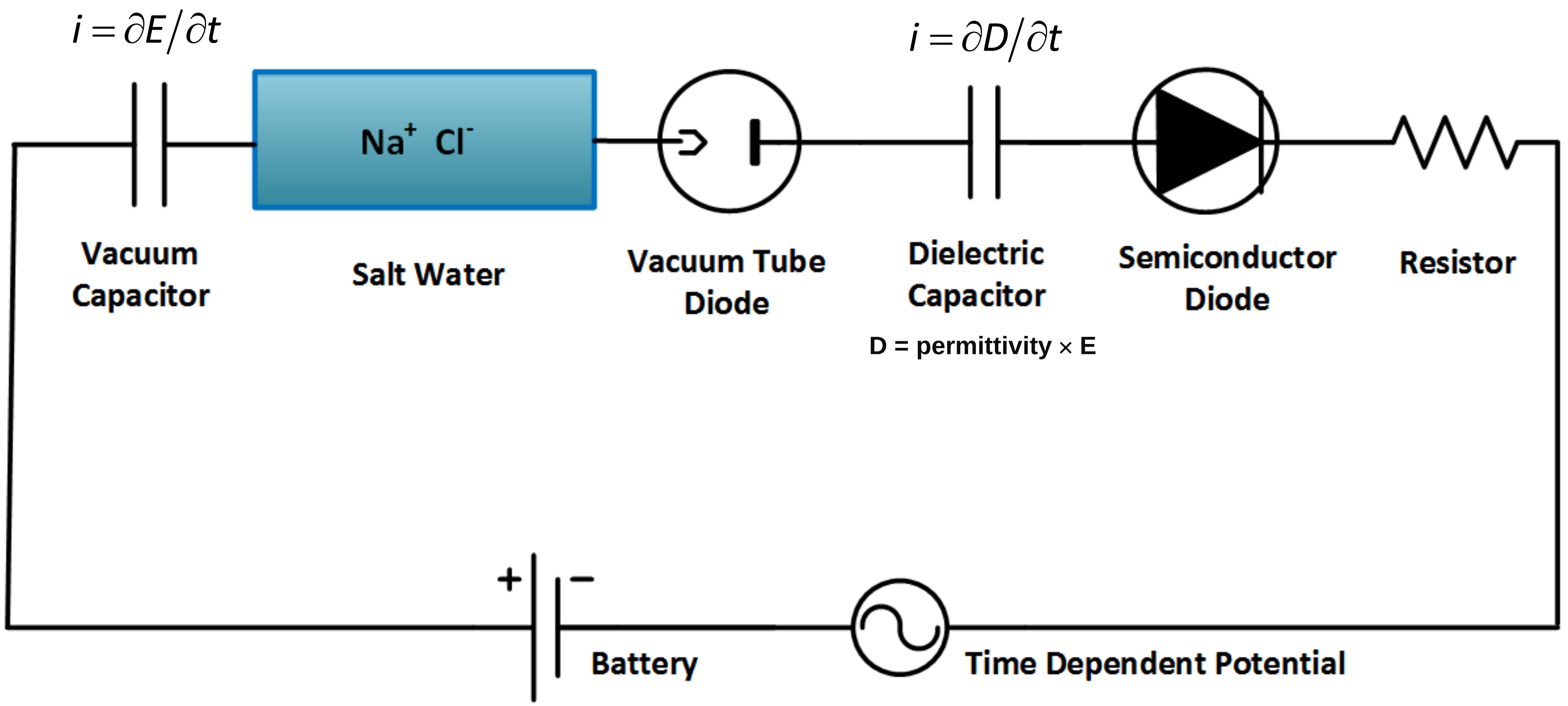
Answer: if the total current is zero in one place
in a series circuit,
it must be zero everywhere!!!

In a series circuit total current is equal everywhere because it has nowhere else it can go. It cannot leave the circuit.
That is what $\text{div}\, \mathbf{J}_{total} = 0$ means.

Question: Why does Current flow in complete circuits?

Answer: in a series circuit total current is equal everywhere

although microphysics is different everywhere



Total Current is a Solenoidal Field

$$\mathbf{div}\ \mathbf{J}_{\mathbf{total}} = \mathbf{0}$$

The total current has no sources or sinks.

$\mathbf{J}_{\mathbf{total}}$ is created only by boundary conditions and dipoles of total current that themselves have zero divergence.

Total Current is a Solenoidal Field

$$\mathbf{div\ J_{total} = 0}$$

1) Charge in solenoidal circuits flows in loops. The loops form circuits for charge movement.

2) Solenoidal flows tend to "slide past" each because of the inherent property of zero divergence:

minimal mixing

3) Circuits in electronic devices isolate the loops.

Isolated loops follow Kirchhoff's laws, generalized to include displacement current.

Solenoidal Flows Greatly Simplify Circuit Design for Devices

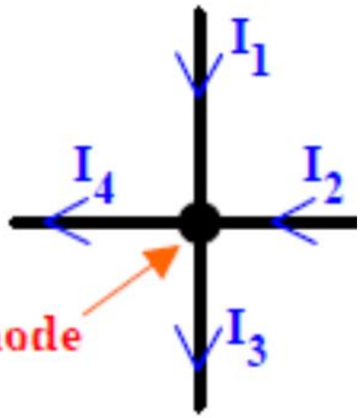
And so Devices work reliably in 0.1 nsec, close to the speed of light.

Light travels about 1 inch in 0.1 nsec

Charge in solenoidal circuits flows in loops
The loops form circuits for charge movement

Circuits in electronic devices
isolate the loops so they interact in simple ways

$$\text{div } \mathbf{J}_{\text{total}} = 0 \text{ is the derivation of Circuit Laws from the Maxwell Equations}$$



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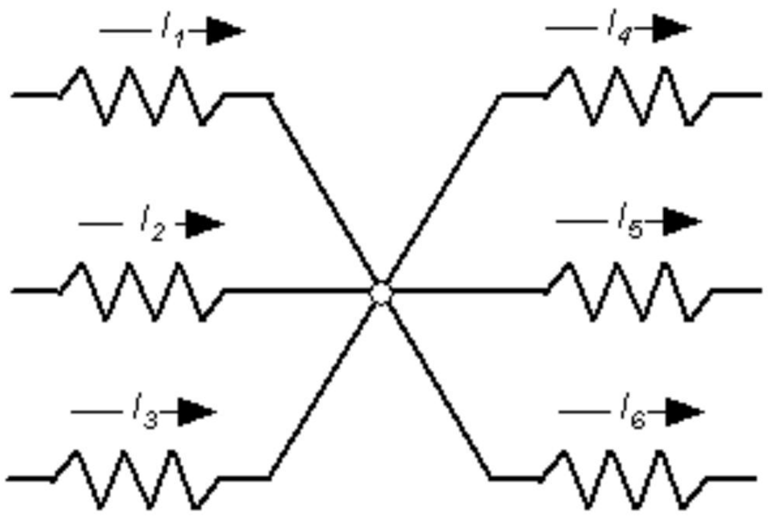


Kirchhoff's Law and $\text{div } \mathbf{J}_{\text{total}} = 0$ become

EQUALITY of Total Current in a Series System

Well known in Electronics

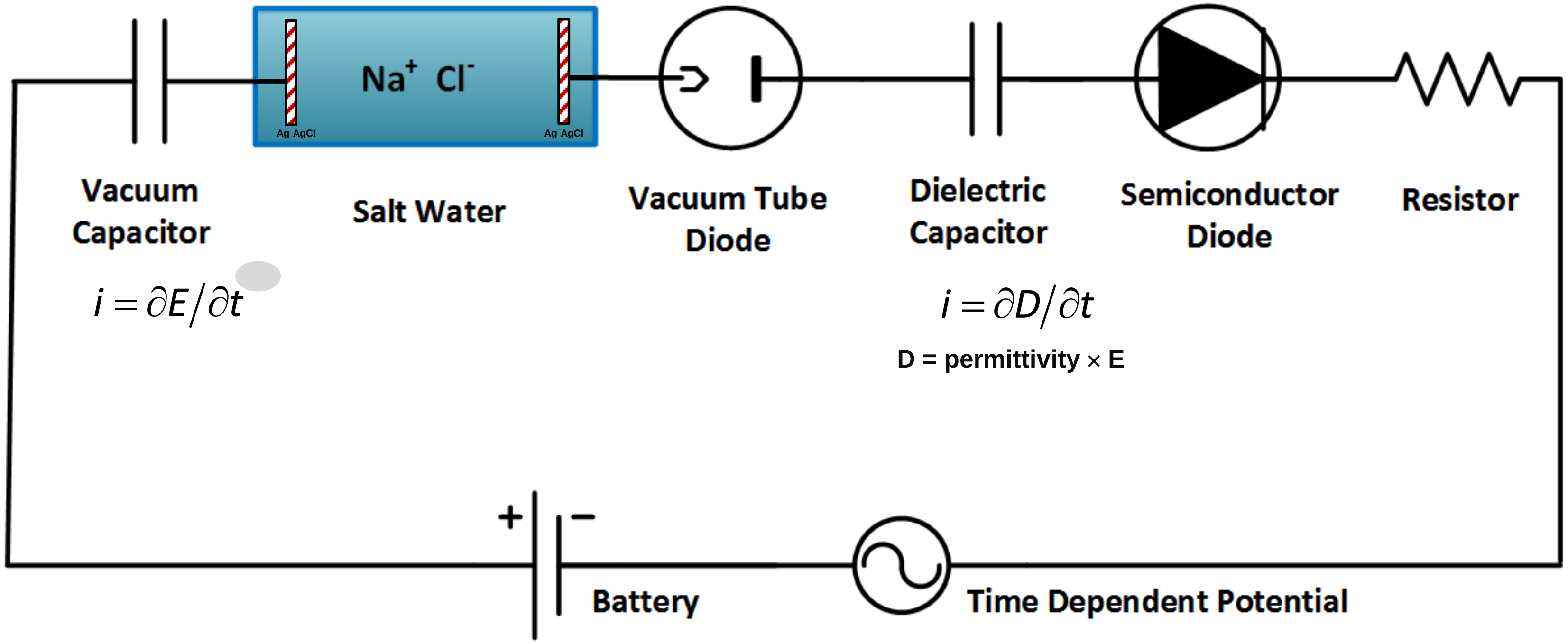
"It is, after all, the **sum** of electron current and **displacement** current which has **no divergence**.
One of those two components can take over from the other."

Landauer (1992) Physica Scripta T42 p 110.

"**Electrodynamic** fields are endowed by **unique** features, including an **exquisite spatial nonlocality**"

Slight paraphrase of Lundeberg *et al* (2017)
Tuning quantum nonlocal effects plasmonics
Science 357:187-191

Flux and Total Current J_{total} have very Different Physics
in Different Systems



but

Continuity of Total Current J_{total} is Exact

$J_{\text{total in Device 1}} = J_{\text{total in Device 2}} = J_{\text{total in Device 3}} \cdots$

no matter what carries the current J
at all times and all locations!

Eisenberg (2016) Mass Action and Conservation of Current.
Hungarian Journal of Industry and Chemistry Posted on arXiv.org with paper ID arXiv:1502.07251 44:1-28.

Conservation of Total Current J_{total} is Exact

even though
Physics of Charge Flow
Varies Profoundly

How can that possibly be?

Electrodynamic Fields

$$\mathbf{E}, \quad \varepsilon_0 \partial \boldsymbol{E} / \partial t, \quad \mathbf{B}$$

take on the

Values that Conserve total Current $\mathbf{J}_{total}$

This is NOT mysterious

E is a force field that moves atoms

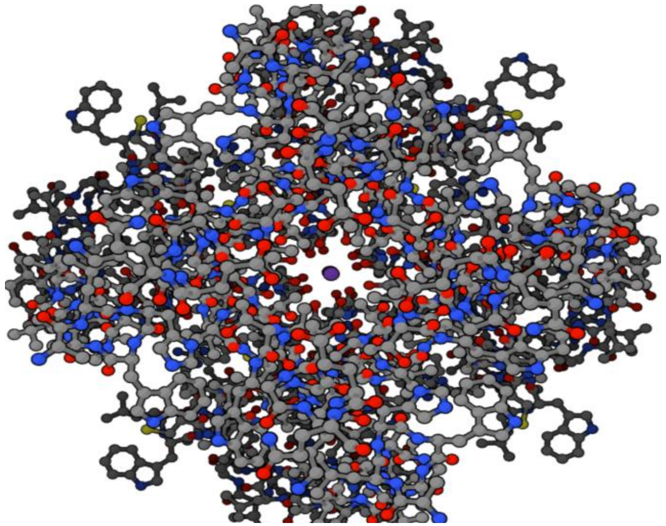
Details and PROOF
including quantum mechanics

Eisenberg, Oriols, and Ferry. 2017. Dynamics of Current, Charge, and Mass.
Molecular Based Mathematical Biology 5:78-115
and arXiv https://arxiv.org/abs/1708.07400

Ferry, Oriols, Eisenberg. 2025.
Displacement Current in Classical and Quantum Systems.
Computation 13, 45 DOI 10.3390/computation13020045

E is a force field that moves atoms

because atoms have charge



Potassium Ion Channel $K_{V_{1.2}}$
***PDB:* 1BL8**

so total current $\mathbf{J}_{total} = \mathbf{J}(x,t) + \varepsilon_0\, \partial \boldsymbol{E}/\partial t$ is always conserved

Details and PROOF including quantum mechanics

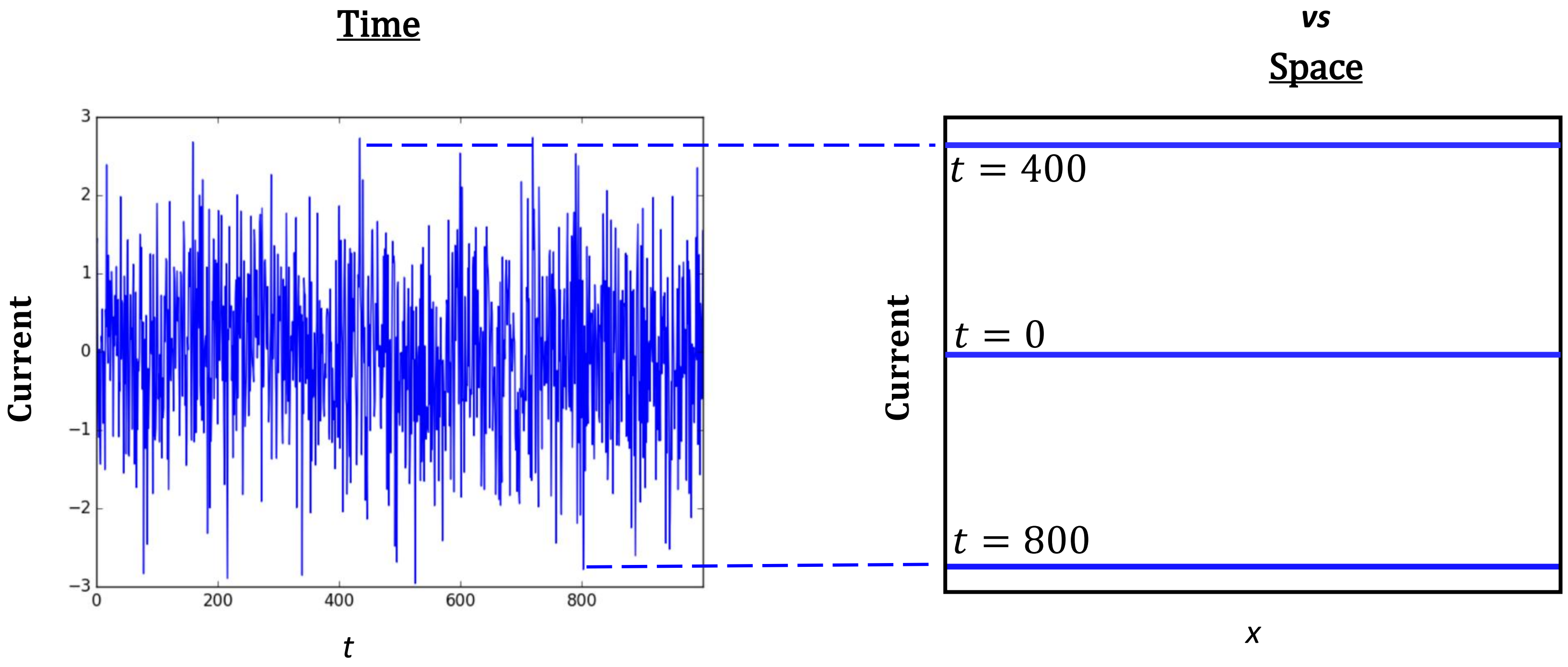
Ferry, Oriols, Eisenberg
Displacement Current in Classical and Quantum Systems.
Computation (2025) 13, 45
DOI 10.3390/computation13020045

EQUALITY of Total Current J_{total} is an Enormous Simplification in Series Systems or Devices

It can create a *Perfect Low Pass Filter*
It can *Convert*
***Chaos* of Brownian Motion**
into a *Constant*

Current Noise J_{total} is Zero in Space

Current Noise
J_{total}
is
HUGE in time



One Dimensional Systems like Channels or Circuit Components

What does this mean for Mathematical Models?

The image of total current flow $\mathbf{J}_{total}$ is very different
VERY SMOOTH in space

Total Current $\mathbf{J}_{total}$ does not vary in space so
<u>Spatial Derivatives are not needed to describe total current</u>
in series systems or devices

Revolution in Biophysics

Total Current flow $\mathbf{J}_{total}$ is equal everywhere in a one dimensional channel

Thermal Motion in Space does <u>not</u> appear in equations for flow of total current $\mathbf{J}_{total}$ in a one dimensional channel

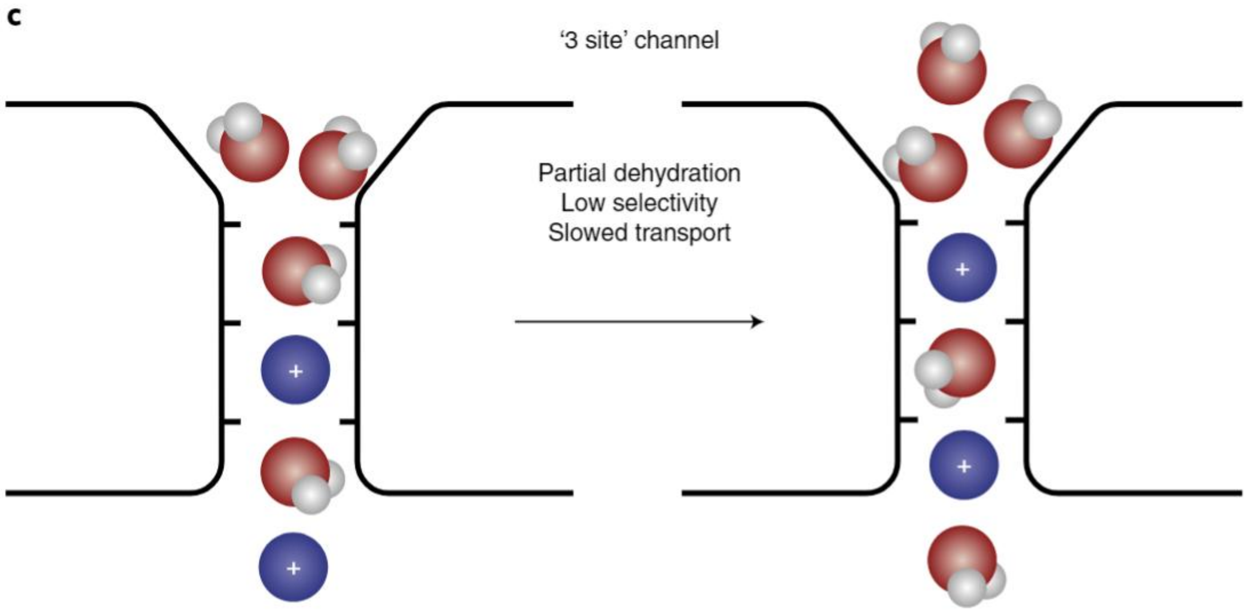
Thermal motion appears ONLY in time

Eisenberg (2020)
Electrodynamics Correlates Knock-on and Knock-off: Current is Spatially Uniform in Ion Channels.
Preprint on arXiv at https://arxiv.org/abs/2002.09012.

What does this mean for Ion Channels?

Knock On and Knock Off of Ions
is
IRRELEVANT for the Total Current J_{total} Through the Channel

Paradigm Change



Corry (2018) The naked truth about K^+ selectivity. Nature Chemistry 10:799-800.

Eisenberg (2020)
Electrodynamics Correlates Knock-on and Knock-off: Current is Spatially Uniform in Ion Channels.
Preprint on arXiv at
https://arxiv.org/abs/2002.09012.

View of Channels has been focused on movements of individual ions in channels,
But
Total Current J_{total} is equal everywhere in a one dimensional channel

Paradigm Change

Position does <u>not</u> appear in equations for total current J_{total} in a one dimensional channel

References and Proofs in

Eisenberg (2019) **Kirchhoff's Law can be Exact**. arXiv: 1905.13574

Eisenberg, Gold, Song, and Huang (2018)
What Current Flows Through a Resistor?
arXiv:1805.04814

From Maxwell Equations to Circuits to Ion Channels

Any Questions??

Slides, References
Bob.Eisenberg@gmail.com
Discussion

Extra Slides

The Electric Field is Strong

If you were standing at arm's length from someone and each of you had

One percent **more electrons than protons,**

the force
would lift the

Entire Earth!

slight paraphrase of third paragraph, p. 1-1 of
Feynman, R. P., R. B. Leighton, and M. Sands. 1963. ***The Feynman: Lectures on Physics, Mainly Electromagnetism and Matter.*** New York: Addison-Wesley Publishing Co.,
also at *http://www.feynmanlectures.caltech.edu/II_toc.html.*

Coulomb's Law in Chemical Units

MANY times larger than thermal energy

$$E_{coulomb} = \frac{560}{\varepsilon_r}\frac{q_i q_j}{r_{ij}}$$

units: $k_B T \cdot N_A$ = **Thermal Energy**

$$E_{coulomb} = 280\frac{q_i q_j}{r_{ij}}$$ *on molecular dynamics time scale* 10^{-15} *sec*

$$E_{coulomb} = 7\frac{q_i q_j}{r_{ij}}$$ *in water*

E in units of thermal energy $RT = k_B T \cdot N_A$; $N_A = 6 \times 10^{23}$

$q_{i,j}$ in units of elementary charge; r_{ij} in Angstrom

ε_r is dielectric coefficient, ~ 80 for pure water

Electric Field is Strong and Important

MANY times the thermal energy
MANY times larger than diffusion

Electric Field is Important

Chemistry and Biology

“... **all forces**

on atomic nuclei in a molecule
can be considered as
purely classical attractions
involving

Coulomb’s law.”

“The electron cloud distribution is prevented from collapsing by obeying Schrödinger’s equation.”

R.P. Feynman (1939)
Forces in Molecules.
Physical Review 56: 340.

Corollaries of Maxwell's Core Equations

Derivation of the Continuity Equation

Linking Flux and Content

$$\mathbf{curl\ B} = \mu_0 \left(\overbrace{\mathbf{J}(x,t)}^{\text{Flux of All Charges}} + \varepsilon_0\, \partial \mathbf{E}/\partial t \right)$$

$$\mathbf{div\ curl\ B} = 0 = \mu_0\, \mathbf{div}(\mathbf{J}(x,t) + \varepsilon_0\, \partial \mathbf{E}/\partial t)$$

$$\mathbf{div\ J}(x,t) = -\,\varepsilon_0\, \mathbf{div}\, (\partial \mathbf{E}/\partial t) = -\,\varepsilon_0\, \boldsymbol{\partial}(\mathbf{div\ E})/\boldsymbol{\partial t}$$

But $\mathbf{div\ E} = \rho/\varepsilon_0$

$$\boxed{\mathbf{div\ J} = -\,\partial\rho/\partial t}$$

Corollaries of Maxwell's Core Equations

Continuity Equation

Linking Flux and Content

$$\mathbf{div\ J} = -\frac{\partial \rho}{\partial t}$$

Question for Students

This is not a useful equation on atomic scale

Why?

Continuity Equation

Linking Flux and Content

Feynman's Hidden Implications

$$\textbf{div J} = -\frac{\partial \rho(x,y,z|t)}{\partial t}$$

Oh Dear!

Often impossible

Must know all charges $\rho(x,y,z|t)$

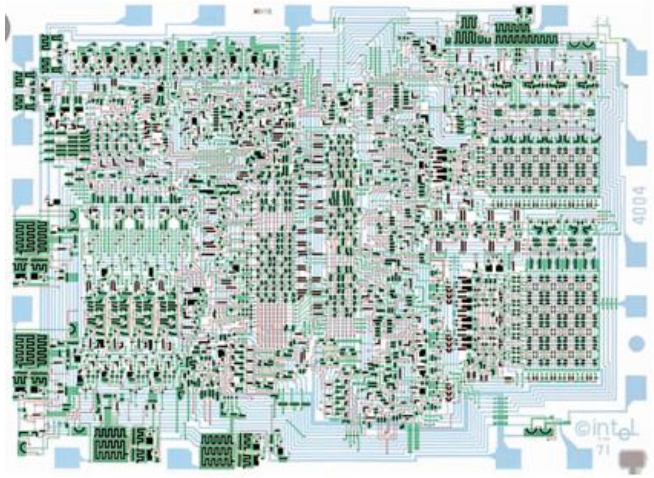
at all times

Continuity Equation

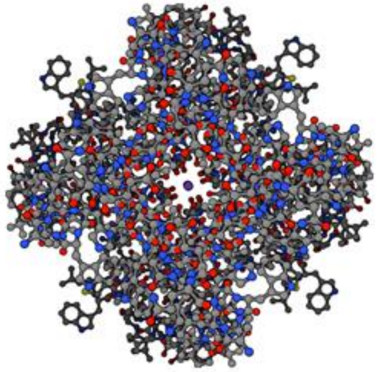
Feynman's Hidden Implications

Must know all charges and how they move

$$\mathbf{div\ J} = -\frac{\partial \rho(x, y, z|t)}{\partial t}$$

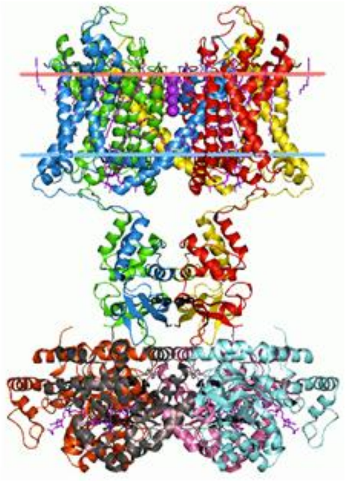


Integrated Circuit



Potassium Ion Channel $K_{V_{1.2}}$

PDB: 1BL8



Source Internet

Hopeless, if one must

"… exhibit in every case all the charges, whatever their origin" at all times

Section 10-4 of Feynman, Leighton, and Sands (1963) vol. 2 ***Electromagnetism and Matter***

Without Conservation of Current

Need to Know ALL charges at all times!!

Hopeless in large systems
where all ions interact with each other!

Charge and Electricity

Gauss Law

$$\mathbf{div\ E} = \frac{\rho}{\varepsilon_0}$$

or

Poisson Equation

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon_0}; \quad -\nabla\varphi = E$$

or

Coulomb Law

$$\mathbf{E} = \frac{560}{\varepsilon_r}\sum\frac{q_i q_j}{r_{ij}}$$

in chemical units kT/e = RT/F

$\mathbf{E}$ is the electric field, $\boldsymbol{\varphi}$ is the electrical potential; thermal energy of diffusion is kT/e = RT/F
q_i, q_j are charges r_{ij} separation of charges
ρ is the the charge density of all types including brief dielectric transients of the **P** and **D** fields
ε_0 is the electrical constant, the permittivity of a vacuum

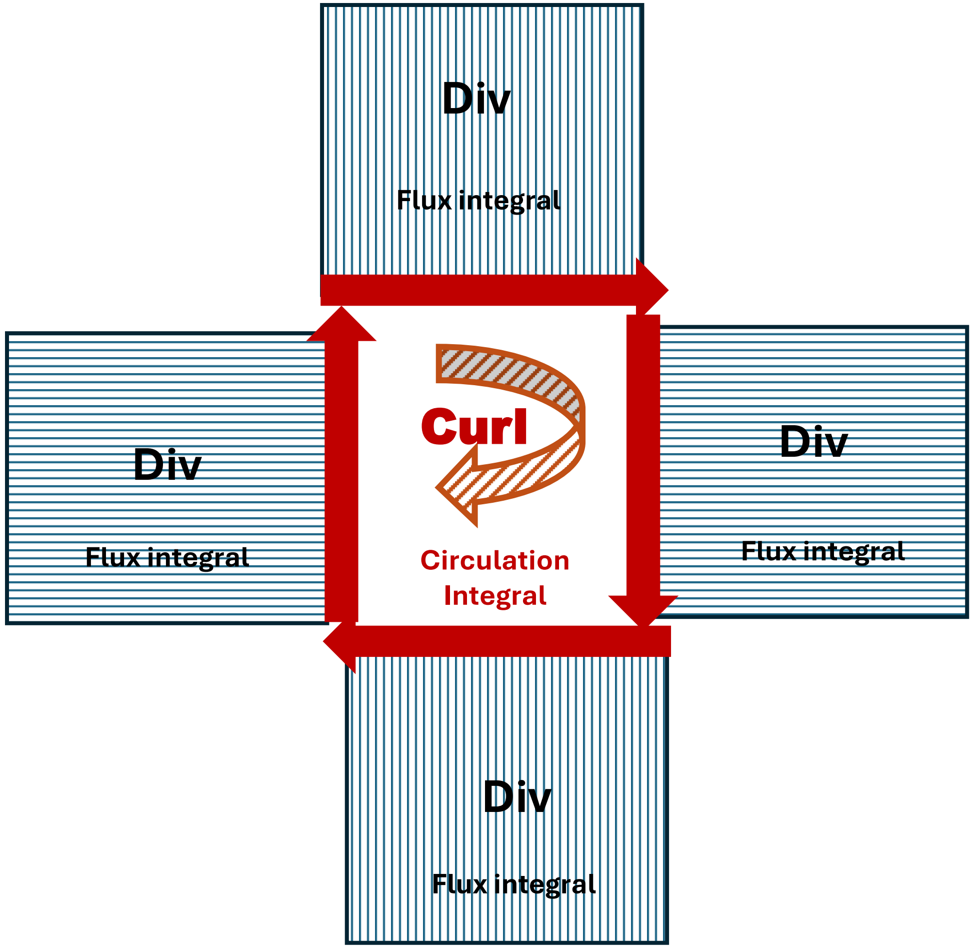
$$For\ Vector\ Field\ \boldsymbol{F}\ with\ Component\ (F_x; F_y; F_z)$$

$$\textbf{Div curl } \boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right); \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial y}\right); \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Schwarz Theorem: $\frac{\partial}{\partial x}\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F_z}{\partial x}; etc.$

$$\textbf{Div curl } \boldsymbol{F} = \mathbf{0}$$

Div Curl = 0
always

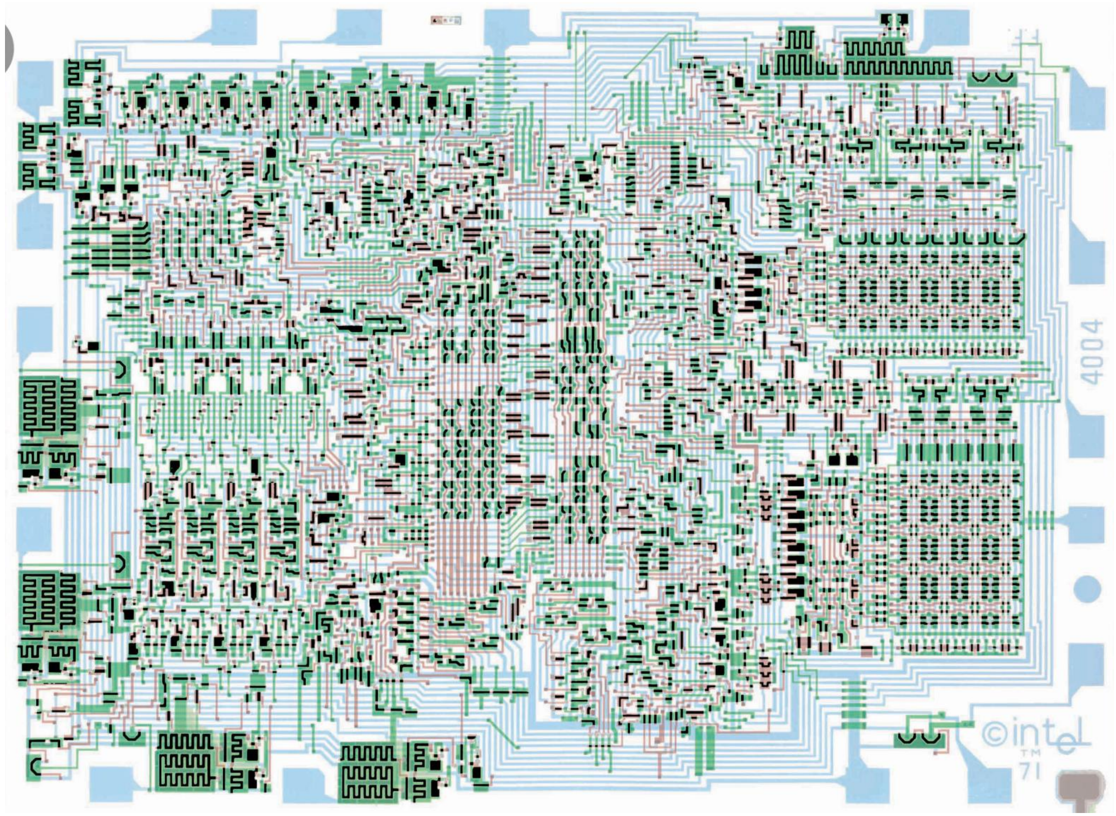


Div Curl = 0
Math identity

EQUALITY of Total Current J_{total} is an Enormous Simplification

Eisenberg, B., N. Gold, Z. Song, and H. Huang. 2018. What Current Flows Through a Resistor? arXiv preprint https://arxiv.org/abs/1805.04814.
Eisenberg, R. S. 2019. Kirchhoff's Law can be Exact. arXiv preprint available at https://arxiv.org/abs/1905.13574.

integrated circuit



Source: textbooks and internet

Seems Hopeless

Fortunately,
it is not hopeless

Current flow is very smooth in spatial coordinate

Differential equation in x is not needed for J_{total}

$$J_{total} = J + \varepsilon_0\, \partial E/\partial t$$

What does this mean for theory and simulations?

Opportunity to Simplify Algorithms and Codes

perhaps dramatically

Spatial Dependence is Already Known

Only have to average the time dependence

Ma, Li and Liu (2016). arXiv:1605.04886; Ma, Li and Liu (2016). arXiv:1606.03625.

Current flow is very smooth in spatial coordinate

Differential equation in x is not needed for $\mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\, \partial E/\partial t$

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Ma, Li and Liu (2016). arXiv:1605.04886; Ma, Li and Liu (2016). arXiv:1606.03625.

Maxwell's Core Equations are Universal and Exact

But they are Complicated Differential Equations

need very complicated mathematics to describe **universal** physics

Electro 'statics'

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Magneto 'statics'

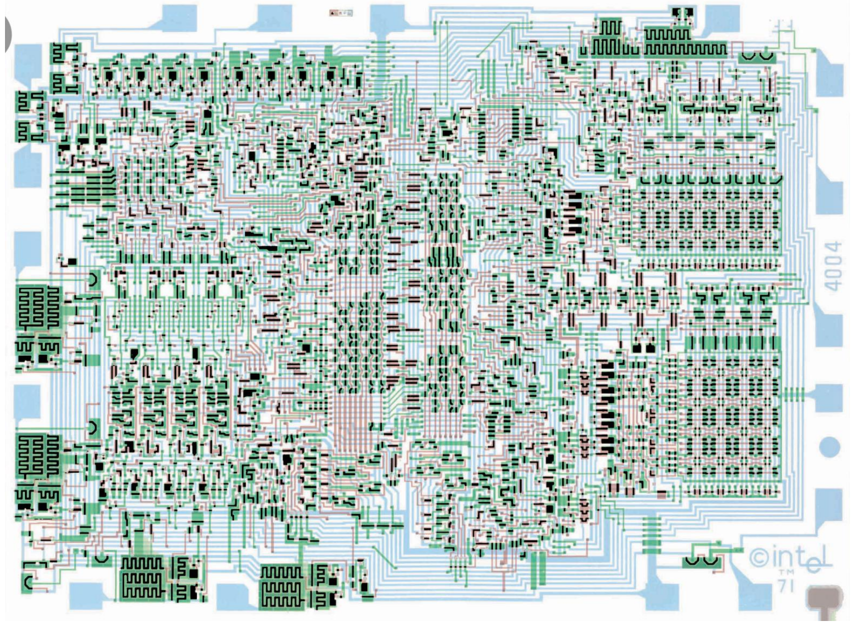
$$\nabla \cdot \mathbf{B} = 0$$

Electrodynamics

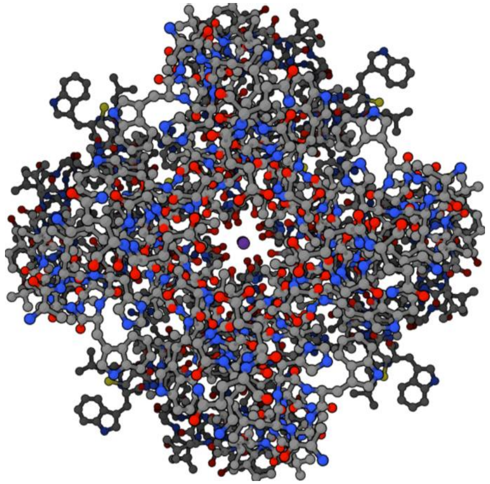
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Magnetodynamics

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



Integrated Circuit



Potassium Ion Channel $K_{V_{1.2}}$

***PDB:* 1BL8**

or