Electric charge motion, induced current, energy balance, and noise

B. Pellegrini
Istituto di Elettronica e Telecomunicazioni, Università di Pisa, Via Diotisalvi 2, 56100 Pisa, Italy
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A general equation is derived which, through an arbitrary irrotational vector field, connects the electric charge carrier motion and the dielectric property of a system with its external currents, voltages, and powers. It is applied to generalize Ramo's theorem to any conducting medium and boundary condition, to compute the components of the energy balance, and to obtain a new, general, and straightforward relationship between carrier velocity and output noise which, to show its potentiality, is used here to evaluate the thermal noise.

For studying an electric system one often needs to know the current induced in a given electrode by the motion of a single or many electric charge carriers occurring in it.

The problem is solved by Ramo's theorem[1] for vacuum tubes and ionization chambers and by its extension, due to Cavalleri, Gatti, Fabbri, and Svelto,[2] for semiconductor detectors in which the electric charge carrier is moving in a space-charge region and, as well in the preceding case, all the n electrodes are maintained at constant potentials V_h $(h=1,\ldots,n)$.

In this work, by computing, over a given region of volume Ω and surface S, the integral of the scalar product $\mathbf{J}\cdot\mathbf{F}$ between the current density $\mathbf{J}$ and an arbitrary irrotational vector $\mathbf{F}$, we obtained a general equation which allows us to determine the currents induced by the carrier motion not only when $V_h(t)$ depends on the time t but also when, for whatever medium, the dielectric properties themselves are functions both of t and of the space $\mathbf{r}$.

By giving suitable values to the arbitrary vector $\mathbf{F}=-\nabla\phi$ or to its potential $\phi=U$ on the boundary surface S, such an equation also allows one to verify the energy balance, to prove Kirchoff's law for the currents, and to obtain a new general relationship suitable for computing the current noise due to the velocity fluctuations of the single carriers.

I. EQUATION DERIVATION

Let the conduction current density $\mathbf{J}_c(\mathbf{r},t)$ be given by

$$\mathbf{J}_c=\sum_{i=1}^{N} q_i\mathbf{v}_i(\mathbf{r}_i)\delta(\mathbf{r}-\mathbf{r}_i) , \tag{1.1}$$

where q_i, $\mathbf{v}_i$, and $\mathbf{r}_i(t)$ are the charge, the velocity, and the position, respectively, of the ith charge carrier of the $N(t)$ carriers contained in Ω at t; $\mathbf{v}_i$ is the group velocity if the carrier is described by a quantum-mechanical wave packet.

Moreover, let $\mathbf{E}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, and $\mathbf{J}_d=\partial\mathbf{D}/\partial t$ be the electric field, electric displacement, and displacement current density, respectively, while for an isotropic medium, after the varible substitution $\mathbf{r}'=\mathbf{r}+\boldsymbol{\rho}$ and $t'=t-\tau$, in the most general form we have

$$\mathbf{D}=\int d\mathbf{r}'\int_{-\infty}^{t}dt'\varepsilon'(\mathbf{r},\mathbf{r}';t,t')\mathbf{E}(\mathbf{r}',t') =\int d\boldsymbol{\rho}\int_0^{\infty}d\tau\,\varepsilon(\mathbf{r},\boldsymbol{\rho};t,\tau)\mathbf{E}(\mathbf{r}+\boldsymbol{\rho},t-\tau)=\Upsilon(\varepsilon\mathbf{E}) , \tag{1.2}$$

$$\mathbf{J}_d=\Upsilon(\partial(\varepsilon\mathbf{E})/\partial t) , \tag{1.3}$$

where, and afterwards, for simplicity, we use the integral operator representation

$$\Upsilon=\int d\boldsymbol{\rho}\int_0^{\infty}d\tau ,\quad \Upsilon_\Omega=\int_\Omega d\mathbf{r}\int d\boldsymbol{\rho}\int_0^{\infty}d\tau , \tag{1.4}$$

Ω being the $\mathbf{r}$ integration region.

Since the electric polarization at $\mathbf{r}$ and t tends to zero when ρ and τ increase, if $\mathbf{E}$, with regard to ε, is a slowly varying function of ρ and τ themselves, we can consider that in the integrals $\varepsilon=\varepsilon_d(\mathbf{r},t)\delta(\boldsymbol{\rho})\delta(\tau)$ and, in particular, if ε is time independent too,

$$\varepsilon(\mathbf{r},\boldsymbol{\rho};t,r)=\varepsilon_0(\mathbf{r})\delta(\boldsymbol{\rho})\delta(\tau) , \tag{1.5}$$

$\varepsilon_0(\mathbf{r})$ being the static, local electric permittivity;[3] in such cases, in particular, from (1.2) and (1.3) we also have $\mathbf{D}=\varepsilon_b E$ and $\mathbf{J}_d=\partial(\varepsilon_b\mathbf{E})/\partial t$ with $\varepsilon_b=\varepsilon_d$ or $\varepsilon_b=\varepsilon_0$.

For the total curent density $\mathbf{J}$ we have finally

$$\mathbf{J}=\mathbf{J}_c+\mathbf{J}_d,\quad \nabla\cdot\mathbf{J}=0 . \tag{1.6}$$

Now let us consider an irrotational vector point function $\mathbf{F}(\mathbf{r},t)$ defined throughout the region Ω by

$$\mathbf{F}=-\nabla\phi\quad(\mathbf{r}\text{ in }\Omega) , \tag{1.7}$$

$$\phi=U(\mathbf{r},t)\quad(\mathbf{r}\text{ in }S) \tag{1.8}$$

for any arbitrary given continuous single-valued function $U(\mathbf{r},t)$ which also depends on t.

The objective is computing, over Ω, the integral

$$\Gamma=\int_\Omega d\mathbf{r}(\mathbf{F}\cdot\mathbf{J}) . \tag{1.9}$$

For this let us assume that $\mathbf{E}=-\nabla\psi$ is irrotational too (ψ being its scalar potential[4]) and let us recall that for two

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vector point functions $\mathbf{G}$ and $\mathbf{L}=\nabla\Lambda$,

$$\mathbf{G}\cdot\mathbf{L}=\mathbf{G}\cdot\nabla\Lambda=\nabla\cdot(\Lambda\mathbf{G})-\Lambda\nabla\cdot\mathbf{G} \ . \quad (1.10)$$

Then by making $\mathbf{L}=\mathbf{F}$ and $\mathbf{G}=\mathbf{J}$ from (1.6)–(1.10) and from the divergence theorem we obtain a first expression for Γ given by

$$\Gamma=-\int_S d\mathbf{S}\cdot(U\mathbf{J})=-\Upsilon_S\cdot(U\mathbf{J}) \ , \quad (1.11)$$

where $d\mathbf{S}$ is the oriented element, with the outward normal, of the boundary surface S of Ω and $\Upsilon_S=\int_S d\mathbf{S}$.

Now let us consider the product

$$\mathbf{F}\cdot\partial(\varepsilon\mathbf{E})/\partial t=-\mathbf{F}\cdot\partial(\varepsilon\nabla\psi)/\partial t \ ,$$

where $\mathbf{F}$ is a function of $\mathbf{r}$ and t, $\mathbf{E}$ and ψ are functions of $\mathbf{r}+\boldsymbol{\rho}$ and $t-\tau$, and $\nabla=\nabla_r$. From (1.10) we have

$$\mathbf{F}\cdot\frac{\partial(\varepsilon\mathbf{E})}{\partial t}=-\nabla\cdot\left[\frac{\partial(\varepsilon\psi)}{\partial t}\mathbf{F}\right]+p(\mathbf{F}) \ , \quad (1.12)$$

$$p=\frac{\partial\psi}{\partial t}\nabla\cdot(\varepsilon\mathbf{F})+\psi\nabla\cdot\left[\frac{\partial\varepsilon}{\partial t}\mathbf{F}\right] =\frac{\partial(\varepsilon\psi)}{\partial t}\nabla\cdot\mathbf{F}+\frac{\partial(\psi\nabla\varepsilon)}{\partial t}\cdot\mathbf{F} \ , \quad (1.13)$$

so that from (1.1), (1.3), (1.4), (1.6), (1.9), and (1.11)–(1.13) we obtain the final equations

$$\int_S d\mathbf{S}\cdot(U\mathbf{J})=-\sum_{i=1}^{N} q_i\mathbf{v}_i(\mathbf{r}_i)\cdot\mathbf{F}(\mathbf{r}_i)+H_S(\mathbf{F})-H_\Omega(\mathbf{F}) \ , \quad (1.14)$$

$$H_S=\Upsilon\left[\Upsilon_S\cdot\left[\frac{\partial(\varepsilon\psi)}{\partial t}\mathbf{F}\right]\right],\ H_\Omega=\Upsilon_\Omega(p(\mathbf{F})) \ , \quad (1.15)$$

which connect $\mathbf{J}$, ψ, and ε to the carrier velocity $\mathbf{v}_i$ though the arbitrary vector $\mathbf{F}$ and its potential $\phi=U$ on S.

II. INFERENCES

A. Induced currents. Ramo's theorem and its extensions

Let us apply Eq. (1.14) to some cases by specifying the value of $\mathbf{F}$ or of $U(\mathbf{r},t)$ over S.

For this let us divide S in n surfaces S_h $(h=1,\ldots,n)$ and, as a first case, let us define $\mathbf{F}=\mathbf{F}_{1h}(\mathbf{r})$ through $\nabla\cdot\mathbf{F}=0$ in Ω and $U=1$ over a given S_h and $U=0$ elsewhere on S. Then from (1.14) we have

$$i_h=\sum_{i=1}^{N} q_i\mathbf{v}_i(\mathbf{r}_1)\cdot\mathbf{F}_{1h}(\mathbf{r}_i)-H_S(\mathbf{F}_{1h})+H_\Omega(\mathbf{F}_{1h}) \ , \quad (2.1)$$

where $i_h=-\Upsilon_{S_h}\cdot\mathbf{J}$ is the current entering Ω across S_h and, according to (1.13) and (1.15), we have

$$H_\Omega=\frac{\partial}{\partial t}[\Upsilon_\Omega(\psi\mathbf{F}_{1h}\cdot\nabla\varepsilon)] \ , \quad (2.2)$$

which, therefore, becomes null for an homogeneous medium.

Moreover, if the surface $S_{(k=1,\ldots,n)}$ is metallized, so that the potential ψ assumes the same value $V_k(t)$ on it, we have

$$H_S=\frac{\partial}{\partial t}\sum_{k=1}^{n}\Upsilon(V_k(t-\tau)\Upsilon_S\cdot(\varepsilon\mathbf{F}_{1h})) \ . \quad (2.3)$$

Equations (2.1)–(2.3) allow one to compute the current i_h induced across S_h by the moving carriers and by the time variations of the medium permittivity, the internal potential, and the electrode voltages in the most general case and, hence, also in that of (1.5). However, in the latter case it is more convenient to define $\mathbf{F}=\mathbf{F}'_{1h}$ through $\nabla\cdot(\varepsilon_0\mathbf{F}'_{1h})=0$, and again $U=1$ over S_h because, according to (1.13)–(1.15), one also has $H_\Omega(\mathbf{F}'_{1h})=0$ for inhomogeneous media and (2.1) becomes

$$i_h=\sum_{i=1}^{N} q_i\mathbf{v}_i(\mathbf{r}_i)\cdot\mathbf{F}'_{1h}(\mathbf{r}_i)-\sum_{k=1}^{n} C_{hk}\frac{\partial V_k}{\partial t} \ , \quad (2.4)$$

$C_{hk}=\Upsilon_{S_k}\cdot(\varepsilon_0\mathbf{F}'_{1h})$ being the capacitance between the surfaces S_h and S_k.

For time-independent and -dependent potentials V_k, (2.4) gives Ramo's theorem[1] and the Schockley result,[5] respectively, extended from the vacuum to any material and electric field for which (1.5) holds true.

As a second case let us now define $\mathbf{F}=\mathbf{F}_v$ through the equation

$$\nabla\cdot\Upsilon(\varepsilon\mathbf{F}_v(\mathbf{r}+\boldsymbol{\rho},t-\tau))=0 \ , \quad (2.5)$$

in Ω and the boundary condition $U=V_h$ over $S_h(h=1,\ldots,n)$, that is, we choose as $\mathbf{F}_v$ the electric field due to the electrodes with their actual potential in the absence of any volume charge distribution except the one due to dielectric polarization if the medium is not uniform. Then from (1.9), (1.11), and (1.13)–(1.15) we get the equation

$$\sum_{h=1}^{n} V_h i_h=\sum_{i=1}^{N}\mathbf{v}_i(\mathbf{r}_i)\cdot\mathbf{F}_v(\mathbf{r}_i)-H_S(\mathbf{F}_v)+H_\Omega(\mathbf{F}_v)=\int_\Omega d\mathbf{r}(\mathbf{F}_v\cdot\mathbf{J}) \ , \quad (2.6)$$

which, as well as (2.1), holds true in the most general conditions.

In the case of (1.5), instead, (2.5) becomes $\nabla\cdot(\varepsilon_0\mathbf{F}_v)=0$, so that we have $\mathbf{F}_v=\sum_{h=1}^{n}V_h\mathbf{F}'_{1h}$. Then, according to (1.13) and (1.15), (2.6) becomes

$$\sum_{h=1}^{n} V_h i_h=\sum_{i=1}^{N} q_i\mathbf{v}_i(\mathbf{r}_i)\cdot\mathbf{F}_v(\mathbf{r}_i)-\sum_{k=1}^{n}\sum_{h=1}^{n} C_{hk}V_h\frac{\partial V_k}{\partial t} \ , \quad (2.7)$$

which, when V_k's are constant, reduces itself to the extension of Ramo's theorem to the semiconductor detectors, due to Cavalleri *et al.*[2]

B. Energy and current balance

It is worthwhile to show that (2.6) expresses the energy balance.

For this we observe that its first member,

$$\sum_{h=1}^{n} V_h i_h=\frac{\partial\Xi}{\partial t} \ , \quad (2.8)$$

represents the work entering the system in unit time, whereas

$$\int_\Omega \left[\mathbf{E}\cdot\mathbf{J}_c + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} \right] d\mathbf{r} = \int_\Omega (\mathbf{E}\cdot\mathbf{J}) d\mathbf{r} = \frac{\partial W}{\partial t} \quad (2.9)$$

is the corresponding variation of its internal energy.

If we put $\mathbf{E}=\mathbf{F}_v+\mathbf{E}_q$ where $\mathbf{F}_v$ has been previously defined and $\mathbf{E}_q=-\nabla\psi_q$ is the electric field due to the actual charge distribution and to the boundary condition $\psi_q(\mathbf{r},t)=0$ for $\mathbf{r}$ in S, from (1.6) and (1.10), where we put $\mathbf{G}=\mathbf{J}$ and $\mathbf{L}=\mathbf{E}_q$, we have

$$\int_\Omega \mathbf{E}_q\cdot\mathbf{J} d\mathbf{r} = -\int_\Omega \nabla\cdot(\psi_q \mathbf{J}) d\mathbf{r} + \int_\Omega \psi_q \nabla\cdot\mathbf{J} d\mathbf{r} = 0 \ . \quad (2.10)$$

Therefore, according to (2.8)–(2.10), in effect (2.6) verifies the energy balance $\partial\Xi/\partial t=\partial W/\partial t$ and it allows one to compute the various components.

Such a balance, according to (2.6) itself, seems to be independent of $\mathbf{E}_q$. However, this is not true because $\mathbf{E}_q$ affects the carrier velocities.

As a further application of (1.14), and as a check, let us assume that $\phi(\mathbf{r},t)=U(\mathbf{r},t)=U(t)$ is independent of $\mathbf{r}$ in Ω and S. In this case from (1.7) we have $\mathbf{F}=0$ and then from (1.13)–(1.15) we get $U(t)\sum i_h(t)=0$, that is (1.14) also allows one to deduce Kirchoff's law for the currents which, of course, according to (1.6), may be obtained in a direct way from $\int_S \mathbf{J}\cdot d\mathbf{S}=\int_\Omega \nabla\cdot\mathbf{J} d\Omega=0$.

C. Noise

The Eq. (1.14) connects the output parameters, currents, and potentials to the corpuscular nature and behavior of the single charge carriers and to their motion, and, in particular, to their velocity fluctuation. For such characteristics (1.14) may be used as a powerful tool to study the noise phenomena of electrical systems.

In order to show this fact it is suitable to choose a vector $\mathbf{F}(\mathbf{r},t)=F(t)\hat{\mathbf{u}}$ which is independent of $\mathbf{r}$, $\hat{\mathbf{u}}$ being a unit vector independent of both $\mathbf{r}$ and t, because in this way the motion contribute to (1.14), unlike in Ramo's and Shockley's equation (2.4), becomes dependent only on the component $v_{ui}(\mathbf{r}_i)=\mathbf{v}_i(\mathbf{r}_i)\cdot\hat{\mathbf{u}}$ along $\hat{\mathbf{u}}$ of the carrier velocity. This result is important for any following statistical computation of fluctuations and noise.

In virtue of such choosing of $\mathbf{F}$, from (1.7), (1.8), and (1.13)–(1.15) we have $\phi=-F(t)\ (\mathbf{r}\cdot\hat{\mathbf{u}}+a)$, a being an arbitrary constant, and

$$\sum_{h=1}^{n} \int_{S_h} d\mathbf{S}\cdot(\mathbf{r}\cdot\hat{\mathbf{u}})\mathbf{J} = \sum_{i=1}^{N} q_i v_{ui} - H_S(\hat{\mathbf{u}}) + H_\Omega(\hat{\mathbf{u}}) \ , \quad (2.11)$$

where, since $\nabla\cdot\hat{\mathbf{u}}=0$, H_Ω and H_S are given again by (2.2) and (2.3) after the substitution of $\mathbf{F}_{1h}$ with $\hat{\mathbf{u}}$ so that also in this case, in particular, we have $H_\Omega=0$ for a homogeneous medium.

In order to see the facilities of the new Eq. (2.11) for analyzing noise phenomena, let us apply it to compute, in the case of (1.5), the thermal noise of a conducting cylinder in which $\hat{\mathbf{u}}$ is chosen parallel to its x axis and the terminal surfaces $S_1=S_2=S$, put at $x=0$ and $x=w$, respectively, unlike the lateral one S_3, are metallized.

From (1.5), (2.2), (2.3), and (2.11), for a sample homogeneous along the x axis and $q_i=q$, the current i_1 across S_1 becomes

$$i_1=\frac{q}{w}\sum_{i=1}^{N} v_{ui} + C\frac{dV}{dt} + \int_{S_3} d\mathbf{S}\cdot(1-x/w)\mathbf{J}_d \ , \quad (2.12)$$

where $V=V_1-V_2$ and $C=w^{-1}\mathbf{Y}_{S_2}\cdot(\varepsilon_0\hat{\mathbf{u}})$ is the sample capacitance.

Let us apply (2.12) just to compute the power spectral density S_i of i_1 when the terminals 1 and 2 are shortcircuited, that is, when it is $V=0$ and, accordingly,

$$i_1=-i_2=i, \quad \mathbf{Y}_{S_3}\cdot\mathbf{J}_d=0, \quad \Delta N(t)=0 \ ,$$

and $\langle v_{ui}\rangle_t=0$.

Therefore, from (2.12), whose second member reduces to its first term when, at least at no high frequencies, the contribution $\mathbf{Y}_{S_3}\cdot(xw^{-1}\mathbf{J}_d)$ can be neglected too, the correlation function of i, since v_{ui}'s are uncorrelated, becomes

$$\langle i(t)i(t+s)\rangle_t = \frac{q^2}{w^2}\sum_{i=1}^{N}\langle v_{ui}^2(t)\rangle_t \exp\left[-\frac{|s|}{\tau_c}\right] = \frac{q^2 N k_B T}{w^2 m}\exp\left[-\frac{|s|}{\tau_c}\right] \ , \quad (2.13)$$

where k_B is Boltzmann's constant, T is the absolute temperature, m is the carrier effective mass, and τ_c is the correlation time.[6]

Since $\mu=q\tau_c/m(1+j\omega)$ and $G=q\mu N/w$ are the carrier mobility and the sample conductance,[6] respectively, from the Wiener-Khintchine theorem applied to (2.13) we obtain

$$S_i=4k_B T\,\mathrm{Re}[G(j\omega)] \ , \quad (2.14)$$

in accordance with Nyquist's theorem; ω is the circular frequency.

The new Eq. (2.12), after adding the inhomogeneous contribution

$$w^{-1}\int_\Omega d\mathbf{r}(\partial\psi/\partial t)(\partial\varepsilon_0/\partial x) \ ,$$

and especially (2.11), are quite general and they may be used as bases to study any noise type[7] in any conducting systems.

Of course in order to study the noise of a system by means of (2.11), when (1.5) does not hold true, as well as to compute the induced current and the components of the internal power $\partial W/\partial t$ by means of (2.1) and (2.6), respectively, the system under analysis has to be specified through $\varepsilon(\mathbf{r},\rho;t,\tau)$.

New results may be reached through such new general tools, especially for the open questions on the noise.

In particular Eqs. (1.14), (2.1), (2.4), (2.6), (2.7), and, especially (2.11) and (2.12), by connecting the motion of the single carriers to the external currents, voltages, and powers of the system, can be directly and usefully applied in the numerical simulations of the electric, transport, and noise phenomena, such as the Monte Carlo method, because they, indeed, compute the microscopic quantities, such as velocity and distributions of the particles, as well as the macroscopic collective ones, directly from the microscopic motion and behavior of the single particles.

Finally, we also observe that the Eqs. (2.11) and (2.12) also hold true when $\mathbf{F}=F(t)\hat{\mathbf{u}}$ becomes a low frequency magnetic field $H(t)\hat{\mathbf{u}}$ which, however, acts on the velocities $\mathbf{v}_i$.

III. CONCLUSIONS

A new general equation has been found which, through an arbitrary irrototional vector and its potential, connects the boundary currents, voltages, and powers of an electric system to the carrier motion and to the dielectric-property variation occurring inside the system itself. It holds true when the vector potential is negligible with respect to the scalar one, that is, according to the system sizes, even up to microwave field.[4]

Through suitable definitions of the arbitrary vector such an equation then allows one to obtain Ramo's theorem, its extension to semiconductor detectors, and its present generalization to any boundary condition and conducting medium characterized by whatever permittivity depending on the space and on the frequency. It also leads to the energy and current balances and to the computation of the internal power components in the most general conditions.

Finally, apart from other applications, the equation allows one to obtain new simple realtionships which, by connecting in a straightforward way the corpuscular nature and motion of the single carriers to the output currents and voltages, give general and efficacious tools to study transport and noise phenomena of electrical systems and to solve open problems both in closed analytical form and through direct numerical simulations such as the Monte Carlo method.

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[2]G. Cavalleri, E. Gatti, G. Fabbri, and V. Svelto, Nucl. Instrum. Methods **92**, 137 (1971).

[3]In general, ε and ε_d may depend on t owing to any physical and chemical cause, e.g., flow of the fluid medium, temperature variations, chemical reactions and so on, which changes the medium dielectric properties.

[4]The vector potential of $\mathbf{E}$ is negligible with respect to the scalar one when the squared maximum size of Ω is much smaller than the squared minimum wavelength of the electromagnetic field in the medium [J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), pp. 23–25], so that the "quasielectrostatic" model which we are proposing may hold true up to the microwave field, e.g., up to tens of Gigahertz in the integrated circuits.

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