end impedance of the quarter-wave line, and θ represents the electrical length of the line. In this figure the solid line is obtained from (11). It will be observed that the experimental points are in close agreement with the theoretical curve, the greatest departure being of the order of 5 per cent. It is interesting to note that there is no indication of the optimum predicted by the previous theoretical results, shown by the dashed line.

We must conclude, therefore, that the short-line calculations based on previous theoretical formulas are in error, and that the theory presented in this paper is adequate for the design of transmission-line circuits.

Currents Induced by Electron Motion*

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Summary—*A method is given for computing the instantaneous current induced in neighboring conductors by a given specified motion of electrons. The method is based on the repeated use of a simple equation giving the current due to a single electron's movement and is believed to be simpler than methods previously described.*

Introduction

In designing vacuum tubes in which electron transit-time is relatively long, it becomes necessary to discard the low-frequency concept that the instantaneous current taken by any electrode is proportional to the number of electrons received by

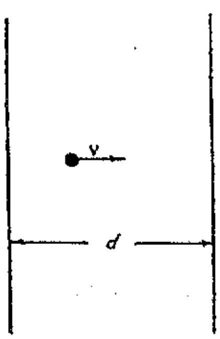

Fig. 1

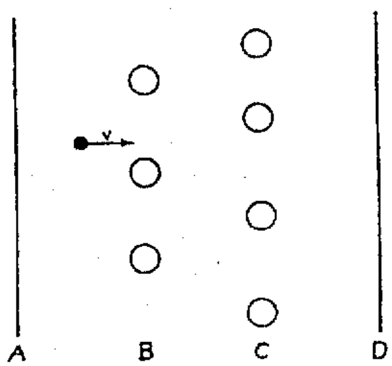

Fig. 2

it per second. Negative grids, it is known, may carry current even though they collect no electrons and current may be noted in the circuit of a collector during the time the electron is still approaching the collector. A proper concept of current to an electrode must consider the instantaneous change of electrostatic flux lines which end on the electrode and the methods given in the literature for computing induced current due to electron flow are based on this concept.

A method of computing the induced current for a specified electron motion is here explained which is believed to be more direct and simpler than methods previously described. In the more difficult cases, in which flux plots or other tedious field-determination methods must be used, only one field plot is needed by the present method while the usual methods require a large number.

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Method of Computation

The method is based on the following equation, whose derivation is given later:

$$i = E_v e v \tag{1}$$

where i is the instantaneous current received by the given electrode due to a single electron's motion, e is the charge on the electron, v is its instantaneous velocity, and E_v is the component in the direction v of that electric field which would exist at the electron's instantaneous position under the following circumstances: electron removed, given electrode raised to unit potential, all other conductors grounded. The equation involves the usual assumptions that induced currents due to magnetic effects are negligible and that the electrostatic field propagates instantaneously.

Simple Example

A simple example is offered in the computation of the instantaneous current due to an electron's motion between two infinite plates (Fig. 1). (The result is a starting point for the analysis of a diode, for example, when the transit-time is long.)

From (1) we obtain immediately

$$i = evE_v = \frac{ev}{d}\cdot$$

In the literature[1] it is stated that this same result is deduced from image theory. This involves the setting up of an infinite series of image charges on each side of the plates for a given position of the electron and a consideration of the total flux crossing one of the planes due to the series of charges, a method which is lengthy and requires no little familiarity with methods of handling infinite series.

The General Case

Consider a number of electrodes, A, B, C, D, in the presence of a moving electron (Fig. 2) whose path and instantaneous velocity are known. A tedious way to find the current induced in, say, electrode

[1] D. O. North, "Analysis of the effects of space charge on grid impedance," Proc. I.R.E., vol. 24, pp. 108–158; February, (1936).

A is to make a flux plot of the lines of force emanating from the electron, when it is at some point of its path, and note the portion of the total lines which end on A. By making a number of such plots it is possible to observe the change in the number of lines ending on A as the electron moves, and consequently to compute the induced current. The accuracy is dependent upon the number of plots made.

It is much simpler to use (1). One plot is made for the case of A at unit potential, B, C, D grounded, and the electron removed, E_v is then known and

$$i = E_v ev.$$

To minimize the induced current in a negative grid, an important consideration in the design of high-frequency amplifiers and oscillators, it may be that (1) will prove helpful to the designer. The equation states that the electrode configuration should be such as to yield minimum E_v. If the electron's path, for example, is made to coincide with an equipotential of the grid (not an equipotential in the field in which the electron is traveling, of course, but an equipotential in that artificial field due to unit potential on the grid, the electron removed, and all else grounded) the induced current will be zero. It will not be possible to realize this for the complete electronic path, since the electron must start at some equipotential surface, but it may be possible to find practical configurations that will approach this condition over a good share of the path.

Derivation of Equation (1)

Consider the electron, of charge e, in the presence of any number of grounded conductors, for one of which, say A, the induced current is desired. Surround the electron with a tiny equipotential sphere. Then if V is the potential of the electrostatic field, in the region between conductors

$$\nabla^2 V = 0$$

where ∇^2 is the Laplacian operator. Call V_e the potential of the tiny sphere and note that $V=0$ on the conductors and

$$-\int_{\text{sphere's surface}} \frac{\partial V}{\partial n} ds = 4\pi e \qquad \text{(Gauss' law)}$$

where $\partial V/\partial n$ indicates differentiation with respect to the outward normal to the surface and the integral is taken over the surface of the sphere.

Now consider the same set of conductors with the electron removed, conductor A raised to unit potential, and the other conductors grounded. Call the potential of the field in this case V', so that $\nabla^2 V' = 0$ in the space between conductors, including the point where the electron was situated before. Call the new potential of this point V_e'.

Now Green's theorem[2] states that

$$\int_{\substack{\text{volume}\\\text{between}\\\text{boundaries}}} [V'\nabla^2 V - V\nabla^2 V']dv = -\int_{\substack{\text{boundary}\\\text{surfaces}}} \left[V'\frac{\partial V}{\partial n} - V\frac{\partial V'}{\partial n}\right]ds. \qquad (2)$$

Choose the volume to be that bounded by the conductors and the tiny sphere. Then the left-hand side is zero and the right-hand side may be divided into three integrals:

(1) Over the surfaces of all conductors except A. This integral is zero since $V = V' = 0$ on these surfaces.

(2) Over the surface of A. This reduces to $-\int_{\text{surface } A} (\partial V)/(\partial n) ds$, for $V' = 1$ and $V = 0$ for conductor A.

(3) Over the surface of the sphere. This becomes

$$-V_e' \int_{\text{sphere's surface}} \frac{\partial V}{\partial n} ds + V_e \int_{\text{sphere's surface}} \frac{\partial V'}{\partial n} ds.$$

The second of these integrals is zero by Gauss' law since $\int(\partial V')/(\partial n)ds$ is the negative of the charge enclosed (which was zero for the second case in which the electron was removed).

Finally, we obtain from (2)

$$0 = -\int_{\text{surface } A} \frac{\partial V}{\partial n} ds - V_e' \int_{\text{sphere's surface}} \frac{\partial V}{\partial n} ds$$

$$= 4\pi Q_A + 4\pi e V_e'$$

or

$$Q_A = -eV_e'$$

$$i_A = \frac{dQ_A}{dt} = -e\frac{dV_e'}{dt} = -e\left[\frac{\partial V_e'}{\partial x}\frac{dx}{dt}\right]$$

where x is the direction of motion.

Now

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{\partial V_e'}{\partial x} = -E_v,$$

so

$$i = evE_v. \qquad (1)$$

[2] J. H. Jeans, "Electricity and Magnetism," page 160, Cambridge, London, England, (1927).