A Necessary Addition to Kirchhoff's Current Law of Circuits
Version 2

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Abstract

Kirchhoff's current law for circuits is about the flux $\mathbf{J}$ of charge, usually electrons, and is derived for systems without significant time dependence of total charge. Yet Kirchhoff's current law is used to design circuits on nanosecond time scales where time dependence is important. Maxwell's equations are general and deal with time dependence. They imply conservation of the source term for the magnetic field. $\mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial t$ is conserved. It does not accumulate. $\mathbf{J}$ accumulates. $\mathbf{J}$ accumulates as charge ρ as the electric field changes. $\mathbf{J}$ is not conserved. It is necessary to add a term $\varepsilon_0\, \partial\mathbf{E}/\partial t$ to reconcile Kirchhoff's law with the Maxwell Ampere equation. But this term is not sufficient to allow a derivation of Kirchhoff's law for general circuits. Additional constraints are needed that depend on details of the real circuit not usually included in simplified circuit diagrams, like the skin effect of current flow in wires and the locations and amount of stray capacitance. Kirchhoff's current law can made general for the simplified circuits used by circuit designers, if the displacement term $\varepsilon_0 \partial\mathbf{E}/\partial t$ is included in a revised definition of current $\mathbf{J}_{total}$ in Kirchhoff's current law $\mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\, \partial\mathbf{E}/\partial t$. Including stray capacitance everywhere automatically adds the needed term $\varepsilon_0\, \partial\mathbf{E}/\partial t$. Perhaps that is why the need for the added terms has not been widely recognized.

Introduction

Kirchhoff's current law is used widely to help design the circuits of our technology that respond in nanoseconds [1-5]. Kirchhoff's law has been used to design much slower circuits for nearly a century [6-9].

Kirchhoff's current law is about the flux $\mathbf{J}$ of charges in circuits, often the flux of electrons, and does not deal with the rate of change of the total charge in the circuit. The rate of change does not appear in a term in the usual formulation of Kirchhoff's current law. But the rates of change of charge and electric field are not small in circuits that respond in nanoseconds. The mechanisms and properties of current flow vary significantly between nanoseconds and seconds in wires [4, 5, 10] and other components of circuits as well [11-18]. If Kirchhoff's current law is derived without time dependence, circuit designers are likely to have concerns about using it to construct modern circuits where potentials change volts in nanoseconds. Their concern is likely to increase when they realize that Kirchhoff's current law (for flux of charge) is incompatible with the conservation law implied by the Maxwell equations, as we shall derive in eq. (2).

Methods

Electrical phenomenon—slow (sec) and fast (nsec), even optical (fsec)—are described by the Maxwell equations [19-23] that do depend on the rate of change of the electric field. Our approach starts with them. Our method is to try to derive Kirchhoff's current law for circuits from the Maxwell equations, specifically from the Maxwell Ampere law.

We show how adding a term $\varepsilon_0\, \partial\mathbf{E}/\partial t$ to the usual Kirchhoff's law—symbols defined after eq. (1)—helps deal with rapidly changing currents, but not enough to allow a derivation of Kirchhoff's current valid for any circuits. Adding the term $\varepsilon_0\, \partial\mathbf{E}/\partial t$ is necessary but other conditions are often needed that depend on the physical properties and layout of the circuit.

Theory

The Maxwell Ampere law is the start of our derivation.

$$\mathbf{curl\,B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{1}$$

Here, μ_0 is the magnetic constant; ε_0 is the electric constant; **E** is the electric field, and **J** is the flux of all charge with mass, however brief or transient, including the polarization charge of dielectrics.

The Maxwell Ampere law includes time. The rate of change of the electric field $\partial \mathbf{E}/\partial t$ appears on the right-hand side of the law as part of the source for **curl B** in eq. (1). Specifying **curl B** (and boundary conditions) is enough to specify **B** according to the Helmholtz decomposition theorem because $\mathbf{div\,B} = 0$, which is another Maxwell equation see [21, 24].

The flux **J** of charges is a source of the magnetic field along with displacement current $\varepsilon_0\, \partial \mathbf{E}/\partial t$ in eq. (1). The displacement term is needed to describe light propagating in a vacuum, as Maxwell himself discovered long ago [19-23]. In a vacuum, the displacement term is the only source of **curl B**. The displacement term $\varepsilon_0\, \partial \mathbf{E}/\partial t$ was once thought to be a property of the aether [19, 25]. Many scientists have been puzzled by its existence in a vacuum devoid of matter [26], so I like to call $\varepsilon_0\, \partial \mathbf{E}/\partial t$ an ethereal current, in deference to its intangible nature as well as its history, with a smile on my face.

Results

Our main results concern the conservation of current.

Conservation of Current. Conservation of current involves the displacement term $\varepsilon_0\, \partial \mathbf{E}/\partial t$ because conservation is described by the divergence operator **div** of vector calculus that can be applied to both sides of eq. (1).

$$\mathbf{div}\,(\mathbf{curl\,B}) = 0 = \mathbf{div} \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{2}$$

Then,

$$\mathbf{div} \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0 \tag{3}$$

or

$$\begin{gathered}\mathbf{div}\,\mathbf{J}_{total} = \mathbf{0} \\ \textbf{where}\quad \mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t\end{gathered} \tag{4}$$

Eq. (3) is a general conservation law true whenever the Maxwell equations are true. $\mathbf{J}_{total} = \mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t$ is conserved in general. $\mathbf{J}_{total}$ never accumulates anywhere. $\mathbf{J}_{total}$ is incompressible to the accuracy of the Maxwell equations themselves, wherever they apply. It describes a perfectly incompressible flow.

The conservation law eq, (3) also implies that the flux of charge $\mathbf{J}$ is ***not*** conserved when the electric field varies in time.

$$\begin{aligned}\mathbf{div}\,\mathbf{J} &\neq 0, \quad \text{when } \partial\mathbf{E}/\partial t \neq 0 \\ &= -\varepsilon_0\,\partial\mathbf{E}/\partial t\end{aligned} \tag{5}$$

This result comes as no surprise: the continuity equation—derived from eq.(1) and Gauss' law—shows that charge ρ is stored; $\mathbf{J}$ is not conserved. $\mathbf{div}\,\mathbf{J}$ changes $\partial\rho/\partial t$ and is stored as charge ρ.

$$\mathbf{div}\,\mathbf{J} = -\,\partial\rho/\partial t \tag{6}$$

ρ includes all charge with mass, including polarization charge on dielectrics.

Kirchhoff's current law for circuits implies that the flux $\mathbf{J}$ of charge is conserved in circuits. Maxwell's equations imply that the flux of of charge is not conserved when electric fields or charges change with time. Conservation of total current $\mathbf{J}_{total}$ is in conflict with Kirchhoff's current law for the flux $\mathbf{J}$ of charge (in circuits). Compare eq. (6) or (5) and (3) or (4). Kirchhoff's law needs to be changed if it is to be used when charges and fields change with time, if it is to be compatible with the Maxwell equations.

It is necessary then to add a term $\varepsilon_0\,\partial\mathbf{E}/\partial t$ to Kirchhoff's current law (for circuits) if it is to be consistent with the Maxwell Ampere law for time varying electric fields. If Kirchhoff's law is applied to the 'total current' $\mathbf{J}_{total}$ defined by eq. (4), it becomes consistent with the Maxwell equations under all conditions (even when fields and charges change with time). Total current $\mathbf{J}_{total}$ is then conserved in circuits just as it is conserved wherever the Maxwell equations apply.

I hasten to say that this definition of total current eq. (4) is ***not*** sufficient to ensure that the revised Kirchhoff's law (for total current in circuits) is consistent with the Maxwell Ampere law. The addition of the term $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is necessary as in eq. (4), but it is not sufficient, a point I have not always understood, e.g., [27, 28].

Circuits must satisfy other conditions besides conservation of total current eq. (4) to be consistent with the Maxwell equation (1). In that sense, Kirchhoff's law cannot be made exact in a general way or for every real circuit. Kirchhoff's current law can made general for the simplified circuits used by circuit designers, displayed in their textbooks [1-3] if the displacement term $\varepsilon_0\partial\mathbf{E}/\partial t$ is included in the definition of current as in [27, 28].

The other conditions needed to make Kirchhoff's current law 'exact', are difficult to define broadly in mathematical form, because the sufficient conditions are as diverse as systems and circuits. Kirchhoff's law requires that the total current (and power [29, 30]) remain in the circuit itself. It requires that total current be reasonably well behaved in components and wires of the circuit (e.g., not dominated by the skin effect [4, 10]). The sufficient requirements and conditions are not apparent in the diagrams of circuits that are analyzed with Kirchhoff's laws, yet the additional requirements may be important. The sufficient conditions depend on the properties of components and the location of the stray capacitances that link everything, including structures outside the circuit itself [5].

Discussion

Stray Capacitance. Real circuits always include stray capacitances C_i [1, 2]. These are typically of the order of 10^{-12} to 10^{-10} farads and are important on time scales something like R_iC_i where R_i is an effective resistance. Without stray capacitance, the idealized circuits of textbooks [1] are accurate when $t \gg \mathrm{R_i}C_i$. Stray capacitances can be important in modern day circuits even when $R_i = 100$ ohms. Stray capacitances are needed when real circuits are designed p. 579–587 of [1].

The stray capacitances of engineering practice describe the displacement current $\varepsilon_r\,\varepsilon_0\partial\mathbf{E}/\partial t$, where ε_r is the dielectric constant $\varepsilon_r \geq 1$. Stray capacitance arising from

$\varepsilon_r\ \varepsilon_0 \partial \mathbf{E}/\partial t$ includes the displacement current $\varepsilon_0 \partial \mathbf{E}/\partial t$ of eq. (1) – (4) as a component, because the dielectric term $\varepsilon_r\ \varepsilon_0 \partial \mathbf{E}/\partial t$ includes ε_0 for all values of $\varepsilon_{r,}$ see p. 9-10 of [2]

Stray capacitances add the displacement current $\varepsilon_0 \partial \mathbf{E}/\partial t$ to the classical $\mathbf{J}$ of Kirchhoff's law as is required by eq. (2).

Historical Remark. Changing Kirchhoff's law of current is not needed if stray capacitances have already been included and analyzed throughout a circuit. Stray capacitances are almost never included in circuit diagrams but are included in special cases as necessary, p. 579–587 of [1].

Kirchhoff's current law can made general for the simplified circuits used by circuit designers, displayed in their textbooks [1-3], if the displacement term $\varepsilon_0 \partial \mathbf{E}/\partial t$ is included in the definition of current, as in [27, 28].The need to ***always*** add displacement terms to Kirchhoff's law (according to eq. (1)-(4)) has not received much attention, probably because the terms are already included if stray capacitances are included everywhere in a circuit.

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